We construct a dictionary between some geometrical notions and some notions from linear algebra.

An element of \((x, y) \in \mathbb{R}^2\) corresponds to an arrow with tail at the origin in \(\mathbb{R}^2\) and head at the point \((x, y)\). Two arrows are the same vector if they have the same length and direction. This works the same way in \(\mathbb{R}^3\) or \(\mathbb{R}^n\) for any \(n\).

We can add vectors algebraically. We have \((x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\). This corresponds to the geometrical operation of placing the vector \((x_2, y_2)\) so that its tail is at the head of \((x_1, y_1)\). The result of adding is the vector with tail at the origin and head at the head of \((x_2, y_2)\). We can interchange the roles of the two vectors and we obtain the same result. Addition works the same in \(\mathbb{R}^3\).

We can multiply a vector \((x, y)\) by a scalar \(a \in \mathbb{R}\). Algebraically we get \(a(x, y) = (ax, ay)\). Geometrically this is the operation of stretching or squeezing the vector \((x, y)\) by a factor of \(a\) if \(a \geq 0\). If \(a < 0\), then we turn the vector around and stretch by a factor of \(|a|\).

We can combine the two operations. Let \(\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\) and \(\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}\). What is the geometrical picture of \(2\vec{u} - \vec{v}\)?

How do we visualize this? Start by sketching examples such as 

\[
2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \text{ or } -2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}.
\]

We can draw a picture of solving linear equations. We can look at each of the equations

\[
\begin{align*}
x - 3y &= 1 \\
2x + y &= -5
\end{align*}
\]

as defining a line, so the solution to the set of equations is the intersection of the lines. A single linear equation in variables \(x, y\) can be thought of as eliminating all the points in the plane except those satisfying the equation. This description allows us to see what happens when we are given two linear equations in two unknowns. Three different things can happen.

- The two corresponding lines can intersect in one point, the unique solution to the set of equations.
- The two lines can be parallel, so that there are no solutions.
- The two lines can be the same line, so there are infinitely many solutions.

We now give examples of these phenomena.

- The set of equations

\[
\begin{align*}
x + 2y &= 1 \\
x - y &= 2
\end{align*}
\]
has a unique solution. Using Gauss elimination, starting from

\[
\begin{pmatrix}
1 & 2 & | & 1 \\
1 & -1 & | & 2
\end{pmatrix}
\]

we obtain

\[
\begin{pmatrix}
1 & 0 & | & 5/3 \\
0 & 1 & | & 1/3
\end{pmatrix}.
\]

• The set of equations

\[
\begin{align*}
x - 2y &= 1 \\
2x - 4y &= 3,
\end{align*}
\]

leads to

\[
\begin{pmatrix}
1 & -2 & | & 1 \\
0 & 0 & | & 1
\end{pmatrix}.
\]

This has no solutions. Geometrically it corresponds to two parallel lines.

• The set of equations

\[
\begin{align*}
x - 2y &= 1 \\
2x - 4y &= 2
\end{align*}
\]

corresponds to two copies of the same line, so that the intersection is again that same line. Thus there are infinitely many solutions. In terms of Gauss elimination we get

\[
\begin{pmatrix}
1 & -2 & | & 1 \\
0 & 0 & | & 0
\end{pmatrix}.
\]

This corresponds to the equation

\[
x - 2y = 1.
\]

The set of solutions to this is

\[
x = 1 + 2s \\
y = s
\]

and \(s\) can take on any value.

We have been thinking of a line as all the points \((x, y)\) that satisfy a linear equation. We can think of this as eliminating all the pairs \((x, y)\) that do not satisfy the equation. There is another algebraic way of describing a line. We can give formulas that allow us to construct all the points on the line. Start with the equation

\[
2x - 3y = 1.
\]
This is associated to the augmented matrix

\[
\begin{pmatrix}
2 & -3 & 1 \\
\end{pmatrix}.
\]

We use Gauss elimination and we obtain

\[
\begin{pmatrix}
1 & -3/2 & 1/2 \\
\end{pmatrix}
\]

and from this we can write down the solutions as

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}.
\]

Here \( s \) is allowed to take on any value.

We interpret this geometrically. When \( s = 0 \), we obtain the point \( \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \) on the line. For each \( s \) we start at this point and travel a distance proportional to \( s \) in the direction of the vector \( \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \).

We look at planes. The set of solutions to a linear equation in three variables \( x, y, z \) is a plane. For example all the solutions to

\[
x - 2y + 3z = 2
\]

is a plane. The set of solutions to a set of linear equations in three variables is just the intersection of the corresponding planes. We look at several ways this works out.

- Consider the three equations

\[
x + y + z = 1 \\
x - y + 2z = 2 \\
2x + 3z = 3.
\]

After using Gauss elimination we obtain the augmented matrix

\[
\begin{pmatrix}
1 & 0 & 3/2 \\
0 & 1 & -1/2 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
3/2 \\
-1/2 \\
0
\end{pmatrix}
\]

and hence the set of solutions is

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3/2 \\ 1/2 \\ 1 \end{pmatrix}
\]

where \( t \) can take on any real value. This is a line. Geometrically we have three planes intersecting in a single line.
• We look at the three equations that give us the augmented matrix

\[
\begin{pmatrix}
1 & 1 & 1 & | & 1 \\
1 & -1 & 2 & | & 2 \\
2 & 0 & 3 & | & 4
\end{pmatrix}
\]

Using Gauss elimination, we obtain

\[
\begin{pmatrix}
1 & 0 & 3/2 & | & 3/2 \\
0 & 1 & -1/2 & | & -1/2 \\
0 & 0 & 0 & | & 1
\end{pmatrix}
\]

We see this has no solutions. How are the three planes configured?

