Notes 8: Kernel, Image, Subspace

Fix a matrix $M$ of size $n \times m$.

**Question.** For what $b \in \mathbb{R}^n$ does the system of equations

$$Fx = b$$

have a solution?

We can ask this same question in another way. The matrix $M$ induces a linear map: $M : \mathbb{R}^m \to \mathbb{R}^n$. By definition the image of $F$ is the set of $b \in \mathbb{R}^n$ so that the equation $F(x) = b$ has a solution. This set of all such $b$ has structure.

**Theorem 1.** Let $F$ be a linear map $\mathbb{R}^m \to \mathbb{R}^n$.

1. Let $y_1, y_2$ be elements in the image of $F$. Then $y_1 + y_2$ is also in the image of $F$.

2. Let $y$ be an element in the image of $F$ and $\lambda \in \mathbb{R}$. Then $\lambda y$ is also in the image of $F$.

**Proof.** Since each of $y_1, y_2$ is in the image of $F$, there are elements $x_1, x_2$ in the domain of $F$, that is, they are elements of $\mathbb{R}^m$, so that

$$F(x_1) = y_1, \quad F(x_2) = y_2.$$

Now

$$F(x_1 + x_2) = (\text{our function is linear}) F(x_1) + F(x_2) = y_1 + y_2.$$

This proves the first statement.

Since $y$ is in the image of $F$, there is an element $x$ in the domain of $F$ so that $F(x) = y$. Then

$$F(\lambda x) = (\text{our function is linear}) \lambda F(x) = \lambda y.$$

This says that $\lambda y$ is in the image of $F$. \qed

**Definition 1.** Let $S$ be a subset of $\mathbb{R}^n$. We say that $S$ is a subspace provided

- If $u, v$ are both elements of $S$, then so is $u + v$.
- If $u \in S$ and if $\lambda$ is any real number, then $\lambda \cdot u \in S$ also.

The above theorem says that the image of a linear map is a subspace. Indeed it is a fact that any subspace of $\mathbb{R}^n$ is the image of a linear map. Subspaces can appear in many other ways besides being images of linear maps.
Note that \( \begin{pmatrix} 0 \\ 0 \\ . \end{pmatrix} \) is always in the image of \( F \) when \( F \) is linear since

\[
F(0 \cdot x) = 0 \cdot F(x) = \begin{pmatrix} 0 \\ 0 \\ . \end{pmatrix}.
\]

For the next three examples we assume that \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is linear.

**Example 2.** Let \( D \) be the set of vectors in \( \mathbb{R}^2 \) that are with a distance of 1 from the origin. Can \( D \) be the image of \( F \)? another way of asking this question is to ask: Is \( D \) a subspace? No. We can find two elements in \( D \) so that if we add them the result is not in \( D \). In addition, if we multiply elements in \( D \) by a scalar they will not necessarily stay in \( D \).

**Example 3.** Let \( L \) be the line \( y = 3x + 1 \). Then \( L \) is not a subspace. In other words \( L \) can not be the image of some linear map \( F \) since \( \vec{0} \) is not an element of \( L \). In addition if we multiply an element in \( L \) by a scalar then it is not, in general, in \( L \).

**Example 4.** Let \( L \) be the line \( y = -2x \). Then \( L \) contains the zero vector, and is closed under addition and scalar multiplication. Hence it is a candidate for being an image of a linear map \( F \). In fact, it is the image of the linear map given by the matrix ( among others)

\[
\begin{pmatrix}
1 & 3 \\
-2 & -6
\end{pmatrix}.
\]

What are all the subsets of \( \mathbb{R}^2 \) that are closed under addition and scalar multiplication? What are all the subsets of \( \mathbb{R}^2 \) which are subspaces? Answer: The set consisting of the zero vector, the whole of \( \mathbb{R}^2 \), and the lines through the origin.

Given a matrix, say

\[
A = \begin{pmatrix}
1 & -1 & 2 \\
0 & 3 & -1 \\
1 & 2 & 1
\end{pmatrix}
\]

there are two ways of describing the image. One way is to look at the equation

\[
Ax = \begin{pmatrix} a \\ b \\ c \end{pmatrix}
\]

and use Gauss elimination to find the conditions, if any, on \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) that insure the equation can be solved.
Another way is based on the observation that

\[ A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + z \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}. \]

**Remark 5.** In general, we observe that a matrix times a vector is a linear combination of the column vectors of the matrix.

