1. (10 points) Use Green's theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = \langle 2y + \cos^2(x), 4x - e^y \rangle \) and \( C \) is given by \( r(t) = \langle \cos(t) + 2, \sin(t) + 1 \rangle \) for \( t \in \left[ \frac{\pi}{4}, \frac{3\pi}{4} \right] \).

To use Green's theorem, I need to add \( C' \) to have a closed curve.

\[
\mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} - \int_{C'} \mathbf{F} \cdot d\mathbf{r}
\]

So, \( C' \) is the portion of the circle from \( t = 0 \) to \( t = \frac{\pi}{4} \) and from \( t = \frac{3\pi}{4} \) to \( t = \pi \).

Using Green's theorem:

\[
\iint_R (N_y - M_x) \, dA = \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}
\]

\[
N_x = \frac{d}{dt} (2t^2 + \sin t - e^{t-1}) = 4t + \cos t - e^{t-1}
\]

\[
M_y = \frac{d}{dt} (-2tx = t - 2 + \cos t = t - 2 + \cos t)
\]

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \iint_R (N_y - M_x) \, dA
\]

Area of \( R \) which is half a disk \( \Rightarrow \frac{\pi}{2} \) of radius 2.

\[
= \pi - \left( 3 \left( \frac{\sqrt{2} \pi}{2} + 2 \right)^2 - 2 \left( \frac{\sqrt{2} \pi}{2} + 2 \right) \sin \left( \frac{\sqrt{2} \pi}{2} + 2 \right) - e^{\frac{\sqrt{2} \pi}{2} + 1} \right)
\]

\[
+ 3 \left( 2 - \frac{\sqrt{2} \pi}{2} \right)^2 - 2 \left( 2 - \frac{\sqrt{2} \pi}{2} \right) \sin \left( 2 - \frac{\sqrt{2} \pi}{2} \right) - e^{1 - \frac{\sqrt{2} \pi}{2}}
\]
2. (10 points) (a) (7 points) Express the area of the region bounded by \( r(t) = (\sin(2t), \sin(t)) \) for \( 0 \leq t \leq \pi \) with an integral of the form \( \int_0^\pi f(t) \, dt \). Do not evaluate. Hint: think about how \( \sin(2t) \) and \( \sin(t) \) behave (which increase? decrease?) when \( t \in [0, \frac{\pi}{4}] \), when \( t \in [\frac{\pi}{4}, \frac{\pi}{2}] \), when \( t \in [\frac{\pi}{2}, \frac{3\pi}{4}] \) and when \( t \in [\frac{3\pi}{4}, \pi] \). Moreover, recall that the area of some region \( R \) is equal to \( \iint_R 1 \, dA \). Finally, let \( D \) be some region and \( C \) its counterclockwise boundary, then Green’s theorem states that, if \( \mathbf{F} \) is defined and differentiable everywhere on \( D \), then \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl}(\mathbf{F}) \, dA \) where \( \text{curl}(\mathbf{F}) = N_x - M_y \) for \( \mathbf{F} = (M, N) \).

(b) (3 points) Using the following trigonometric identities, evaluate the integral you found in (a):

\[
\sin^3(t) = \frac{3\sin(t) - \sin(3t)}{4}, \sin(2t) = 2\sin(t)\cos(t), \cos(2t) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t).
\]

Area \( (R) = \iint_R 1 \, dA \)

Area \( (R) = \iint_R 1 \, dA \)

How can I set up the bounds for \( R \)? I only know \( C \) as parametric equation!

Use Green’s theorem: need to figure out what \( \mathbf{F} \) so that it holds.

\[
\iint_R 1 \, dA = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C x \, dy = \oint_0^\pi \sin(2t) \cdot \cos(t) \, dt \]

\[
\text{Green if curl } \mathbf{F} = 1
\]

Many possible \( \mathbf{F} : \)

\[
\left< \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right>
\]

\[
\left< -\frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right>
\]

\[
\left< -\frac{\partial}{\partial y} \right.
\]

Can pick any!
3. (10 points) Let \( \mathbf{r}(t) = (2 \cos(t), \sin(2t)) \) for \( 0 \leq t \leq \pi \). Express the mass of the region between \( \mathbf{r}(t) \) and the \( x \)-axis with density equal to twice the distance from the \( y \)-axis with integrals of the form \( \int_{0}^{\pi} f(t) \, dt \). Make sure \( f(t) \) is free of absolute values. **Do not evaluate.**

\[
S = 2 \cdot 1 \cdot 1
\]

Note that the mass of the right upper region is the same as the mass of the lower left region since the contribution of point \( \mathbf{r}(t^*) \) is \( 2 \cdot |2 \cos(t^*)| \) and so is the contribution of point \( \mathbf{r}(\pi - t^*) \) since \( 2 \cdot |2 \cos(\pi - t^*)| = 2 \cdot |2 \cdot (-\cos(t^*))| = 2 \cdot 2 \cdot \cos(t^*)| \) and for any \( t^* \in [0, \pi] \), \( \pi - t^* \) is also in \([0, \pi]\). Therefore, the mass is

\[
2 \cdot \int_{0}^{\pi} 2 \cdot \cos(t) \, dt = 4 \int_{0}^{\pi} \cos(t) \, dt
\]

Here again, I don’t know how to set the bounds, so I need to use Green’s theorem. To do so, I need to close \( C \) with \( C' \)

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \nabla \times \mathbf{F} \cdot d\mathbf{a}
\]

**Green’s Theorem**

\[
\text{curl} \mathbf{F} = \nabla \times \mathbf{F} = (0, \frac{2}{x^2})
\]

\[
\mathbf{F} = (0, \frac{2}{x^2})
\]

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \nabla \times \mathbf{F} \cdot d\mathbf{a} = 4 \int_{0}^{\pi} \left[ -\int_{2}^{0} (2 \cos(t), \sin(2t), 2 \cdot (-\sin(t))) \, dt + \int_{0}^{2} t \cdot 0 \cdot t \, dt \right]
\]

**Physics and Math:**

\[
\text{mass} = 16 \int_{0}^{\pi} 2 \cos(t) \sin(2t) \sin(2t) \, dt
\]