An introduction to some aspects of functional analysis, 6: Weak and weak* convergence

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Abstract

Some basic properties of weak and weak* topologies are discussed, especially in connection with convergence of sequences.

Contents

1 Seminorms

Let V be a vector space over the real or complex numbers. A *seminorm* on V is a nonnegative real-valued function $N(v)$ on V such that

$$N(t\,v) = |t|\,N(v) \tag{1.1}$$

for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and

$$N(v + w) \le N(v) + N(w) \tag{1.2}$$

for every $v, w \in V$. Here $|t|$ denotes the absolute value of $t \in \mathbf{R}$, or the modulus of $t \in \mathbf{C}$. If $N(v) > 0$ when $v \ne 0$, then N is said to be a *norm* on V.

Suppose that $\mathcal{N}$ be a nonempty collection of seminorms on V. This leads to a natural topology on V, in which $U \subseteq V$ is an open set if for each $v \in U$ there are finitely many seminorms $N_1, \ldots, N_l$ in $\mathcal{N}$ and finitely many positive real numbers $r_1, \ldots, r_l$ such that

$$\{w \in V : N_j(v - w) < r_j \text{ for } j = 1, \ldots, l\} \subseteq U. \tag{1.3}$$

If $N \in \mathcal{N}$, $v \in V$, and $r > 0$, then it is easy to see that

$$B_N(v, r) = \{w \in V : N(v - w) < r\} \tag{1.4}$$

is an open set in V with respect to this topology, using the triangle inequality. By construction, the collection of all of these open balls with respect to elements of $\mathcal{N}$ is a sub-base for the topology on V just defined. If $\mathcal{N}$ consists of a single norm N, then this topology is the same as the one associated to the metric

$$d(v, w) = N(v - w) \tag{1.5}$$

corresponding to N on V.

Let us say that $\mathcal{N}$ is a *nice* collection of seminorms on V if for each $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that $N(v) > 0$. This implies that the topology on V just defined is Hausdorff, and also regular. By standard arguments, addition and scalar multiplication on V are continuous with respect to the topology on V associated to $\mathcal{N}$, so that V becomes a topological vector space. More precisely, V is a locally convex topological vector space, because the balls (1.4) are convex sets in V. It is well known that the topology on any locally convex topological vector space is determined by a collection of seminorms in this way.

Let λ be a linear functional on V, which is to say a linear mapping from V into $\mathbf{R}$ or $\mathbf{C}$, as appropriate. Suppose that λ is continuous with respect to the topology determined on V by a collection $\mathcal{N}$ of seminorms on V. Thus

$$U = \{w \in V : |\lambda(w)| < 1\} \tag{1.6}$$

is an open set in V that contains 0. This implies that there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and positive real numbers $r_1, \ldots, r_l$ such that

$$\{w \in V : N_j(w) < r_j \text{ for } j = 1, \ldots, l\} \subseteq U, \tag{1.7}$$

as in (1.3) with $v = 0$. Using this, one can check that

$$|\lambda(v)| \le C \, \max(N_1(v), \ldots, N_l(v)) \tag{1.8}$$

for every $v \in V$, where $C = \max(1/r_1, \ldots, 1/r_l)$. Conversely, if there are finitely many seminorms $N_1, \ldots, N_l$ in $\mathcal{N}$ and a nonnegative real number C such that (1.8) holds for every $v \in V$, then λ is continuous on V with respect to the topology determined by $\mathcal{N}$. Of course, (1.8) implies that

$$|\lambda(v) - \lambda(w)| = |\lambda(v - w)| \le C \, \max(N_1(v - w), \ldots, N_l(v - w)) \tag{1.9}$$

for every $v, w \in V$, because λ is linear.

2 Weak topologies

Let V be a vector space over the real or complex numbers again. If λ is any linear functional on V, then

$$N_\lambda(v) = |\lambda(v)| \tag{2.1}$$

defines a seminorm on V. Suppose that Λ is a nonempty collection of linear functionals on V, and consider the corresponding collection

$$\mathcal{N}(\Lambda) = \{N_\lambda : \lambda \in \Lambda\} \tag{2.2}$$

of seminorms on V. The topology on V determined by $\mathcal{N}(\Lambda)$ as in the previous section is known as the weak topology on V associated to Λ. Each $\lambda \in \Lambda$ is automatically continuous with respect to this topology, and in fact this topology is the weakest topology on V with this property.

