Taylor's theorem (3.2).

$f(x)$ is a smooth function

$f: \mathbb{R} \to \mathbb{R}$.

Taylor's thm for one variable:

$$f(x_0+h) = f(x_0) + f'(x_0)\cdot h + f''(x_0)\cdot \frac{h^2}{2} + \dots + \frac{f^{(k)}(x_0)}{k!} h^k + \boxed{R_k(x_0,h)}$$

$\longleftarrow$ reminder

$$R_k(x_0,h) = \int_{x_0}^{x_0+h} \frac{(x_0+h-\tau)^k}{k!} f^{(k+1)}(\tau)\, d\tau.$$

If $\left|f^{(k+1)}(\tau)\right| \leq M \Rightarrow$

$$\left|R_k(x_0,h)\right| \leq M \int_{x_0}^{x_0+h} \frac{(x_0+h-\tau)^k}{k!} d\tau = M \frac{h^{k+1}}{k!}$$

Idea of the proof: use integration by parts.

$$f(x_0+h) = f(x_0) + \int_{x_0}^{x_0+h} f'(h)\, d\tau$$

$$d\tau = -d(x_0+h-\tau) \Longrightarrow$$

$$-\int_{x_0}^{x_0+h} f'(\tau)\, d(x_0+h-\tau) = -f'(\tau)(x_0+h-\tau)\Big|_{x_0}^{x_0+h} +$$

$$\int_{x_0}^{x_0+h} f''(\tau)(x_0+h-\tau)\, d\tau = f'(x_0)h +$$

$R_1(x_0;h)$.

Next we apply the integration by parts trick to $R_1(x_0,h)$.

$$(x_0+h-\tau)\, d\tau = -\frac{d(x_0+h-\tau)^2}{2}$$

$$-\int_{x_0}^{x_0+h} f''(\tau)\, \frac{d(x_0+h-\tau)^2}{2} = \frac{f''(\tau)(x_0+h-\tau)^2}{2}\Big|_{x_0}^{x_0+h} -$$

$$+\frac{1}{2}\int_{x_0}^{x_0+h} f''''(\tau)(x_0+h-\tau)^2\,d\tau.$$

We can iterate this method to get the general Taylor formula

Examples $f(x) = \sin(x)$

$x_0 = 0$ $f(0)=0$ $f'(0) = \cos(0) = 1$

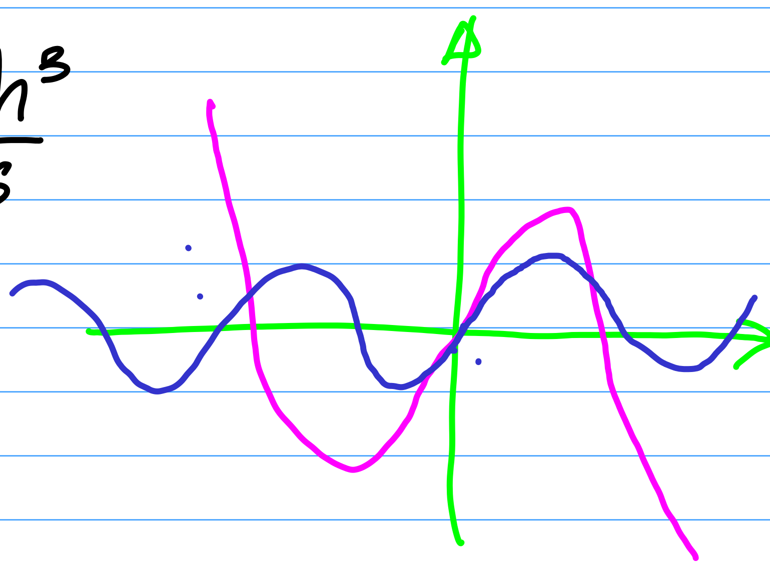
$f''(0) = -\sin(0) = 0$ $f'''(0) = -\cos(0) = -1$

$f(h) = \sin(h) = 0 + 1\cdot h + 0\cdot\frac{h^2}{2} - \frac{1}{3!}h^3 + R_3(h)$. Since $|f''''| \leq 1$

we have an estimate

$$|R_3(h)| \leq \frac{1}{4!}h^4.$$

$$\sin(h) \approx h - \frac{h^3}{6}$$



Funny function $f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

Multivariable Taylor's

Thm 2 (1st order)

$$f(x_0 + \vec{h}) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + R_1(x_0, \vec{h}), \quad \frac{R_1(x_0, \vec{h})}{\|\vec{h}\|} \xrightarrow[\vec{h} \to 0]{} 0.$$

Thm 3 (Second order)

$$f(x_0 + \vec{h}) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + \frac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + R_2(x_0, \vec{h}),$$

$$\frac{\|R_2(x_0,\vec{h})\|}{\|\vec{h}\|^2} \longrightarrow 0 \qquad h \to 0.$$

<u>Reminder</u> $\|\vec{h}\| = \sqrt{h_1^2 + \dots + h_n^2}$

Idea of proof.

Consider function

$$g(t) = f(x_0 + t\vec{h})$$

and write Taylor's formula for $g(t)$. Use chain rule:

$$g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0 + t\vec{h})\, h_i$$

Similarly for the second derivatives.

<u>**Example**</u> $f(x,y) = \sin(x+2y)$

Second order Taylor formula at $x_0 = (0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \cos(0+2\cdot 0) = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \cos(0+2\cdot 0)\cdot 2 = 2.$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0)$$

$$f(\vec{h}) = h_1 + 2h_2 + R_2(0,\vec{h})$$

$$R_2(0,\vec{h}) \approx \|\vec{h}\|^3$$

Other way: just use Taylor expansion for $\cos(x)$.

Example $f(x,y) = e^x \cos y$

$x_0 = 0 \quad y_0 = 0.$

Second order Taylor expansion

$f(0,0) = 1 \qquad \frac{\partial f}{\partial x}(0,0) = 1 \quad \frac{\partial f}{\partial y}(0,0) = 0$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 1, \quad \frac{\partial^2 f}{\partial y^2}(0,0) = -1, \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0.$$