Annie’s Survival Kit 6 - Math 324

1. (10 points) Use Green’s theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = \langle 2y + \cos(x), 4x - e^y \rangle \) and \( C \) is given by \( \mathbf{r}(t) = \langle \cos(t) + 2, \sin(t) + 1 \rangle \) for \( t \in [\frac{\pi}{4}, \frac{5\pi}{4}] \).

2. (10 points) (a) (7 points) Express the area of the region bounded by \( \mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle \) for \( 0 \leq t \leq \pi \) with an integral of the form \( \int_{t_0}^{t_1} f(t) \, dt \). **Do not evaluate.** Hint: think about how \( \sin(2t) \) and \( \sin(t) \) behave (which increases? decreases?) when \( t \in [0, \frac{\pi}{4}] \), when \( t \in [\frac{\pi}{4}, \frac{\pi}{2}] \), when \( t \in [\frac{\pi}{2}, \frac{3\pi}{4}] \) and when \( t \in [\frac{3\pi}{4}, \pi] \). Moreover, recall that the area of some region \( R \) is equal to \( \iint_R 1 \, dA \).

Finally, let \( D \) be some region and \( C \) its counterclockwise boundary, then Green’s theorem states that, if \( \mathbf{F} \) is defined and differentiable everywhere on \( D \), then \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl} (\mathbf{F}) \, dA \) where \( \text{curl} (\mathbf{F}) = N_x - M_y \) for \( \mathbf{F} = \langle M, N \rangle \).

(b) (3 points) Using the following trigonometric identities, evaluate the integral you found in (a):

\[
\sin^3(t) = \frac{3 \sin(t) - \sin(3t)}{4}, \quad \sin(2t) = 2 \sin(t) \cos(t), \quad \cos^2(t) - \sin^2(t) = 1 - 2 \sin^2(t).
\]

3. (10 points) Let \( \mathbf{r}(t) = \langle 2 \cos(t), \sin(2t) \rangle \) for \( 0 \leq t \leq \pi \). Express the mass of the region between \( \mathbf{r}(t) \) and the \( x \)-axis with density equal to twice the distance from the \( y \)-axis with integrals of the form \( \int_{t_0}^{t_1} f(t) \, dt \). Make sure \( f(t) \) is free of absolute values. **Do not evaluate.**