Suppose that μ is a linear functional on V that is continuous with respect to the topology determined by $\mathcal{N}(\Lambda)$. As in the previous section, this implies that there are finitely many elements $\lambda_1, \ldots, \lambda_n$ of Λ such that

$$|\mu(v)| \le C \max(|\lambda_1(v)|, \ldots, |\lambda_n(v)|) \tag{2.3}$$

for some $C \ge 0$ and every $v \in V$. Put

$$W = \{v \in V : \lambda_j(v) = 0 \text{ for each } j = 1, \ldots, n\}, \tag{2.4}$$

which is a linear subspace of V contained in the kernel of μ, by (2.3).

Under these conditions, it is well known that W has codimension less than or equal to n in V, which is to say that the quotient vector space V/W has dimension less than or equal to n. Using the linear functionals $\lambda_1, \ldots, \lambda_n$, one can define a linear mapping from V into $\mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate, with kernel equal to W. This leads to an injective linear mapping from V/W into $\mathbf{R}^n$ or $\mathbf{C}^n$. If $\lambda_1, \ldots, \lambda_n$ are linearly independent, then one gets an isomorphism between V/W and $\mathbf{R}^n$ or $\mathbf{C}^n$. It is easy to reduce to this case, by dropping any λ_j that can be expressed as a linear combination of the rest.

Because W is contained in the kernel of λ_j for each j, λ_j can be expressed as the composition of the canonical quotient mapping from V onto V/W with a linear functional $\widehat{\lambda}_j$ on V/W for every $j = 1, \ldots, n$. Similarly, μ can be expressed as the composition of the canonical quotient mapping from V onto V/W with a linear functional $\widehat{\mu}$ on V/W. Of course, every linear functional on $\mathbf{R}^n$ or $\mathbf{C}^n$ is a linear combination of the coordinate functions, and in fact every linear functional on any subspace of $\mathbf{R}^n$ or $\mathbf{C}^n$ can be expressed in this way too. This implies that every linear functional on V/W can be expressed as a linear combination of $\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n$, and in particular that $\widehat{\mu}$ can be expressed in this way. It follows that μ can be expressed as a linear combination of $\lambda_1, \ldots, \lambda_n$ on V.

Any linear combination of finitely many elements of Λ is continuous on V with respect to the topology determined by $\mathcal{N}(\Lambda)$, since every element of Λ is continuous on V with respect to this topology. Thus the space of continuous linear functionals on V with respect to the topology determined by $\mathcal{N}(\Lambda)$ is spanned by Λ, by the argument in the previous paragraphs.

Let us say that Λ is a *nice* collection of linear functionals on V if for each $v \in V$ with $v \ne 0$ there is a $\lambda \in \Lambda$ such that $\lambda(v) \ne 0$. This is the same as saying that $\mathcal{N}(\Lambda)$ is a nice collection of seminorms on V, so that the corresponding topology on V is Hausdorff. This is also equivalent to the condition that Λ separate points in V.

Let $\mathcal{N}$ be any collection of seminorms on V, let $\{v_j\}_{j=1}^\infty$ be a sequence of vectors in V, and let v be a vector in V. It is easy to see that $\{v_j\}_{j=1}^\infty$ converges to v with respect to the topology on V determined by $\mathcal{N}$ as in the previous section if and only if

$$\lim_{j \to \infty} N(v_j - v) = 0 \tag{2.5}$$

for each $N \in \mathcal{N}$. In particular, this implies that

$$\lim_{j \to \infty} N(v_j) = N(v) \tag{2.6}$$

for each $N \in \mathcal{N}$, by standard arguments. If Λ is a collection of linear functionals on V again, then $\{v_j\}_{j=1}^\infty$ converges to v with respect to the topology determined by $\mathcal{N}(\Lambda)$ if and only if

$$\lim_{j \to \infty} \lambda(v_j) = \lambda(v) \tag{2.7}$$

for each $\lambda \in \Lambda$.

