Notes 21: Eigenvalues, Eigenvectors
Lecture December 1, 2009

**Definition 1.** Let $F : V \to V$ be a linear map. An eigenvalue for $F$ is a number, $\lambda$, real or complex, so that there exists a non-zero vector $v \in V$ so that $F(v) = \lambda v$. The vector $v$ is an eigenvector for $F$ with eigenvalue $\lambda$.

Our goal is to find the eigenvalues, eigenvectors of a given matrix. Let $A$ be an $n \times n$ matrix. If $\lambda$ is an eigenvalue for $A$, then there is a non-zero vector $v \in \mathbb{R}^n$ so that

$$Av = \lambda v \text{ or } (A - \lambda I_n)v = 0.$$ 

This says that the matrix $A - \lambda I$ has a non-trivial kernel. We know that an $n \times n$ matrix has a non-trivial kernel if and only if its determinant is zero. Thus $\lambda$ is an eigenvalue for the matrix $A$ if and only if $\det(A - \lambda I) = 0$. This gives us an algorithm for finding eigenvalues.

**Algorithm 1.** Let $A$ be an $n \times n$ matrix. A number $\lambda \in \mathbb{R}$ or $\mathbb{C}$ is an eigenvalue for $A$ if and only if $\det(A - \lambda I) = 0$. Thus to find the eigenvalues we compute the polynomial $\det(A - \lambda I)$ and find its roots. The roots are the eigenvalues.

**Definition 2.** The polynomial $\chi_A(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of $A$.

We compute the characteristic polynomial of an arbitrary $2 \times 2$ matrix since we do this often. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We find that

$$\chi_B(\lambda) = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - Tr(B)\lambda + \det(B).$$

Recall that $Tr(B)$ is the sum of the terms on the main diagonal of $B$.

**Example 1.** Let $A = \begin{pmatrix} 11/7 & -9/7 \\ -6/7 & -4/7 \end{pmatrix}$, so $\chi_A(\lambda) = \lambda^2 - \lambda - 2$. This has roots $\lambda = -1, \lambda = 2$. These are the eigenvalues of $A$. We now find the eigenvectors.

**Algorithm 2.** Let $\lambda$ be an eigenvector of a matrix $A$. We wish to find all the non-zero vectors $v$ so that $Av = \lambda v$. This is the same as finding all the solutions to the equation

$$(A - \lambda I)v = 0.$$ 

This is the same as finding the kernel of the matrix $A - \lambda I$. We have an algorithm for finding the basis of the kernel.

We apply this to our example. For the eigenvalue $\lambda = -1$ we need to find a basis for the kernel of the matrix

$$A - (-1)I = \begin{pmatrix} 11/7 + 1 & -9/7 \\ -6/7 & -4/7 + 1 \end{pmatrix} = \begin{pmatrix} 18/7 & -9/7 \\ -6/7 & 3/7 \end{pmatrix}.$$ 

1
A basis for the kernel is \((1 \ 2)\). Since the kernel of a matrix is a subspace and the set of eigenvectors for a specific eigenvalue is a kernel, we know that the set of eigenvectors is a subspace. This means that any linear combination of eigenvectors for a specific eigenvalue is again an eigenvector for that eigenvalue.

**Definition 3.** Let \(A\) be an \(n \times n\) matrix and let \(\lambda\) be an eigenvalue of \(A\). Then the set of eigenvectors for \(\lambda\) is called the eigenspace of \(\lambda\).

We have a second eigenvalue, namely, \(\lambda = 2\). We proceed as above and find that \((-3 \ 1)\) is a basis of the eigenspace for the eigenvalue \(\lambda = 2\).

Note that the two eigenvectors form a basis \(B = \{f_1 = (-3 \ 1), f_2 = (1 \ 2)\}\) for \(\mathbb{R}^2\). We compute the matrix of \(A\) with respect to the basis \(B\) in two ways.

- Note that \(A(f_1) = 2f_1\) says that
  \[
  A \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B = \begin{pmatrix} 2 \\ 0 \end{pmatrix}_B.
  \]

  Similarly \(A(f_2) = -1f_2\) says that
  \[
  A \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_B.
  \]

  Thus the matrix of \(A\) with respect to the basis \(B\) is
  \[
  A_B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.
  \]

- We know that \(1_{E-B} = \begin{pmatrix} -3 & 1 \\ 1 & 2 \end{pmatrix}\) and \(X = 1_{B-E}\) is the inverse to this matrix. We have
  \[
  A_B = 1_{B-E} A 1_{E-A} = X A X^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.
  \]

This says that \(A\) is similar to a diagonal matrix.

**Definition 4.** We say that an \(n \times n\) matrix \(A\) is diagonalizable if there exists a matrix \(X\) so that
\[
X A X^{-1}
\]
is a diagonal matrix.

The above calculation shows that if we can find a basis of \(\mathbb{R}^n\) of eigenvectors for an \(n \times n\) matrix \(A\), then the matrix \(A\) is diagonalizable.
Example 2. Let \( A = \begin{pmatrix} -1 & 9 \\ -1 & 5 \end{pmatrix} \). We find the eigenvalues and eigenvectors of \( A \). We genin with the eigenvalues. The characteristic equation of \( A \) is
\[
\chi_A(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.
\]
Thus the only eigenvector is \( \lambda = 2 \).

We now find all the eigenvectors of the matrix \( A \). they are the solutions to
\[
Av = 2v \text{ or } AV = 2Iv \text{ or } (A - 2I)v = 0.
\]
Thus we are looking for a basis of the kernel of the matrix
\[
A - 2I = \begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix}.
\]
A basis of the kernel is \( \begin{pmatrix} 3 \\ 1 \end{pmatrix} \).

Note that the eigenvectors do not span all of \( \mathbb{R}^2 \). Thus we can not use the same trick to diagonalize as we did in the first example.

Lemma 3. let \( a \) be an \( n \times n \) matrix. Then \( A \) is diagonalizable if and only if there is a basis of \( \mathbb{R}^n \) consisting of eigenvectors for the matrix \( A \).

Example 4. Let \( A = \begin{pmatrix} 3/5 & -4/5 \\ 26/5 & 7/5 \end{pmatrix} \). What are the eigenvalues and eigenvectors of \( A \). The characteristic polynomial is
\[
\chi_A(\lambda) = \lambda^2 - 2\lambda + 5.
\]
This is no real roots. It does have complex ones, namely, \( 1 \pm 2i \). We can find vectors with complex coefficients which are eigen vectors using the usual algorithm.

Example 5. Let \( A = \begin{pmatrix} 13/2 & -7/2 \\ 21/2 & -15/2 \end{pmatrix} \). The eigenvalues are \( \lambda = 3, \lambda = -4 \). A basis for the eigenspace corresponding to the eigenvalue \( 3 \) is \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). A basis for the eigenspace corresponding to the eigenvalue \( \lambda = -4 \) is \( \begin{pmatrix} 1 \\ 3 \end{pmatrix} \).