Problem 33.

Let us call denote by

\[ I = \int f \, dx \quad a_n = \int f_n \, dx \quad \text{and} \quad A = \lim \inf \int f_n \, dx \]

We wish to prove that \( I \leq A \)

By assumption we know

1. \( f_n \geq 0 \) (in particular \( a_n \geq 0 \) also)
2. \( f_n \to f \) in measure.

Now since \( A \) is the smallest of all limit points, there must exist a subsequence of \( a_{n_k} \) of \( a_n \) that converges to \( A \); i.e.

\[ \lim_{k \to \infty} a_{n_k} = A. \]

In particular note that

\[ A = \lim_{k \to \infty} \int f_{n_k} \, dx \quad \text{and that} \quad f_{n_k} \to f \] in measure as well

Then for every subsequence \( a_{n_{k_j}} \) of \( a_{n_k} \) we also have that

\[ A = \lim_{j \to \infty} a_{n_{k_j}} = \lim_{j \to \infty} \int f_{n_{k_j}} \, dx \]

and \( f_{n_{k_j}} \to f \) in measure as well.

In particular then, there exists one such subsequence for which we have a.e. convergence to \( f \). By abuse of notation let us refer to this particular subsequence as \( f_{n_{k_j}} \) once again. Now, by Fatou’s lemma we have that

\[ I = \int f = \int \lim f_{n_{k_j}} = \int \lim \inf f_{n_{k_j}} \leq \lim \inf a_{n_{k_j}} = \lim_{j \to \infty} a_{n_{k_j}} = A \]

as desired. \( \square \)

Problem 38 Part b).

First note that :

\[ f_ng_n - fg = (f_n - f)(g_n - g) + f(g_n - g) + g(f_n - f) \]

Then prove that if a function is finite a.e. and \( \mu(X) < \infty \) then the function is almost bounded: i.e.
∀\varepsilon > 0 \text{there exists } M = M(\varepsilon) > 0 \mu(\{x \in X : f(x) > M\}) < \varepsilon

To see this look at the sequence of sets $E_n = \{x \in X : f(x) > n\}$ and use the continuity from above of the measure $\mu$. Note $\mu(E_1) \leq \mu(X) < \infty$.

To find a counterexample in the case $\mu(X) = \infty$ consider $X = \mathbb{R}$ and $\mu$ Lebesgue measure. Define a sequence of functions $f_n(x) = a_n$ for all $x$ in $\mathbb{R}$ where $\{a_n\}$ is any sequence of positive real numbers you care to choose such that $a_n \to 0$ as $n \to \infty$. And define $g_n(x) = g(x)$ for all $n \geq 1$ where $g(x)$ is any function you care to choose over $\mathbb{R}$ such that $g(x) \to \pm \infty$ as $x \to \pm \infty$. Show $f_n$ converges in measure to zero, $g_n$ converges in measure to $g$ but $f_n g_n$ does not converge in measure to zero.

**Problem 41.**

The argument below might need some fine tuning: check it carefully.

Since $X$ is $\sigma$-finite we can write

$$X = \bigcup_{i=1}^{\infty} X_i \quad \mu(X_i) < \infty \quad X_i \cap X_{i'} = \emptyset, \quad i \neq i'$$

Then there exists $F_i \subset X_i$, $\mu(F_i) < \frac{\varepsilon}{2^i}$ such that

$$f_m \to f \quad \text{uniformly on } X_i \setminus F_i$$

Let $\varepsilon > 0$ be given and fixed. For each $i \geq 1$ we will apply Egoroff’s theorem with $\frac{\varepsilon}{2^i}$.

Note that $F_i \cap F_{i'} = \emptyset$ \quad $i \neq i'$.