3 Dual spaces

Let V be a real or complex vector space, equipped with a norm $\|v\|$. Thus we get a topology on V determined by the norm, and we let V^* be the space of continuous linear functionals on V. This is also a vector space over the real or complex numbers, as appropriate, with respect to pointwise addition and scalar multiplication. The *weak topology* on V is defined to be the topology associated to $\Lambda = V^*$ as in the previous section. This definition also makes sense when the topology on V is determined by a collection $\mathcal{N}$ of seminorms on V, or when V is any topological vector space, but we shall be especially concerned with the case where the topology on V is determined by a single norm here.

The weak topology on V is automatically weaker than the original topology on V, in the sense that every open set in V with respect to the weak topology is an open set in V with respect to the original topology. This follows from the fact that every element of V^* is continuous on V with respect to the original topology on V. However, if V is infinite-dimensional, and the original topology on V is determined by a norm $\|v\|$, then every open ball in V with respect to $\|v\|$ is an open set with respect to the original topology, but not with respect to the weak topology. This is because every nonempty open set in V with respect to the weak topology contains a translate of a linear subspace of V of finite codimension, which cannot be bounded in V with respect to the norm when V is infinite-dimensional. More precisely, any translate of any nontrivial linear subspace W of V is unbounded with respect to the norm, and every linear subspace W of V with finite codimension is nontrivial when V is infinite-dimensional.

As in Section 1, a linear functional λ on V is continuous with respect to the topology determined by a norm $\|v\|$ if and only if λ is bounded, in the sense that

$$|\lambda(v)| \le C\,\|v\| \tag{3.1}$$

for some $C \ge 0$ and every $v \in V$. Put

$$\|\lambda\|_* = \sup\{|\lambda(v)| : v \in V,\ \|v\| \le 1\}, \tag{3.2}$$

which is the same as the smallest value of $C \ge 0$ for which (3.2) holds. One can check that (3.2) defines a norm on V^*, which is the dual norm associated to $\|v\|$ on V.

Let $N(v)$ be a seminorm on a real or complex vector space V, let W be a linear subspace of V, and let λ be a linear functional on W that satisfies

$$|\lambda(v)| \le C\, N(v) \tag{3.3}$$

for some $C \ge 0$ and every $v \in W$. Under these conditions, the Hahn–Banach theorem implies that there is an extension of λ to V that also satisfies (3.3), with the same constant C. Using this, one can show that V^* separates points in V when the topology on V is defined by a norm, or by a nice collection of seminorms. If $v \in V$, $v \ne 0$, and $\|\cdot\|$ is a norm on V, then one can use this to show that there is a $\lambda \in V^*$ such that

$$\lambda(v) = \|v\| \tag{3.4}$$

and $\|\lambda\|_* = 1$.

4 Some convergence theorems

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{f_j\}_{j=1}^\infty$ be a sequence of functions in $L^p(X)$ for some p, $1 \le p \le \infty$. Suppose that the L^p norms of the f_j's are bounded L^p, so that

$$\|f_j\|_p \le C \tag{4.1}$$

for some $C \ge 0$ and every $j \ge 1$. Suppose also that $\{f_j\}_{j=1}^\infty$ converges to a function f pointwise almost everywhere on X. This implies that $f \in L^p(X)$ too, with $\|f\|_p \le C$.

If $p > 1$ and $\mu(X) < +\infty$, then it is well known that $\{f_j\}_{j=1}^\infty$ converges to f with respect to the L^1 norm under these conditions. This follows from the bounded convergence theorem when $p = \infty$, and otherwise one can use an analogous argument. The main point is that

$$\int_A |f_j|\, d\mu \le C \mu(A)^{1-(1/p)} \tag{4.2}$$

for each measurable set $A \subseteq X$, by Hölder's inequality, and similarly for f. If $p > 1$, then $1 - (1/p) > 0$, and it follows that the integral of $|f_j|$ or $|f|$ is small when $\mu(A)$ is small. The same proof also works when $\{f_j\}_{j=1}^\infty$ converges to f in measure on X, instead of pointwise almost everywhere.

