Definition 1. A function consists of two sets, the domain and the target, and a rule which assigns to every element in the domain an element of the target.

Sometimes we use the word range instead of target.

Example 1. We define a function that we call $f$. Let our domain be $\mathbb{R}$, let the target be $\mathbb{R}$, and let our rule associate to $x \in \mathbb{R}$ the number $x^2$. We can write this as

$$f : \mathbb{R} \rightarrow \mathbb{R}$$
$$f : x \mapsto x^2.$$

Example 2. Define a function called name that associates to each person in the class their name. Here the domain is the set of people in the class and the target is the set of strings of letters.

We can write the idea of solving an equation using the concept of function.

Example 3. Let

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

be given by

$$x \mapsto x^2 + 3.$$

Solving the equation

$$x^2 + 3 = 7$$

means find an $x \in \mathbb{R}$ so that when we apply the function $f$ to it, we get out 7.

For the above function $f$ we can ask for what $a \in \mathbb{R}$ does the equation $f(x) = a$ have a solution? The answer is $\{a|a \geq 3\}$. This motivates the

Definition 2. Let $f : S \rightarrow T$ be a function with domain $S$ and target $T$. The set of $a \in T$ such that $f(x) = a$ has a solution is the image of $f$.

Example 4. Let $f : \mathbb{R} \rightarrow \mathbb{R}^2$ be given by the rule

$$x \mapsto x \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Then the target of $f$ is $\mathbb{R}^2$ and the image of $f$ is the line through the origin with slope 2.

Example 5. Consider the function

$$F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + 2y + 3z \\ -y + z \\ 2x + 3y + 7z \end{pmatrix}.$$
What is the image of \( F \)?

Answer: We want to find the elements \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \) so that the equations

\[
\begin{align*}
x + 2y + 3z &= a \\
-y + z &= b \\
2x + 3y + 7z &= c
\end{align*}
\]

have a solution. We use Gauss elimination. Starting with

\[
\begin{pmatrix}
1 & 2 & 3 & a \\
0 & -1 & 1 & b \\
2 & 3 & 7 & c
\end{pmatrix}
\]

we obtain

\[
\begin{pmatrix}
1 & 2 & 3 & a \\
0 & 1 & -1 & b \\
0 & 0 & 0 & c - 2a - b
\end{pmatrix}.
\]

We see this has a solution if and only if \( c - 2a - b = 0 \). Thus the image of \( F \) is the subset of \( \mathbb{R}^3 \) consisting of all \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) that satisfy this equation.

## Matrix Multiplication

A matrix is a rectangular array of numbers. If it has \( n \) rows and \( m \) columns we say that it has size \( n \times m \). The matrix

**Example 6.** The matrix \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) has size \( 3 \times 1 \).

**Example 7.** The matrix \( \begin{pmatrix} 2 & 2 \\ -2 & 0 \\ 1/2 & 8 \end{pmatrix} \) has size \( 3 \times 2 \).

We can multiply a matrix times a column vector provided the sizes are compatible. We must have that the number of columns of the matrix has to equal the number of rows of the column vector. We can multiple the matrix times the column vector in the first example but not in the second example.

**Example 8.** \( \begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \)

**Example 9.** \( \begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \)
We give a formula for performing this operation in the case of a $2 \times 3$ matrix times a column vector of length 3.

\[
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix}
= \begin{pmatrix}
a_1x + a_2y + a_3z \\
b_1x + b_2y + b_3z \\
\end{pmatrix}
\]

**Example 10.** Let \( A = \begin{pmatrix}
-2 & 1 \\
2 & 7 \\
1 & -3 \\
\end{pmatrix} \) and let \( x = \begin{pmatrix}
1 \\
-3 \\
\end{pmatrix} \). Then

\[
Ax = \begin{pmatrix}
-5 \\
-19 \\
10 \\
\end{pmatrix}.
\]

We can write down the function

\[
F : \mathbb{R}^3 \rightarrow \mathbb{R}^3
\]

given by

\[
\begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix}
\mapsto
\begin{pmatrix}
x + 2y + 3z \\
-y + z \\
2x + 3y + 7z \\
\end{pmatrix}.
\]

using matrix multiplication. It is the function

\[
\begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix}
\mapsto
\begin{pmatrix}
1 & 2 & 3 \\
0 & -1 & 1 \\
2 & 3 & 7 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix}.
\]

If we multiply a \( n \times m \) matrix times a column vector of length \( m \), then the output is a column vector of length \( n \). Using matrix notation we can write systems of linear equations as matrix equations. For example the system

\[
x + 2y + 3z = -4 \\
-y + z = 2 \\
2x + 3y + 7z = 0
\]

can be written as

\[
AX = \begin{pmatrix}
4 \\
2 \\
0 \\
\end{pmatrix}
\]

with

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
0 & -1 & 1 \\
2 & 3 & 7 \\
\end{pmatrix}, 
X = \begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix}.
\]