Notes 16: Vector Spaces: Bases, Dimension, Isomorphism
Lecture November 5, 2009

Let $V$ be a vector space.

**Definition 1.** Let $v_1, v_2, \ldots, v_m \in V$. A linear combination of the elements $v_i$ is any element of $V$ of the form $\sum_1^m a_i v_i$, $a_i \in \mathbb{R}$.

**Definition 2.** Let $S \subset V$. The span of $S$ is the set of all linear combinations of elements of $S$. If $W \subset V$ is a subspace of $V$, we say that $S$ spans $W$ if the span of $S$ is $W$.

**Definition 3.** Let $S \subset V$. We say that $S$ is linearly independent if, whenever $\sum_1^m a_i v_i = 0$, $a_i \in \mathbb{R}, v_i \in V$ all of the $a_i = 0$.

**Definition 4.** Let $W$ be a subspace of $V$ and $S$ a subset of $W$. We say $S$ is a basis of $W$ provided $S$ spans $W$ and $S$ is linearly independent.

**Theorem 1.** Let $S$ be a basis of a subspace $W$ of $V$. We can write every element $w \in W$ uniquely as $w = \sum a_i v_i$, $a_i \in \mathbb{R}, v_i \in S$. We call the coefficients $a_i$ the coordinates of $w$ with respect to $S$.

**Proof.** Since $S$ spans $W$, we can write $w = \sum a_i v_i$, $a_i \in \mathbb{R}, v_i \in S$.

Assume that we an write $w$ in two different ways:

$$w = \sum a_i v_i = \sum b_i v_i, a_i, b_i \in \mathbb{R}, v_i \in S.$$ 

We obtain

$$0 = w - w = \sum (a_i - b_i) v_i.$$ 

Since $S$ is linearly independent, $a_i = b_i$. Thus the expressions for $w$ are the same.

**Definition 5.** Let $W$ be a subspace of $V$. The dimension of $W$ is the number of elements in a basis of $W$. If a basis of $W$ is infinite we say that the dimension is infinite.

Dimension in abstract vector spaces satisfies the same properties as it does in $\mathbb{R}^n$.

- Every basis of a vector space has the same number of elements.
- Let $W \subset V$ be a subspace of $V$. Then $\text{dim}(W) \leq \text{dim}(V)$ and equality only occurs if $V = W$.

**Example 2.** Let $V = \mathbb{R}^{m \times n}$. Then a basis of $V$ consists of the matrices with all zero entries except in one position. The entry in that position should be 1. Since there are $mn$ distinct positions, the dimension of $V$ is $mn$.

**Example 3.** Let $V = \mathbb{C}$, the complex numbers. A basis of $C$ is $\{1, i\}$. The dimension of $\mathbb{C}$ as a real vector space is 2.

**Example 4.** The set of polynomials $P_2$ of degree $\leq 2$ is a vector space. One basis of $P_2$ is the set $1, t, t^2$.  

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Example 5. Let \( P \) denote the set of all polynomials of all degrees. Then \( P \) is a vector space. It has a basis \( \{1, t, t^2, \cdots \} \). It is infinite dimensional.

Example 6. \( F(\mathbb{R}, \mathbb{R}) \) denotes the set of all functions from \( \mathbb{R} \) to \( \mathbb{R} \). It is infinite dimensional.

Example 7. Let \( C \) denote the set of all infinitely differentiable functions from \( \mathbb{R} \) to \( \mathbb{R} \). Then \( C \) is a subspace of \( F(\mathbb{R}, \mathbb{R}) \) and it is infinite dimensional.

Example 8. Let \( W \subset P_2 \) be all the functions \( \{f \in P_2 | f'' - 2f' = 0\} \). Here 0 means the 0 polynomial. We verify that this is a subspace and then we find a basis and thus we find its dimension.

Let \( f, g \in W \) so \( f'' - 2f' = 0 \) and \( g'' - 2g' = 0 \). We need to show that \( f + g \in W \). We have \((f + g)'' - 2(f + g)' = f'' + g'' - 2f' - 2g' = f'' - 2f' + g' - 2g' = 0 + 0 = 0\).

We also need to show that \( \lambda f \in W, \lambda \in \mathbb{R} \). We have \((\lambda f)'' - 2(\lambda f)' = \lambda f'' - \lambda 2f' = 0\).

Thus \( W \) is a subspace.