Now let $1 \le q \le \infty$ be the exponent conjugate to p, so that $1/p + 1/q = 1$, and let $g \in L^q(X)$ be given. If $p = \infty$, then $\{f_j\, g\}_{j=1}^\infty$ converges to $f\, g$ with respect to the L^1 norm, by the dominated convergence theorem. The same conclusion also holds when $1 < p < \infty$, by an analogous argument. As before,

$$\int_A |f_j|\, |g|\, d\mu \le C \Big(\int_A |g|^q\, d\mu \Big)^{1/q} \tag{4.3}$$

for every measurable set $A \subseteq X$, by Hölder's inequality, and similarly for f in place of f_j. This implies that the integrals of $|f_j|\,|g|$ and $|f|\,|g|$ over A are small when $\mu(A)$ is small, and that these integrals are small for some measurable sets $A \subseteq X$ such that $\mu(X \setminus A) < \infty$. If X is σ-finite with respect to μ, then it suffices to ask that $\{f_j\}_{j=1}^\infty$ converge to f in measure on measurable sets of finite measure instead of pointwise almost everywhere. Of course, the set of $x \in X$ such that $g(x) \neq 0$ is automatically σ-finite when $g \in L^q(X)$ for some $q < \infty$.

If $g \in L^q(X)$ and $1 \leq p, q \leq \infty$ are conjugate exponents, then Hölder's inequality implies that

$$\lambda_g(f) = \int_X f\, g\, d\mu \tag{4.4}$$

defines a bounded linear functional on $L^p(X)$. If $1 < p < \infty$, then it is well known that every bounded linear functional on $L^p(X)$ is of this form for some $g \in L^q(X)$. In this case, the discussion in the previous paragraph gives a criterion for a sequence in $L^p(X)$ to converge with respect to the weak topology. One can also give examples of sequences in $L^p(X)$ that satisfy these conditions but do not converge with respect to the L^p norm.

5 The weak* topology

Let V be a real or complex vector space equipped with a norm $\|v\|$, and let V^* be the corresponding dual space of countinuous linear functionals on V, as in Section 3. Put

$$L_v(\lambda) = \lambda(v) \tag{5.1}$$

for each $v \in V$ and $\lambda \in V^*$, which defines a linear functional on V^* for every $v \in V$. As in Section 2, the collection of all of these linear functionals on V^* defines a topology on V^*, known as the *weak* topology* on V^*. This definition also makes sense when the original topology on V is defined by a collection of seminorms, or when V is any topological vector space, but we shall again be especially concerned here with the case where the original topology on V is determined by a single norm. If $\lambda \in V^*$ and $\lambda \neq 0$, then there is a $v \in V$ such that $\lambda(v) \neq 0$, so that the collection of linear functionals on V^* of the form L_v for some $v \in V$ automatically separates points in V^*.

Of course,

$$|L_v(\lambda)| = |\lambda(v)| \leq \|\lambda\|_* \, \|v\| \tag{5.2}$$

for every $v \in V$ and $\lambda \in V^*$, by definition of the dual norm $\|\lambda\|_*$ on V^*. This implies that L_v is a continuous linear functional on V^* for each $v \in V$, with respect to the topology defined on V^* by the dual norm $\|\lambda\|_*$. In particular, every open set in V^* with respect to the weak* topology is also an open set with respect to the topology defined by the dual norm.

If $(X, \mathcal{A}, \mu)$ is a σ-finite measure space, then it is well known that every bounded linear functional on $L^1(X)$ is of the form (4.4) for some $g \in L^\infty(X)$. The dual norm of such a linear functional λ_g on $L^1(X)$ is equal to the L^∞ norm of g, so that the dual of $L^1(X)$ is isometrically isomorphic to $L^\infty(X)$. Suppose that $\{g_j\}_{j=1}^\infty$ is a sequence of elements of $L^\infty(X)$ with uniformly bounded L^∞ norms, which converges pointwise almost everywhere on X to a function g.

This implies that $g \in L^\infty(X)$ too, and it would be enough to ask instead that $\{g_j\}_{j=1}^\infty$ converges to g in measure on measurable subsets of X of finite measure. If $f \in L^1(X)$, then the dominated convergence theorem implies that $\{f\, g_j\}_{j=1}^\infty$ converges to $f\, g$ with respect to the L^1 norm. In particular,

$$\lim_{j \to \infty} \int_X f\, g_j \, d\mu = \int_X f\, g \, d\mu \tag{5.3}$$

for every $f \in L^1(X)$, which means that $\{g_j\}_{j=1}^\infty$ converges to g with respect to the weak* topology on $L^\infty(X)$ as the dual of $L^1(X)$ under these conditions. It is easy to give examples of sequences in $L^\infty(X)$ that satisfy these conditions but do not converge with respect to the L^∞ norm.

