We give an algorithm to find out if a set of linear equations has a solution, and, if it does have a solution, how to find all of the solutions. First we rewrite the set of linear equations dropping much of the redundant information. If we have the equations

\[
\begin{align*}
x - 2y + 3z &= 4 \\
y - 7z &= 14 \\
2x - z &= 0,
\end{align*}
\]

then we write this as

\[
\begin{pmatrix}
1 & -2 & 3 & | & 4 \\
0 & 1 & -7 & | & 14 \\
2 & 0 & -1 & | & 0
\end{pmatrix}.
\]

Note that we separate out the coefficients from the constants. We call such an array an augmented matrix. A matrix is just a rectangular array of numbers with none of the columns or rows set out. For example

\[
\begin{pmatrix}
3 & 4 & -6 \\
23 & -7 & 0
\end{pmatrix}
\]

is a matrix.

To (try) to solve this set of equations we perform special operations on the equations, or rather we perform equivalent operations on the augmented matrix. These operations do not change the set of solutions to our initial equations. We can do three kinds of row operations.

1. We can add a multiple of a row to any other different row.
2. We can multiply a row by a non-zero scalar.
3. We can reorder the rows.

By using these operations we can put the left side of an augmented matrix into a special form called reduced row echelon form. Once this has been done we can read off the solutions to the initial set of equations.

**Definition 1.** A matrix is in reduced row-echelon form provided

1. If a row has non-zero entries the first entry is 1. This is called a leading 1.
2. If a column has a leading 1 in it, all the other entries are zero.
3. If a row has a leading 1, then each row above that row has a leading one to the left of the leading one in our original row.
The first three matrices below are NOT in reduced row-echelon form. The fourth example is in reduced row-echelon form.

\[
\begin{pmatrix}
1 & 2 & 3 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 & 0 \\
1 & 2 & -1 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

We now show how to find the solutions, if any, of a matrix in reduced row-echelon form.

1. Assume that we have used row operations and turned an augmented matrix into the reduced row-echelon form

\[
\begin{pmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & 14 \\
0 & 0 & 1 & -3 \\
\end{pmatrix}.
\]

This corresponds to the set of equations

\[
\begin{align*}
x &= 4 \\
y &= 14 \\
z &= -3.
\end{align*}
\]

We conclude that this set of equations has a unique solution.

**Definition 2.** If a set of equations has a solution we say that the set of equations is consistent.

**Definition 3.** The number of leading ones of a reduced row echelon matrix is called its rank. The rank of an arbitrary matrix is the number of leading ones when it is put into reduced row-echelon form.

The rank of this example is 3.

2. This time assume that after performing row operations, we end up with the augmented matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & c \\
\end{pmatrix}.
\]

This corresponds to the equations

\[
\begin{align*}
x &= 2 \\
z &= 3 \\
0x + 0y + 0z &= c
\end{align*}
\]

This has a solution if \(c = 0\). If \(c \neq 0\), then it has no solution.
Definition 4. A set of equations is inconsistent if it has no solutions.

In case $c \neq 0$, this set of equations is inconsistent.

3. Consider the augmented matrix

$$
\begin{pmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

This corresponds to the equations

$$
x + 2y = 3 \\
z = -5.
$$

We solve starting with the last variable. We must have $z = 0$. Note that in the equation $x + 2y = 3$ the coefficient of $x$ is a leading one. Hence if we choose $y$ then $x$ is determined. It is $3 - 2y$. Thus we can choose $y$ freely, say $y = s$. Then we get $x = 3 - 2s$. We have many solutions. This is a consistent set of equations since there is a solution. The rank is 2.

The solutions are

$$
x = 3 - 2s \\
y = s \\
z = 5.
$$

We can write this as

$$
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
3 \\
0 \\
5
\end{pmatrix} + s
\begin{pmatrix}
-2 \\
1 \\
0
\end{pmatrix}.
$$

Here $s$ can take on any real number.

4. This time the reduced row-echelon matrix is

$$
\begin{pmatrix}
0 & 1 & -2 & 0 & 4 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix}
$$

This corresponds to the equations

$$
x_2 - 2x_3 = 3 \\
x_4 = 2
$$

There are two leading ones, one corresponds to the variable $x_2$ and the other corresponds to the variable $x_4$. Thus the rank is 2. From this we see that we can choose
the other variables $x_1, x_3$ freely. Thus the set of solutions is

\[
x_1 = t \\
x_2 = 3 + 2s \\
x_3 = s \\
x_4 = 2.
\]

We can write this as

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Here $s, t$ can take on any real value.