Notes 10: Monomial Ideals

Let \( f = xy^2 - x, \ f_1 = xy + 1, \ f_2 = y^2 - 1 \). When we divide \( f \) by \( f_1, f_2 \) we obtain
\[
xy^2 - x = y(xy + 1) + 0(y^2 - 1) + (-x - y).
\]
When we divide \( f \) by \( f_2, f_1 \), we obtain
\[
xy^2 - x = y(xy + 1) + 0.
\]

From the second computation we conclude that \( f \in I = \langle f_1, f_2 \rangle \), and from this we see that the remainder from the first calculation, namely \((-x - y)\), is in \( I \) also. We want to make a guess as to why the first calculation failed, and from that guess see how to improve the division algorithm. The difficulty is that neither of the leading terms of \( f_1, f_2 \) divides either of the terms in the remainder \(-x - y\). Our plan is to adjoined to the list \( f_1, f_2 \) more terms in the ideal \( f_1, f_2 \) so that we have sufficiently many leading terms. For example if we adjoin the term \(-x - y\) (which is in \( I \)) to the list \( f_1, f_2 \), then we would not have run into this difficulty.

**Example 1.** We use the lexicographic order with \( x > y \) in \( k[x, y] \). Let \( I = \langle f_1 = x3y^2 + y, f_2 = xy^3 - x \rangle \). The element \( g = yf_1 - xf_2 = x^2 - xy \) is an element of \( I \), but neither of its leading terms is divisible by either of the leading terms of \( f_1 \) or \( f_2 \). If we applied the division algorithm to \( g \) we would get \( g = 0f_1 + 0f_2 + x^2 - xy \). The division algorithm fails to detect that \( g \in I \). Again the problem is the paucity of leading terms appearing in our list of generators.

Our plan is to add more terms to the set of generators of a given ideal, so that we have more leading terms. The first step in carrying out this program is to look at 'leading terms' more carefully.

**Definition 1.** An ideal \( I \) that can be written
\[
I = \{ \sum a_\alpha x^\alpha | \alpha \in A, a_\alpha \in k \}
\]
for some \( A \subseteq \mathbb{Z}_{\geq 0}^n \) is a monomial ideal.

**Example 2.** Let \( I = \langle x^2 \rangle \). Then \( I \) is a monomial ideal and \( A = \{ n \in |Z, n \geq 2 \} \).

**Example 3.** Let \( I = \langle xy^3, x^4y \rangle \). We have the notation:
\[
(a, b) + \mathbb{Z}_{\geq 0}^2 = \{(a, b) + (x, y) | (x, y) \in \mathbb{Z}^2 \text{ such that } x \geq 0, y \geq 0 \}.
\]
Let
\[
A = ((1, 3) + \mathbb{Z}_{\geq 0}^2) \cup ((4, 1) + \mathbb{Z}_{\geq 0}^2) .
\]
Then \( I \) is the monomial ideal \( < x^\alpha, \alpha \in A > \).

It is often easier to use a figure to see exactly what the set \( A \) looks like.

**Note.** Let \( I \) be a monomial ideal. If \( f = \sum a_\alpha x^\alpha, h \in k \) is an element of \( I \), then each of the terms \( h_\alpha x^\alpha \) is also in \( I \).
Note. Let $B \subseteq \mathbb{Z}_{\geq 0}^n$. The ideal $J = \langle x^\alpha, \alpha \in B \rangle$ is a monomial ideal. Indeed if we set

$$A = \bigcup_{\alpha \in B} (\alpha + \mathbb{Z}_{\geq 0}^n),$$

then $J = \{ \sum_{\alpha \in A} h_\alpha x^\alpha \}$.

Corollary 1. Two monomial ideals are the same iff they contain the same monomials.

Theorem 1. Dickson’s Lemma Let $I = \langle x^\alpha, \alpha \in A \rangle$ be a monomial idea. Then there exists a finite subset $B \subseteq A$ so that $I = \langle x^\alpha, \alpha \in B \rangle$.

Proof. We prove the theorem by induction on the number of variables. If there is a single variable $x = x_1$, then $I = \langle x^\alpha, \alpha \in A \cap \mathbb{Z}_{\geq 0} \rangle$. Let $\beta$ be the smallest element in this list. Then for any $\alpha \in A$, $x^\alpha = x^\beta x^\gamma$ for some $\gamma \in \mathbb{Z}_{\geq 0}$. This says that $I$ is generated by the single element $x^\beta$.