**Definition 2.** Let \( S = \{v_1, v_2, \ldots, v_m\} \) be a set of elements of \( \mathbb{R}^n \) and let \( a_i \in \mathbb{R} \). Then an element \( v \in \mathbb{R}^n \) of the form

\[ v = \sum_{i=1}^{m} a_i v_i \]

is a linear combination of the elements in \( S \).

We can rephrase our observation as

**Theorem 6.** The image of a matrix \( A \) is the set of all linear combinations of the column vectors of \( A \).

Denote the column vectors of \( A \) by \( A_i \), \( i = 1, 2, 3 \). Observe that \( 5/3 A_1 + (-1/3) A_2 = A_3 \). Thus the image is the set of all linear combinations of just \( A_1 \) and \( A_2 \).

**Definition 3.** Let \( S \) be a set of elements of \( \mathbb{R}^n \). Then the span of \( S \) is the set of all linear combinations of the elements of \( S \).

We can restate what we have just said by saying that the image of \( A \) is the span of the three vectors \( A_1, A_2, A_3 \). The image of \( A \) is the span of just \( A_1 \) and \( A_2 \) alone.

![Diagram](image_url)

**The Kernel**

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**Definition 4.** Let $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear function. The kernel of $F$ is the subset of all vectors $x \in \mathbb{R}^m$ such that $F(x) = 0$. We denote the kernel of $F$ by $\ker(F)$.

- The zero vector is always an element of the kernel of $F$ since $F(\vec{0}) = F(0 \cdot \vec{0}) = 0 \cdot F(\vec{0}) = \vec{0}$.
- The kernel of $F$ is closed under addition. If $x, y \in \ker(F)$, then
  \[ F(x + y) = F(x) + F(y) = \vec{0} + \vec{0} = \vec{0}. \]
  The statement $F(x + y) = \vec{0}$ says that $x + y$ is an element of $\ker(F)$.
- The kernel of $F$ is closed under scalar multiplication. If $x \in \ker(F), \lambda \in \mathbb{R}$, then
  \[ F(\lambda x) = \lambda F(x) = \lambda \vec{0} = \vec{0}. \]

Let $S$ be a set of vectors in $\ker(F)$, then the above two statements imply that any linear combination of elements in $S$ is also in $\ker(F)$.

We give an algorithm that constructs a small set of elements so that any element in $\ker(F)$ is an linear combination of the set we have constructed. Let

\[
A = \begin{pmatrix}
1 & -1 & 2 & 1 \\
2 & 1 & -3 & 0
\end{pmatrix}.
\]

We perform row operations as usual to find the solutions to the equation $Ax = 0$. If we write down the initial right column of the augmented matrix, it will be all zeros, but it is not necessary to record the right column of the augmented matrix since no matter what row operations we perform, the last column will always remain all zeros. In this example we get

\[
\begin{pmatrix}
1 & 0 & (-1/3) & 1/3 \\
0 & 1 & (-7/3) & (-2/3)
\end{pmatrix}.
\]

Thus we get the solutions to our system of linear equations are

\[
\begin{aligned}
x &= (1/3)s - (1/3)t \\
y &= (7/3)s + (2/3)t \\
z &= s \\
w &= t
\end{aligned}
\]

where $s, t \in \mathbb{R}$ maybe freely chosen. We can write this as

\[
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} = s \begin{pmatrix}
1/3 \\
7/3 \\
1 \\
0
\end{pmatrix} + t \begin{pmatrix}
-1/3 \\
2/3 \\
0 \\
1
\end{pmatrix}.
\]
This says that every solution to the equation $Ax = 0$, that is, every element in $ker(A)$ can be written as a linear combination of the two vectors

$$\begin{pmatrix} 1/3 \\ 7/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 2/3 \\ 0 \\ 1 \end{pmatrix}.$$ 

Thus these two vectors span $ker(F)$. 