Let us denote by $G_i = X_i \setminus F_i$ then since

$$G_i^c = X \setminus G_i = \bigcup_{j \neq i} X_j \cup F_i$$

where all unions are disjoints one can prove (homework: prove it ! ) that

$$\bigcap_{i=1}^{\infty} G_i^c \subseteq \bigcup_{i=1}^{\infty} F_i$$

(again last union is disjoints union ). Hence

$$\mu((\bigcup_{i=1}^{\infty} G_i)^c) = \mu(\bigcap_{i=1}^{\infty} G_i^c) \leq \sum_{i=1}^{\infty} \mu(F_i) \leq \varepsilon$$

For later use we now denote by

$$H_1 = (\bigcup_{i=1}^{\infty} G_i)^c = X \setminus (\bigcup_{i=1}^{\infty} G_i) \quad \text{and} \quad \mathcal{E}_1 = (\bigcup_{i=1}^{\infty} G_i)$$
Now consider the sequence \( \varepsilon_n = 2^{-n}, \ n \geq 1 \).

Let \( n = 1 \) and run the argument above with \( \varepsilon = \varepsilon_1 = 1/2 \). We get \( \mathcal{E}_1 \) and \( H_1 \) such that

1. \( X = \mathcal{E}_1 \cup H_1 \) where the union is disjoint and \( \mu(X \setminus \mathcal{E}_1) = \mu(H_1) < 1/2 \)
2. \( f_m \to f \) uniformly on each set \( G_i \) in \( \mathcal{E}_1 \)
3. \( f_m \to f \) a.e in \( X \); hence in particular, \( f_m \to f \) a.e in \( H_1 \)

For \( n = 2 \) we now consider as full space \( X = H_1 \) and apply Egoroff’s theorem with \( \varepsilon_2 = 1/4 \). Then there exists a set \( \mathcal{E}_2 \) such that

1. \( X = \mathcal{E}_2 \cup H_2 \), where the union is disjoint, \( H_2 = H_1 \setminus \mathcal{E}_2 \) and \( \mu(H_1 \setminus \mathcal{E}_2) = \mu(H_2) < 1/4 \)
2. \( f_m \to f \) uniformly in \( \mathcal{E}_2 \)
3. \( f_m \to f \) a.e in \( H_1 \); hence in particular, \( f_m \to f \) a.e in \( H_2 \)

Repeat the step \( n = 2 \) above inductively for all \( n \geq 3 \) applying Egoroff to \( X = H_{n-1} \) and \( \varepsilon = \varepsilon_n = 2^{-n} \) to get sets \( \mathcal{E}_n \) and \( H_n \) such that

1. \( X = \mathcal{E}_n \cup H_n \), where the union is disjoint, \( H_n = H_{n-1} \setminus \mathcal{E}_n \) and \( \mu(H_{n-1} \setminus \mathcal{E}_n) = \mu(H_n) < 2^{-n} \)
2. \( f_m \to f \) uniformly in \( \mathcal{E}_n \)
3. \( f_m \to f \) a.e in \( H_{n-1} \); hence in particular, \( f_m \to f \) a.e in \( H_n \)

Note that \( \mu(H_1) < \infty \) and \( \ldots \subseteq H_n \subseteq H_{n-1} \ldots \subseteq H_2 \subseteq H_1 \). Then

\[
X = \bigcup_{n=1}^{\infty} \mathcal{E}_n \cup H \quad \text{where} \quad H = \left( \bigcup_{n=1}^{\infty} \mathcal{E}_n \right)^c \quad \text{and}
\]

\[
\mu(H) = \mu(X \setminus \bigcup_{n=1}^{\infty} \mathcal{E}_n) = \mu\left( \bigcap_{n=1}^{\infty} H_n \right) = \lim_{n \to \infty} \mu(H_n) = 0
\]

On the other hand by relabeling all the \( G_i \) in \( \mathcal{E}_1 \) and all the \( \mathcal{E}_n \), \( n \geq 2 \) as \( E_m, \ m \geq 1 \) (note that the union of two countable families of sets is countable) for example by sending \( G_i, \ i \geq 1 \) to \( E_{2k}, \ k \geq 1 \) and \( \mathcal{E}_n, \ n \geq 2 \) to \( E_{2k+1}, \ k \geq 0 \) we obtained the desired conclusion. \( \square \)