We compute a basis of \( W \). Let \( f = at^2 + bt + c \). Then \( f'' - 2f' = -4at + (2a - 2b) \). This is the zero polynomial if and only if

\[-4a = 0 \text{ and } 2a - 2b = 0.\]

Hence \( W \) consists of all the polynomials in \( P_2 \) such that the coefficients \( a, b \) are both zero. Hence \( W \) consists of all of the polynomials of degree zero. A basis of this set is the polynomial 1. The dimension of \( W \) is 1. Notice that our work led us to finding solutions to a system of linear equations

\[
\begin{align*}
4a &= 0 \\
2a - 2b &= 0.
\end{align*}
\]

Example 9. Let \( L \) be the set of lower triangular \( 2 \times 2 \) matrices, that is, matrices of the form

\[
\begin{pmatrix}
a & 0 \\
b & c
\end{pmatrix}.
\]

A basis for \( L \) consists of the three matrices

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
\]

The dimension of \( L \) is 3.

Definition 6. Let \( V, W \) be vector spaces. We say that a function

\[
F : V \longrightarrow W
\]

\[
F : v \mapsto F(v)
\]

is linear provided \( F(v_1 + v_2) = F(v_1) + F(v_2),, v_1, v_2 \in V \) and \( F(\lambda v) = \lambda F(v), \lambda \in \mathbb{R}, v \in V \).

Definition 7. Let \( F : V \longrightarrow W \) be a linear map of vector spaces. Then the kernel of \( F \) is \( \text{ker}(F) = \{v \in V | F(v) = 0\} \) and the image of \( F \) is \( \text{im}(F) = \{y \in W | y = F(v), \text{ for some } v \in V\} \).
Theorem 10. Let $F : V \rightarrow W$ be a linear map of vector spaces. Then $\text{im}(F)$ is a subspace of $W$ and $\ker(F)$ is a subspace of $V$.

Example 11. Let $C$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ with infinitely many derivatives. Define $D : C \rightarrow C$ by $f \mapsto f'' - f$. Notice that $D(e^x) = 0$. Here $0$ is the 0-function in $C$. Notice that $D(e^{-x}) = 0$. Notice that $D$ is linear. Thus the kernel of $D$ is a subspace. We conclude that $D(ae^x + be^{-x}) = 0$ for all $a, b \in \mathbb{R}$.

Definition 8. Let $F : V \rightarrow W$ be a linear map of vector spaces. The rank of $F$ is the dimension of the subspace $\text{im}(F)$ of $W$. The nullity of $F$ is the dimension of $\ker(F)$.

Theorem 12. Let $F : V \rightarrow W$ be a linear map of vector spaces. Assume that $\dim(V)$ is finite. Then

$$\dim(V) = \text{rk}(F) + \text{nullity}(F).$$

Example 13. Let

$$F : P_2 \rightarrow \mathbb{R}^3$$

$$F : at^2 + bt + c \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$ We check that $F$ is linear. Notice that $F$ has an inverse. In particular, $\ker(F) = 0$ and $\text{im}(F) = \mathbb{R}^3$.

Definition 9. Let $F$ be a linear map of vector spaces $F : V \rightarrow W$. We say that $F$ is an isomorphism if $F$ has an inverse.

If $F$ is a map of finite dimensional vector spaces of the same dimension and $\ker(F)$ is trivial, then $F$ is an isomorphism. To see this we use the rank nullity theorem. This does not hold if the dimensions are infinite.

Example 14. Let $P$ denote the vector space of all polynomials. Define $F : P \rightarrow P$ by $f \mapsto tf$ (Here $t$ is the variable). This has trivial kernel but the image is not all of $P$.

Example 15. We show how to use an isomorphism to turn a problem about a challenging vector space into a problem about $\mathbb{R}^n$. Find all the polynomials $f$ of degree $\leq 2$ so that $f'' - 3f' + f = 0$ (Here 0 is the 0 polynomial).

We use the isomorphism from the previous example: $F : P_2 \rightarrow \mathbb{R}^3, at^2 + bt + c \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Define a map $D : P_2 \rightarrow P_2, f \mapsto f'' - 3f' + f$. Define a map $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$G : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ (b - 6a) \\ (2a - 3b + c) \end{pmatrix}.$$ We have the following correspondence:

$$D : at^2 + bt + c \mapsto at^2 + (b - 6a)t + (2a - 3b + c)$$

$$G : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ (b - 6a) \\ (2a - 3b + c) \end{pmatrix}.$$