We do the induction step. We assume that any monomial ideal in $n - 1$ variables generated by a subset $A \subseteq \mathbb{Z}_{\geq 0}^{n-1}$

- is generated by a finite set of monomials, and
- we may take these monomials from the set $A$.

We write $x_n = y$ so $k[x_1, \ldots, x_{n-1}, y] = k[x_1, x_2, \ldots, x_n]$. We can write any monomial in the form $x^\alpha y^m$ for some $\alpha \in \mathbb{Z}_{\geq 0}^{n-1}$. Let $J \subseteq k[x_1, x_2, \ldots, x_{n-1}]$ be the ideal of
Let $m$ be the maximum among the integers $m_\alpha$ so that $x^\alpha x^{m_\alpha} \in B^*$. Now let $B = \{x^\alpha y^{m_\alpha}|\alpha \in B^*\}$.

For each $k = 0, 1, \cdots < m$, let $J_k$ denote the ideal of $k[x_1, x_2, \cdots x_{n-1}]$ generated by $x^\alpha$ so that $x^\alpha y^k \in I$. By induction this ideal is finitely generated by elements $x^{\alpha_ik}, i = 1, \cdots, s_k$.

Let $B_k = \{x^{\alpha_ik}y^k\}$.

We claim that $I$ is generated by $B \cup B_0 \cup B_1 \cup \cdots \cup B_{m-1}$.

We prove the claim: Let $x^\beta y^t \in I$. If $t \geq m$, the there is an $\alpha \in B^*$ so that $x^\alpha$ divides $x^\beta$, furthermore, since $t \geq m$ we have $x^\alpha y^{m_\alpha}$ divides $x^\beta y^t$. One the other hand, suppose that $t < m$. In this case there is an element of $B_t$ that divides $x^\beta y^t$. We have shown that the ideal $I$ has a finite set of generators. Call this set of generators $C$.

We show that we can find a finite set of generators in the set $x^\alpha, \alpha \in A$. We have implicitly changed notation here. The notation $x^\alpha$ now refers to a monomial in the complete set of variables $x_1, x_2, \cdots, x_n$ and not just the first $n-1$ variables. For each element $x^\alpha$ in $C$ there is some element $x^\beta$ in $A$ so that $x^\beta$ divides $x^\alpha$. The set of such $x^\beta$ generates $I$ and is a subset of $x^\beta, \beta \in A$.

The above proof has a puzzling point. Can we proceed as follows: Instead of taking generators of the form $x^\alpha x^{m_\alpha}, \alpha \in B^*, m = max\{m_\alpha\}$, why not take as generators $x^\alpha y^{m_\alpha}$. Does this allow us to avoid dealing with the ideals $J_k$?

The answer is no. We look at an example. Let $I = \langle x y^3, x^4 y \rangle$.

- The ideal $J$ is generated by a single element $x$.
- This comes from the monomial $x y^3$.
- The monomial $x y^3$ does not generate $I$.
- We have $J_0 = (0), J_1 = x^4, J_2 = x^4$.
- Hence we add to the generator $x y^3$ coming from $J$, the generator $x^4 y$.

**Corollary 2.** Consider an total order on the monomials in $x_1, x_2, \cdots, x_n$. Assume it satisfies the property:

$$x^\alpha > x^\beta \iff x^{\alpha+\gamma} > x^{\beta+\gamma}.$$  

Then this is a well ordering if and only if $x^\alpha > x^0$ for all non-zero $\alpha$. 


Proof. We have already done the proof in the direction $\implies$. We use the theorem above to prove the implication $\iff$.

Let $A \subseteq \mathbb{Z}_{\geq 0}^n$ and let $I$ be the monomial ideal generated by $A$. By the theorem above the ideal has a finite set of generators, $\{x^{\alpha_1}, x^{\alpha_2}, \ldots x^{\alpha_t}\}$. Relabeling if necessary, we may assume that $x^{\alpha_1} < x^{\alpha_2} < \cdots < x^{\alpha_t}$. It is easy to see that $x^{\alpha_1}$ is the smallest element in the set $A$.