On a Counterexample to Fubini

Let $W$ be a well ordered set;

$$j : [0, 1] \to W$$

and let $Q$ be defined as the set of pairs $(x, y) \in [0, 1] \times [0, 1]$ such that $j(x)$ preceds $j(y)$ in $W$ (here preceds means with respect to the total order in $W$).

We need that

1. $Q_x$ should contain all of $[0, 1]$ except for a countable set in $[0, 1]$.
2. $Q_y$ should contain at most a countable set in $[0, 1]$.

Hence note in particular both sections would be Borel sets.

Then the counterexample follows by considering $f(x, y) = \chi_{Q(x, y)}$ and observing that the integrals over $[0, 1]$ w.r.t. Lebesgue measure of $f_x$ and $f_y$ (which would be Borel functions themselves) were different (first one = 1; second one = 0).

The reason why Fubini doesn’t work is because $f$ itself is not measurable w.r.t the product Borel sigma algebra.

Proving 2) above maybe tricky if $j(x)$ is “too general”. One needs to actually assume a few additional things:

a) Well Ordering Principle: every nonempty set $X$ can be ”well ordered”. (this relies on an equivalent form of Zorn’s lemma which states that every partially ordered set has a maximally linearly ordered set i.e. given $X$ there exists a subset $E$ of $X$ for which the partial order in $X$ becomes a total or linear order in $E$ and $E$ is maximal in the sense that no other subset of $X$ that is also totally ordered with respect to the same partial order strictly contains $E$ )

b) The Continuum Hypothesis : there is no set whose cardinality is strictly bigger than that of the natural numbers and strictly smaller than that of the reals. (In other words if a set is uncountable then its cardinality is bigger or equal than that of the reals).

As a consequence of a) we have the existence of uncountable well order sets. We need then to assume our $W$ above is such a set.

As a consequence of b) and the fact $W$ has cardinality at least that of $[0, 1]$ we can choose $j(x)$ to be one-to-one function. Now if $j(x)$ is chosen as above in a one-to-one fashion then this guarantees that 1) and 2) above holds as desired.