Math 235 Practice Midterm 1.

Instructions: Exam time is 2 hours. You are allowed one sheet of notes (letter-size paper, both sides). Calculators, the textbook, and additional notes are not allowed. Justify all your answers carefully.

Q1.

(a) Compute the reduced row echelon form of the matrix

\[
A = \begin{bmatrix}
1 & 3 & -1 & 1 \\
1 & 5 & -3 & 1 \\
-2 & -4 & 0 & -1 \\
3 & 5 & 1 & 7
\end{bmatrix}
\]

(b) Is the matrix \( A \) invertible?

(c) Which entries are pivot entries?

(d) When solving the equation \( A \mathbf{x} = \mathbf{b} \) with this \( A \), which of the following are possible: there are no solutions, there is one unique solution, there are infinitely many solutions. Justify your answer.

Q2. Let \( A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -3 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 1 \end{bmatrix} \).

(a) Find the general solution of the equation \( A \mathbf{x} = \mathbf{b} \). Write your solution in vector form.

(b) Using your answer to part (a) or otherwise, find the general solution of the equation \( A \mathbf{x} = \mathbf{0} \).

(c) Does the equation \( A \mathbf{x} = \mathbf{c} \) have a solution for every vector \( \mathbf{c} \) in \( \mathbb{R}^4 \)? Justify your answer carefully.

Q3. Consider the vectors

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}
\]
(a) Are the vectors $v_1, v_2, v_3$ linearly independent?

(b) Do the vectors $v_1, v_2, v_3$ span $\mathbb{R}^3$?

Justify your answers carefully.

**Q4.**

(a) Let $S: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects onto the $y$-axis and then rotates clockwise by $\pi/4$ radians. Find the standard matrix of $S$. Is $S$ one-to-one? Is $S$ onto?

(b) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation that maps $T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Write the matrix corresponding to $T$. Is $T$ one-to-one? Is $T$ onto?

**Q5.**

(a) Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 7 \\ -1 & 1 & 5 \end{bmatrix}$$

(b) Using your answer to part (a) or otherwise, solve the system of linear equations

$$x_1 + x_2 + 2x_3 = 2$$
$$2x_1 + 3x_2 + 7x_3 = 3$$
$$-x_1 + x_2 + 5x_3 = 5$$

**Q6.** Consider the equation $BC(2A - 3X)DE = F$ for an unknown $n \times n$ matrix $X$. Assume that $A$, $B$, $C$, $D$, $E$, and $F$ are all invertible $n \times n$ matrices.

(a) Write a solution $X$ in terms of $A$, $B$, $C$, $D$, $E$, and $F$. Is this solution unique? Explain why or why not.

(b) Can we allow any of the matrices $A$, $B$, $C$, $D$, $E$, or $F$ to be singular and still guarantee that a solution $X$ exists? Justify your answer.