6 Sines and cosines

Let f be a real or complex-valued function on the unit interval $[0,1]$ that is integrable with respect to Lebesgue measure, and put

$$\widehat{f}_s(j) = \int_0^1 f(x)\, \sin(2\pi j x)\, dx \tag{6.1}$$

and

$$\widehat{f}_c(j) = \int_0^1 f(x)\, \cos(2\pi j x)\, dx \tag{6.2}$$

for each positive integer j. Note that

$$|\widehat{f}_s(j)|,\ |\widehat{f}_c(j)| \le \int_0^1 |f(x)|\, dx \tag{6.3}$$

for each j, because $|\sin y|$, $|\cos y| \le 1$ for every $y \in \mathbf{R}$. It is well known that

$$\lim_{j \to \infty} \widehat{f}_s(j) = \lim_{j \to \infty} \widehat{f}_c(j) = 0 \tag{6.4}$$

for every $f \in L^1([0,1])$. To see this, it suffices to check that (6.4) holds for all f in a dense subset of $L^1([0,1])$, using (6.3) to estimate the effect on $\widehat{f}_s(j)$ and $\widehat{f}_c(j)$ of approximations of f with respect to the L^1 norm. If $f \in L^2([0,1])$, for instance, then one can show that $\sum_{j=1}^\infty |\widehat{f}_s(j)|^2$ and $\sum_{j=1}^\infty |\widehat{f}_c(j)|^2$ converge, so that their terms tend to 0 as $j \to \infty$ in particular. This implies that (6.4) holds for every $f \in L^1([0,1])$, beause $L^2([0,1])$ is dense in $L^1([0,1])$. Alternatively, if f is the characteristic or indicator function of a subinterval of $[0,1]$, then one can check directly that (6.4). Thus (6.4) holds when f is a step function on $[0,1]$, by linearity, and hence when f is any integrable function on $[0,1]$, since step functions are dense in $L^1([0,1])$.

Equivalently, (6.4) says that $\sin(2\pi j x)$ and $\cos(2\pi j x)$ converge to 0 as $j \to \infty$ with respect to the weak* topology on $L^\infty([0,1])$ as the dual of $L^1([0,1])$. If we restrict our attention to $f \in L^q([0,1])$ for some $q > 1$, then it follows that $\sin(2\pi j x)$ and $\cos(2\pi j x)$ converge to 0 as $j \to \infty$ with respect to the weak topology on $L^p([0,1])$ when $1 \le p < \infty$, where p and q are conjugate exponents.

7 Finite-dimensional spaces

Let V be a real or complex vector space, and let $N(v)$ be a seminorm on V. Observe that

$$N(v) \le N(w) + N(v - w) \tag{7.1}$$

and

$$N(w) \le N(v) + N(v - w) \tag{7.2}$$

for every $v, w \in V$, which implies that

$$|N(v) - N(w)| \le N(v - w) \tag{7.3}$$

for every $v, w \in V$. In particular, it follows that N is continuous as a real-valued function on V with respect to the topology determined by any collection $\mathcal{N}$ of seminorms on V such that $N \in \mathcal{N}$.

Suppose that $V = \mathbf{R}^n$ or $\mathbf{C}^n$ for some positive integer n, and let $\|v\|$ be the standard Euclidean norm on V, for instance. If N is any seminorm on V again, then it is easy to see that

$$N(v) \le C\,\|v\| \tag{7.4}$$

for some $C \ge 0$ and every $v \in V$. More precisely, this can be obtained by expressing v as a linear combination of the standard basis vectors for V, and then using the triangle inequality for N. This implies that N is continuous with respect to the standard topology on V, because of (7.3).

