

## PROBLEMS ON QUADRATIC FORMS

ALEXEI OBLOMKOV

The quadratic form is a homogeneous function of degree 2. That is function on  $\mathbb{R}^n$ :

$$Q(\vec{x}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j x_i, \quad \vec{x} \in \mathbb{R}^n.$$

**Theorem** There is a linear change of variables

$$y_i = \sum_{j=1}^n b_{ij} x_j, \quad 1 \leq i \leq n,$$

such that

$$Q(\vec{y}) = \sum_{i=1}^n \epsilon_i y_i^2$$

where  $\epsilon_i$  is either 0 or 1 or  $-1$ . The sequence of  $\epsilon_i$  is called the *signature of the quadratic form*. We also call the change of variables in the theorem *simplifying change of variables*.

For example if

$$Q(y_1, y_2, y_3, y_4) = y_1^2 - y_3^2 + y_4^2,$$

then the corresponding quadratic form has signature  $(+, 0, -, +)$ . The change of variables in the theorem is not unique but the signature of the quadratic form is unique! That is the number of 0, + and  $-$  does not depend on the change of variables you choose.

In class we discussed a method for finding a linear change of variables from the theorem. The problems below give chance to practice this method and derive some conclusions about the quadratic forms.

**Problem 1.** Find a simplifying change of variables and the signature for the quadratic form:

$$Q(x_1, x_2) = 4x_1^2 - 4x_1x_2 + 5x_2^2.$$

The point  $(0, 0)$  is the critical point of  $Q$ , is it maximum, minimum or neither?

**Problem 2.** Find a simplifying change of variables and the signature for the quadratic form:

$$Q(x_1, x_2) = x_1^2 + 2x_1x_2 - 20x_2^2.$$

The point  $(0, 0)$  is the critical point of  $Q$ , is it maximum, minimum or neither?

**Problem 4.** Find a simplifying change of variables and the signature for the quadratic form:

$$Q(x_1, x_2, x_3) = 2x_1^2 - x_1x_2 + 2x_2^2 - x_2x_3 + 2x_3^2.$$

The point  $(0, 0, 0)$  is the critical point of  $Q$ , is it maximum, minimum or neither?

**Problem 5.** Find a simplifying change of variables and the signature for the quadratic form:

$$Q(x_1, x_2, x_3, x_4) = x_1^2 - 4x_1x_2 + 5x_2^2 + x_3^2 + x_3x_4 - 16x_4^2.$$

The point  $(0, 0, 0, 0)$  is the critical point of  $Q$ , is it maximum, minimum or neither?