Let \( V, W \) be vector spaces whose dimension is finite. Let \( F : V \rightarrow W \) be a linear map. We show how to associate to \( F \) a matrix. We cannot do this just from the map \( F \). We need more information. To do this we need to have bases for \( V \) and \( W \). We begin by reviewing how we find matrices of linear maps when our vector spaces are \( \mathbb{R}^m \) and \( \mathbb{R}^n \).

Let \( e_i \) denote the vector of length \( m \) all of whose entries are zero, except in the \( i \)-th position. The entry in the \( i \) position is 1.

Problem: Let \( M \) be a linear map \( \mathbb{R}^m \rightarrow \mathbb{R}^n \). How do we construct a matrix \( M \) that induces the same function as \( M \)?

**Observation 1.** Let \( v_1, v_2, \ldots, v_m \) be \( m \) vectors in \( \mathbb{R}^n \); then the matrix whose columns are \( v_1, \ldots, v_m \) maps \( e_i \) to \( v_i \) for \( i = 1 \cdots i = m \).

**Observation 2.** Let \( F : \mathbb{R}^m \rightarrow \mathbb{R}^n \) be a linear map. Then \( F \) is determined by the vectors \( F(e_i) \in \mathbb{R}^n \).

**Proof.** Let \( x \in \mathbb{R}^m \). We can write \( x = \sum_{i=1}^{m} x_i e_i \). Then

\[
F(x) = F(\sum_{i=1}^{m} x_i e_i) = \sum_{i=1}^{m} x_i F(e_i).
\]

We have found a formula for \( F(x) \) using our knowledge of \( x \) and the values \( F(e_i) \). \( \square \)

**Algorithm 1.** Let \( \mathcal{M} \) be a linear map \( \mathbb{R}^m \rightarrow \mathbb{R}^n \).

- Define a matrix \( M \) whose columns are the vectors \( F(e_i), i = 1, \ldots, m \). Then \( M e_i = \mathcal{M}(e_i) \).
- By the second observation we see that \( M(x) = \mathcal{M}(x) \) for all \( x \in \mathbb{R}^m \).

Problem: Let \( V, W \) be finite dimensional vector spaces. Let \( \mathcal{M} \) be a linear map \( V \rightarrow W \). How do we construct a matrix \( M \) that induces the same function as \( \mathcal{M} \)?

First note that matrix multiplication does not mean anything in the context of an abstract vector space. For example consider the vector space \( P_2 \), the set of all polynomials in one variable of degree \( \leq 2 \). We cannot multiply a polynomial \( f \in P_2 \) by a matrix. The problem is not solvable as stated. To solve the problem we need to introduce bases for \( V, W \) and use the bases to write vectors in terms of coordinates.

**Example 1.** Let \( \mathcal{M} : P_2 \rightarrow P_2 \) be the map \( f \mapsto f' - 3f \). We choose \( B = \{1, t, t^2\} \) as a basis of \( P_2 \). This allows us to write elements of \( P_2 \) as column vectors. For example, we have

\[
f = 7 - 2t + 3t^2 \mapsto (f)_B = \begin{pmatrix} 7 \\ -3 \\ 3 \end{pmatrix}_B.
\]
Note that

\[ \mathcal{M}(1) = -3 \cdot 1 \]
\[ \mathcal{M}(t) = 1 - 3t \]
\[ \mathcal{M}(t^2) = 2t - 3t^2 \]

In terms of coordinates with respect to the basis \(B\), we have

\[ \mathcal{M} : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_B \mapsto \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}_B \]
\[ \mathcal{M} : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_B \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_B \]
\[ \mathcal{M} : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_B \mapsto \begin{pmatrix} 2 \\ -3 \end{pmatrix}_B \]

If we set \(M_B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & -3 \end{pmatrix}\), then the matrix \(B_M\) and the function \(\mathcal{M}\) are the same in terms of the \(B\)-coordinates. It is very important to note that the matrix \(M_B\) depends on the choice of coordinates.

**Example 2.** We consider the map \(\mathcal{F} : \mathbb{C} \to \mathbb{C}\) that maps

\[ x + iy \mapsto (4 - 3i)(x + iy) = (4x + 3y) + i(-3x + 4y). \]

To obtain a matrix representation for \(\mathcal{F}\) we need to choose a basis for \(\mathbb{C}\). We choose \(A = \{1, i\}\). We have

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix}_A = 1 \mapsto 4 - 3i = \begin{pmatrix} 4 \\ -3 \end{pmatrix}_A \]
\[ \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A = 1 \mapsto 3 + 4i = \begin{pmatrix} 3 \\ 4 \end{pmatrix}_A. \]

We see that the linear maps \(\mathcal{F}\) can be represented as the matrix

\[ F_A = \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}. \]

**Example 3.** Let \(V = \mathbb{R}^{2 \times 2}\) be the vector space of all \(2 \times 2\) matrices. Define

\[ \mathcal{F} : V \to V \]
\[ \mathcal{F} : M \mapsto \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} M. \]

We choose a basis

\[ B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \]
for $V$. We compute

$$
\mathcal{F} : \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}_B.
$$

**Exercise 1.** Let $A = \begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix}$. Consider

$$
F : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}
$$

$$
F : M \mapsto AM.
$$

Is this map linear? If so, is it an isomorphism? If it is linear and it is not an isomorphism, find a basis for the kernel and image of $F$.

We check that it is linear. We have

$$
F(M + N) = A(M + N) = AM + AN = F(M) + F(N), M, N \in \mathbb{R}^{2 \times 2},
$$

and

$$
F(\lambda M) = A(\lambda M) = \lambda AM = \lambda F(M).
$$

We choose a basis for $\mathbb{R}^{2 \times 2}$. A convenient choice is

$$
B = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \}.
$$

We find the matrix for the map $F$ wrt the basis above and then use row operations to find a basis for the kernel and image. Note

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}_B.
$$

This says that the first column of the matrix for $F$ wrt the basis $B$ is

$$
\begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}.
$$

We find the other columns of the matrix in the same way. Eventually we obtain the matrix of $F$:

$$
F_B = \begin{pmatrix} 2 & 0 & -3 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 3 & 0 \\ 0 & -2 & 0 & 3 \end{pmatrix}.
$$
Using row operations we obtain

\[
\begin{pmatrix}
 2 & 0 & -3 & 0 \\
 0 & 2 & 0 & -2 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

From this we can see that the kernel is non-trivial. This implies that \( F \) is not invertible and hence \( F \) is not an isomorphism. We can calculate a basis of the kernel using our usual algorithm. We find that

\[
\left\{ \begin{pmatrix} 3/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3/2 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

is a basis of the kernel of \( F_B \) in \( B \)-coordinates. We really want a basis of \( F \) in the vector space \( \mathbb{R}^{2\times2} \). A basis of \( F \) is \( \left\{ \begin{pmatrix} 3/2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3/2 \\ 0 & 1 \end{pmatrix} \right\} \). The image is dealt with similarly.