If N is a norm on V, then we also have that

$$N(v) \ge c\,\|v\| \tag{7.5}$$

for some $c > 0$ and every $v \in V$, which means that the topology on V determined by N is the same as the standard topology. To see this, it suffices to show that

$$N(v) \ge c \tag{7.6}$$

for some $c > 0$ and every $v \in V$ with $\|v\| = 1$, because of the homogeneity properties of norms. It is well known that

$$\{v \in V : \|v\| = 1\} \tag{7.7}$$

is compact with respect to the standard topology, because it is closed and bounded. This implies that $N(v)$ attains its minimum on (7.7), since N is continuous with respect to the standard topology on V. Of course, $N(v) > 0$ for each $v \in V$ with $v \ne 0$, so that the minimum of N on (7.7) is positive, as desired.

If V is any real or complex vector space with finite positive dimension n, then V is isomorphic as a vector space to $\mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate. The preceding argument implies that any two norms on V determine the same topology on V, which corresponds to the standard topology on $\mathbf{R}^n$ or $\mathbf{C}^n$.

Let V be any real or complex vector space again. If N is a seminorm on V, then it is easy to see that

$$\{v \in V : N(v) = 0\} \tag{7.8}$$

is a linear subspace of V.

Suppose that V is finite-dimensional, and that $\mathcal{N}$ is a nice collection of seminorms on V. If $V \neq \{0\}$, then there is an $N_1 \in \mathcal{N}$ such that $N_1(v) > 0$ for some $v \in V$. Put

$$V_1 = \{w \in V : N_1(w) = 0\}, \tag{7.9}$$

which is a proper linear subspace of V. If $V_1 = \{0\}$, then N_1 is a norm on V, and we stop. Otherwise, if $V_1 \neq \{0\}$, then there is an $N_2 \in \mathcal{N}$ such that $N_2(v) > 0$ for some $v \in V_1$. In this case,

$$V_2 = \{w \in V_1 : N_2(w) = 0\} \tag{7.10}$$

is a proper linear subspace of V_1. If $V_2 = \{0\}$, then the maximum of N_1 and N_2 is a norm on V, and we stop. Otherwise, if $V_2 \neq \{0\}$, then we continue the process. After l steps, we have l seminorms $N_1, \ldots, N_l \in \mathcal{N}$, and we take

$$V_l = \{w \in V : N_j(v) = 0 \text{ for each } j = 1, \ldots, l\}. \tag{7.11}$$

If $V_l = \{0\}$, then the maximum of $N_1, \ldots, N_l$ is a norm on V, and we stop. The process has to stop after a finite number of steps less than or equal to the dimension of V, because V_1 is a proper linear subspace of V, and V_l is a proper linear subspace of V_{l-1} when $l \geq 2$.

Thus there are finitely many seminorms $N_1, \ldots, N_l$ in $\mathcal{N}$ such that

$$N(v) = \max(N_1(v), \ldots, N_l(v)) \tag{7.12}$$

is a norm on V. Every other seminorm on V is bounded by a constant multiple of N, by the earlier arguments. This implies that the topology on V determined by the collection $\mathcal{N}$ is the same as the topology determined by N in this case.

8 Metrizability

Let V be a real or complex vector space, and let $\mathcal{N}$ be a nice collection of seminorms on V. If $\mathcal{N}$ consists of a single element N, then N is a norm on V, and the topology on V determined by $\mathcal{N}$ is the same as the one defined by the metric corresponding to N. Similarly, if $\mathcal{N}$ consists of finitely many seminorms on V, then their maximum is a norm on V that determines the same topology.

Suppose that $\mathcal{N}$ consists of an infinite sequence of seminorms $N_1, N_2, N_3, \ldots$ on V. Put

$$d_j(v, w) = \min(N_j(v - w), 1/j) \tag{8.1}$$

for every $v, w \in V$ and $j \geq 1$, and

$$d(v, w) = \max_{j \geq 1} d_j(v, w) \tag{8.2}$$

for every $v, w \in V$. More precisely, if $v = w$, then $d_j(v, w) = 0$ for every j, and hence $d(v, w) = 0$. Otherwise, if $v \neq w$, then $d_l(v, w) > 0$ for some $l \geq 1$, and

