Geometry of Linear Functions and Multiplication of Matrices

Let \( v = (1, 2) \in \mathbb{R}^2 \). Define the function

\[
proj_v : \mathbb{R}^2 \to \mathbb{R}^2
\]

to be the orthogonal projection onto the line through \( v \). What is the matrix representing this function?

By definition \( proj_v(u) = av \) where \( u = av + x, a \in \mathbb{R}^2 \) and \( x \) is orthogonal to \( v \). This means that

\[
<v, u> = <v, av> + 0,
\]

so that \( a = \frac{<u, v>}{<v, v>} \). Writing \( u = (u_1, u_2) \), we see that

\[
a = \frac{1}{5}u_1 + \frac{2}{5}u_2,
\]

and hence

\[
proj_v(u_1, u_2) = av = (u_1/5 + (2/5)u_2, (2/5)u_1 + (4/5)u_2).
\]

From here there are two ways of reading off the matrix of \( proj_v \). Note that the function \( (x, y) \mapsto (ax + by, cx + dy) \)

is given by the matrix

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix},
\]

so that the matrix of \( proj_v \) is

\[
\begin{pmatrix}
1/5 & 2/5 \\
2/5 & 4/5
\end{pmatrix}.
\]

another way to proceed is to calculate \( proj_v(e_1), proj_v(e_2) \). These give the columns of our sought for matrix.

Products of Matrices

Let \( f : S \to T, g : T \to U \) be two functions. The composition of \( f \) and \( g \) is denoted by \( g \circ f \) and is a function with domain \( S \) and target \( U \). It acts on an element \( s \) of \( S \) by

\[
x \mapsto f(s) = t \mapsto g(t).
\]

**Example 1.** Let \( f : \mathbb{R} \to \mathbb{R} \) be given by \( x \mapsto x^2 - 3x \). Let \( g : \mathbb{R} \to \mathbb{R} \) be given by \( x \mapsto 3x - 1 \). Then

\[
g \circ f : \mathbb{R} \to \mathbb{R}
\]

\[
x \mapsto x^2 - 3x = u \mapsto 3u - 1 = 3(x^2 - 3x) - 1 = 3x^2 - 9x - 1.
\]

**Example 2.** Let \( f : \mathbb{R}^2 \to \mathbb{R}^2, f : (x, y) \mapsto (3x - y, 2x + 5y) \) and let \( g : \mathbb{R}^2 \to \mathbb{R}^2, g : (x, y) \mapsto (2x + y, -x - y) \). We compute \( g \circ f \). We get

\[
(x, y) \mapsto (3x - y, 2x + 5y) = (u, v) \mapsto (2u + v, -u - v)
\]

\[
= (2(3x - y) + (2x + 5y), -(3x - y) - (2x + 5y)) = (8x + 3y, -5x - 4y).
\]
The point is that this is a mess. We have a better way of doing this. The functions $f$ and $g$ are associated to the matrices

$$F = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix},$$

and

$$G = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

We can compute the matrix of the composition of $f$ and $g$ from $F$ and $G$ easily. This operation is called matrix multiplication.

**Example 3.**

$$GF = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 + 1 \cdot 2 & 2 \cdot (-1) + 1 \cdot 5 \\ -1 \cdot 3 + (-1) \cdot 2 & (-1) \cdot (-1) + (-1) \cdot 5 \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ -5 & -4 \end{pmatrix}.$$

Note that this is the same result we got from calculating the composition of $f$ and $g$ directly.

Let $A$ and $B$ be matrices. We can multiply them provided their sizes are compatible. If $A$ is size $n \times m$ and $B$ is size $s \times t$, then we can multiply them if and only if $m = s$.

**Example 4.** Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 9 \\ 8 & 7 & 6 \end{pmatrix}, C = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}.$$

We can multiply $AB, BA, CB, AC$. We can’t multiply $BC$ or $CA$.

Let $A$ be a matrix of size $n \times m$ so that $I_n A$ and $A I_m$ make sense. When we perform the multiplication we get that both of these products are equal to $A$.

Multiplication of matrices, like composition of functions is not commutative. Very occasionally two matrices will commute.

**Example 5.** Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}, BA = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}.$$

We give some examples of matrix products that have geometric content.

**Example 6.** Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so $A$ is reflection across the $x$ axis. Then $A^2 = I_2$. We can see this both algebraically and geometrically.
Example 7. Let \( a = \cos(\theta), b = \sin(\theta) \), \( A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \), so that \( A \) is rotation by angle \( \theta \).

Let \( B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \), so that \( B \) represents stretching by a factor of \( \lambda \). Arguing geometrically we see that the composition of rotating and stretching produces a 'spiral' motion given by the matrix

\[
AB = \begin{pmatrix} \lambda a & \lambda(-b) \\ \lambda b & \lambda a \end{pmatrix}.
\]

Example 8. Let \( L \) be the line through the origin making angle \( \theta \) with the \( x \)-axis. What is the matrix representing reflection across the line \( L \)?

If \( L \) were the \( x \)-axis then this problem is easy. The matrix is \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

We write the reflection as a composition of motions.

- First we rotate the plane about the origin so that \( L \) comes to rest on the \( x \)-axis. this is rotation by angle \(-\theta\).
- Second we reflect about the \( x \)-axis.
- Third we rotate back so that the \( x \)-axis moves to the line \( L \).

We need to find the matrix \( R \) which rotates by angle \( \theta \) and the matrix \( S \) which rotates by angle \(-\theta\). Let \( a = \cos(\theta), b = \sin(\theta) \), then

\[
R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.
\]

Note that \( \cos(-\theta) = \cos(\theta) = a \) and \( \sin(-\theta) = -\sin(\theta) = -b \). Hence

\[
S = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.
\]

Since the reflection across \( L \) is the composition of rotation by \(-\theta\), reflection across the \( x \)-axis, and rotation by \( \theta \), the matrix of reflection across \( L \) is

\[
RAS = \begin{pmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{pmatrix}.
\]

Remark 9. Using trig identities we can express \( a^2 - b^2 \) and \( 2ab \) in terms of \( \sin(2\theta), \cos(2\theta) \).

Remark 10. The composition of rotating by \(+\theta\) and \(-\theta\) is the identity transformation. Verify that the product of the corresponding matrices \( S \) and \( R \) is the identity matrix.