

FUN WITH DIRICHLET FUNCTION.

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Let us define Dirichlet function by:

$$D(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

Problem 1 Show that the Dirichlet function is not integrable on $[0, 1]$.

Let us define modified Dirichlet function on $(0, 1]$ by:

$$D_m(x) = \begin{cases} 1/q, & x = p/q \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases},$$

where in the definition we assume that p, q are co-prime.

Problem 2. This problem is a bit challenging. Show that D_m is integrable on $[1/2, 1]$.

Problem 3. Let us define $f(x, y) = xD(y)$. Show that the iterated integral:

$$\int_0^1 \int_{-1}^1 f(x, y) dx dy$$

is well-defined. On the other hand show that

$$\int_{-1}^1 \int_0^1 f(x, y) dy dx$$

is not well-defined.