• Look at the equations corresponding to the augmented matrix

\[
\begin{pmatrix}
1 & 1 & 1 & | & 1 \\
1 & -1 & 2 & | & 2 \\
2 & 1 & 3 & | & 4
\end{pmatrix}
\]

Using Gauss elimination (check all these calculations), we obtain

\[
\begin{pmatrix}
1 & 0 & 0 & | & -13/3 \\
0 & 1 & 0 & | & 11/3 \\
0 & 0 & 1 & | & 5/3
\end{pmatrix}
\]

Geometrically we have three planes intersecting in a single point.

We introduce the dot product or inner product: Let \( \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}, \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} \in \mathbb{R}^n \). We define

\[
\vec{v} \cdot \vec{u} = \langle \vec{v}, \vec{u} \rangle = \sum_{1}^{n} v_i u_i.
\]

This has the following properties.

1. It is symmetric.

2. It is linear in each variable separately.

3. \( \langle \vec{u}, \vec{u} \rangle \geq 0 \) unless \( \vec{u} = \vec{0} \), and, in particular, when \( u \in \mathbb{R}^2 \) or \( u \in \mathbb{R}^3 \), \( \langle \vec{u}, \vec{u} \rangle \) is the square of the length of \( \vec{u} \).

4. Define the length of a vector \( \vec{u} \) to be the square root of \( \langle \vec{u}, \vec{u} \rangle \). Denote the length of a vector \( \vec{u} \) by \( ||u|| \). Then \( \langle \vec{u}, \vec{v} \rangle = ||u|| ||v|| \cos \theta \) where \( \cos \theta \) is the angle between the vectors \( \vec{u}, \vec{v} \). In particular,

\[
\langle \vec{u}, \vec{v} \rangle = 0 \iff \vec{u} \perp \vec{v} \iff \vec{u} \text{ is orthogonal to } \vec{v}.
\]
Problem: Construct the set of all vectors orthogonal to $\vec{u} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$.

Solution: We want all the vectors $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so that $< \vec{u}, \vec{v} > = 0$ or $2x - 2y + 3z = 0$. We use Gauss elimination. Starting from
\[
\begin{pmatrix} 2 & -2 & 3 & | & 0 \\
\end{pmatrix},
\]
we obtain
\[
\begin{pmatrix} 1 & -1 & 3/2 & | & 0 \end{pmatrix}.
\]
The solutions are
\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ -3/2 \end{pmatrix}.
\]

Problem: Fix a vector $\vec{v} \in \mathbb{R}^3$. For an arbitrary vector $\vec{u} \in \mathbb{R}^3$ we want to find $a \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^3$ so that
\[
\vec{u} = a\vec{v} + \vec{x},
\]
and
\[
< \vec{x}, \vec{v} > = 0.
\]
We call $a\vec{v}$ the projection of $\vec{u}$ on the line through $\vec{v}$ or the projection of $\vec{u}$ on $\vec{v}$.

Solution: Write $\vec{x} = \vec{u} - a\vec{v}$. We have
\[
0 = < \vec{x}, \vec{v} >= < \vec{u} - a\vec{v}, \vec{v} >
\]
so we must have
\[
< \vec{u}, \vec{v} >= < a\vec{v}, \vec{v} >.
\]
Thus
\[
a = \frac{< \vec{u}, \vec{v} >}{< \vec{v}, \vec{v} >}.
\]

Now we can determine $\vec{x}$ from the formula $\vec{x} = \vec{u} - a\vec{v}$.

Notice that the definition of orthogonal is given algebraically. From that algebraic definition of dot product we develop the properties of the dot product. We can use our intuition about the geometry of the dot product to guide us, but it is not a logical necessity. Since we are not tied to the geometry we can use these ideas in a wide variety of situation, everything from statistics to quantum mechanics.

The last part of the notes concerns two problems that lead to linear equations.

Given data of the form $x = x_i$ at $t = t_i$, how do you find a polynomial of sufficiently low degree that matches the data, that is, how do you find a polynomial $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots$ so that $f(t_i) = x_i$?
Problem: Assume that $x = 1$ at $t = 1$, $x = 4$ at $t = 2$, and $x = 8$ at $t = 3$. Find a polynomial $f(t)$ of degree 2 so that $f(1) = 1, f(2) = 4, f(3) = 8$.
Solution: Let $f(t) = a_2 t^2 + a_1 t + a_0$. We require that

\begin{align*}
a_2 + a_1 + a_0 &= 1 \\
4a_2 + 2a_1 + a_0 &= 4 \\
9a_2 + 3a_1 + a_0 &= 8.
\end{align*}

In this case we see how the techniques of linear algebra are applied to dealing with non-linear functions.

Problem: Find a polynomial of degree 3 that passes through the point $(x = 0, y = 0)$ and has horizontal tangent line at $x = 0$. In addition, we want this curve to go through the point $(x = 5, y = 6)$ and have horizontal tangent line at $x = 5$.
Solution: Write $f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$. That $f$ passes through the given points puts conditions on the coefficients of $f$. In particular, we require

\begin{align*}
a_0 &= 0 \\
125a_3 + 25a_2 + 5a_1 + a_0 &= 6.
\end{align*}

The requirement that $f$ have horizontal tangents at $x = 0$ and $x = 5$ imposes the conditions

\begin{align*}
a_1 &= 0 \\
75a_3 + 10a_2 + a_1 &= 0.
\end{align*}