$$d_j(v, w) \leq 1/j < d_l(v, w) \tag{8.3}$$

for all but finitely many j. This shows that the definition (8.2) of $d(v, w)$ makes sense for all $v, w \in V$, and that $d(v, w) > 0$ when $v \neq w$. It is easy to see that $d(v, w)$ is symmetric in v and w, because of the analogous property of $d_j(v, w)$ for each j. One can also check that $d_j(v, w)$ satisfies the triangle inequality for each j, and hence that $d(v, w)$ satisfies the triangle inequality as well. Thus $d(v, w)$ defines a metric on V, which is invariant under translations on V by construction. It is not too difficult to verify that the topology on V defined by $d(v, w)$ is the same as the one determined by $\mathcal{N}$ under these conditions. The main point is that the open ball in V centered at $v \in V$ with radius r is the same as the intersection of the open balls in V centered at v with radius r with respect to N_j for the finitely many j such that $1/j \geq r$.

Conversely, suppose that the topology on V determined by a nice collection of seminorms $\mathcal{N}$ has a countable local base at 0. In this case, there is a subset of $\mathcal{N}$ consisting of only finitely or countably many seminorms on V that determines the same topology on V. This uses the fact that for each open set $U \subseteq V$ with $0 \in U$, there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and finitely many positive real numbers $r_1, \ldots, r_l$ such that

$$\bigcap_{j=1}^{l} B_{N_j}(0, r_j) \subseteq U. \tag{8.4}$$

More precisely, if there is a countable local base for the topology of V at 0, then it follows that only finitely or countably many elements of $\mathcal{N}$ are needed to get such a base.

Now let Λ be a nice collection of linear functionals on V. If Λ has only finitely or countably many elements, then the corresponding weak topology on V is also determined by a translation-invariant metric on V, by the previous discussion. Conversely, if there is a countable local base for the topology determined on V by Λ at 0, then there is a subset Λ_0 of Λ with only finitely or countably many elements that determines the same topology on V, as before. Every element of Λ is continuous on V with respect to this topology, which implies that every element of Λ can be expressed as a linear combination of finitely many elements of Λ_0 under these conditions, as in Section 2.

9 Completeness

As usual, a metric space M is said to be *complete* if every Cauchy sequence of elements of M converges to an element of M. Let V be a real or complex vector space equipped with a norm $\|v\|$, and let V^* be the dual space of continuous linear functionals on V, equipped with the dual norm $\|\lambda\|_*$. It is well known that V^* is automatically complete with respect to the metric associated to $\|\lambda\|_*$.

To see this, let $\{\lambda_j\}_{j=1}^\infty$ be a Cauchy sequence in V^* with respect to the dual norm. Thus for each $\epsilon > 0$ there is an $L(\epsilon) \geq 1$ such that

$$\|\lambda_j - \lambda_k\|_* < \epsilon \tag{9.1}$$

for every $j, k \geq L(\epsilon)$. This implies that

$$|\lambda_j(v) - \lambda_k(v)| \leq \|\lambda_j - \lambda_k\|_* \, \|v\| \leq \epsilon \, \|v\| \tag{9.2}$$

for every $v \in V$ and $j, k \geq L(\epsilon)$. In particular, this means that $\{\lambda_k(v)\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $v \in V$. It follows that $\{\lambda_k(v)\}_{k=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$, because the real and complex numbers are complete with respect to their standard metrics. Put

$$\lambda(v) = \lim_{k \to \infty} \lambda_k(v) \tag{9.3}$$

for each $v \in V$, which defines a linear functional on V, since λ_k is linear for each k. Taking the limit as $k \to \infty$ in (9.2), we get that

$$|\lambda_j(v) - \lambda(v)| \leq \epsilon \, \|v\| \tag{9.4}$$

for every $v \in V$ and $j \geq L(\epsilon)$. In particular,

$$|\lambda(v)| \leq \|v\| + \|\lambda_j(v)\| \leq (1 + \|\lambda_j\|_*) \, \|v\| \tag{9.5}$$

for every $v \in V$ and $j \geq L(1)$, which implies that λ is also a countinuous linear functional on V. This permits (9.4) to be reformulated as saying that

$$\|\lambda_j - \lambda\|_* \leq \epsilon \tag{9.6}$$

for every $j \geq L(\epsilon)$, so that $\{\lambda_j\}_{j=1}^\infty$ converges to λ with respect to the dual norm on V^*, as desired.

