BORDERED HESSIANS: DISCUSSION.

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Reminder: the Hessian Hess(F) of a function $F = F(x_1, \dots, x_n)$ of n variables is the $n \times n$ matrix of the second partial derivatives. The gradient $\nabla F(x_1, \dots, x_n)$ is the n-dimensional vector of the first partial derivatives.

Problem 1 Let $g_1(x,y,z) = y, g_2(x,y,z) = z$ and $f(x,y,z)$ is a differential function. We introduce

$$F(x,y,z,\lambda,\mu) = f(x,y,z) - \lambda g_1(x,y,z) - \mu g_2(x,y,z).$$

- Show that the Lagrange system for the critical points of f with constraints $g_1(x,y,z) = g_2(x,y,z) = 0$:
$$\nabla F(x_0, y_0, z_0, \lambda_0, \mu_0) = (0,0,0,0,0)$$
is equivalent to the one-dimensional critical point equation:
$$\frac{df}{dx}(x_0, 0, 0) = 0, \quad y_0 = z_0 = 0.$$
- Let $(x_0, y_0, z_0, \lambda_0, \mu_0)$ be a solution of the Lagrange system. Compute the determinant of 5×5 matrix that is the Hessian Hess$(F)(x_0, y_0, z_0, \lambda_0, \mu_0)$.
- What can you say about the relation between the second derivative $\frac{d^2 f}{dx^2}(x_0, 0, 0)$ and the determinant from the previous question. In particular, when the determinant is strictly negative, what is the type of the critical point?

Problem 2 Let $g(x,y,z) = z$ and $f(x,y,z)$ is a differential function. We introduce

$$F(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z).$$

- Show that the Lagrange system for the critical points of f with constraints $g(x,y,z)$=0:
$$\nabla F(x_0, y_0, z_0, \lambda_0) = (0,0,0,0,0)$$
is equivalent to the one-dimensional critical point equation:
$$\frac{\partial f}{\partial x}(x_0, y_0, 0) = 0, \quad \frac{\partial f}{\partial y}(x_0, y_0, 0) = 0, \quad z_0 = 0.$$
- Let $(x_0, y_0, z_0, \lambda_0)$ be a solution of the Lagrange system. Compute the determinant of 4×4 matrix that is the Hessian Hess$(F)(x_0, y_0, z_0, \lambda_0)$.
- What can you say about the relation between the determinant of the Hessian Hess$(\tilde{f})(x_0, y_0)$ two variable function $\tilde{f}(x,y) = f(x,y,0)$ at (x_0, y_0) and the determinant from the previous question. In particular, when the (x_0, y_0) is a saddle point of $\tilde{f}$ what is the sign of the determinant?