Let M is a metric space, E be a dense subset of M, and f be a uniformly continuous mapping from E into another metric space N. If N is complete, then it is well known that there is a unique extension of f to a uniformly continuous mapping from M into N. Now let V be a real or complex vector space with a norm $\|v\|$ again, and let W be a dense linear subspace of V. Also let λ be a continuous linear functional on W, with respect to the restriction of the norm $\|v\|$ to $v \in V$. It is easy to see that λ is uniformly continuous as a mapping from W into $\mathbf{R}$ or $\mathbf{C}$, as appropriate, with respect to the metric on W that corresponds to the norm. Thus λ has a unique extension to a uniformly continuous mapping from V into $\mathbf{R}$ or $\mathbf{C}$, as appropriate, because the real and complex numbers are complete with respect to their standard metrics. One can check that this extension is also linear, and that the dual norm of this extension on V is the same as the dual norm of λ on W.

A real or complex vector space with a norm is said to be a *Banach space* if it is complete with respect to the metric associated to the norm. It is well known that any metric space can be isometrically embedded onto a dense subset of a complete metric space, and that such a completion is unique up to isometric equivalence. Similarly, a real or complex vector space V with a norm $\|v\|$ has a linear isometric embedding onto a dense linear subspace of a Banach space, and such a completion of V is unique up to a linear isometry. The dual of V and its completion are essentially the same, by the remarks in the previous paragraph.

Let M be a metric space again, and let E be a subset of M. If E is complete as a metric space with respect to the restriction of the metric on M to E, then E is a closed set in M. This is because any sequence of elements of E that converges to an element of M is a Cauchy sequence in E, and hence converges to an element of E when E is complete. If M is complete and E is a closed set in M, then E is also complete as a metric space with respect to the restriction of the metric on M to E.

Of course, $\mathbf{R}^n$ and $\mathbf{C}^n$ are complete with respect to their standard metrics for any positive integer n. This implies that $\mathbf{R}^n$ and $\mathbf{C}^n$ are complete with respect to the metric associated to any norm on these spaces, because of the equivalence with the standard norm, as in Section 7. It follows that any finite-dimensional real or complex vector space with a norm is complete, since it is either trivial or isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$ for some positive integer n. This also implies that finite-dimensional linear subspaces of a real or complex vector space V with a norm are closed subsets of V, by the remarks in the preceding paragraph.

Let V be an infinite-dimensional real or complex Banach space, and suppose for the sake of a contradiction that there is a sequence $v_1, v_2, v_3, \ldots$ of vectors in V such that every $v \in V$ can be expressed as a linear combination of finitely many v_j's. Let W_n be the linear subspace of V spanned by $v_1, \ldots, v_n$ for each positive integer n, so that W_n has finite dimension less than or equal to n for each n, and

$$V = \bigcup_{n=1}^{\infty} W_n. \tag{9.7}$$

As in the previous paragraph, W_n is a closed set in V for each n, and it is easy to see that W_n cannot contain a nonempty open set in V for any n, because $W_n \neq V$, since V is supposed to be infinite-dimensional. This contradicts the Baire category theorem, since V is also supposed to be complete.

Let V be an infinite-dimensional real or complex vector space with a norm again, and let V^* be the corresponding dual space, with the dual norm. As in Section 3, the Hahn–Banach theorem implies that V^* separates points in V, so that V^* must also be infinite-dimensional. We have seen that V^* is automatically complete with respect to the dual norm, which implies that V^* cannot be spanned by a countable set, as in the previous paragraph. It follows that the weak topology on V cannot be described by only countably many elements of V^*. In particular, there is no countable local base for the weak topology on V at 0.

If V is an infinite-dimensional real or complex Banach space, then V cannot be spanned by a countable set, as before. This implies that the weak* topology on V^* cannot be described by countably many elements of V. More precisely, this uses the fact that for each $v \in V$ with $v \neq 0$, there is a $\lambda \in V^*$ such that

$L_v(\lambda) = \lambda(v) \neq 0$, by the Hahn–Banach theorem, so that the mapping $v \mapsto L_v$ from V to linear functionals on V^* is injective. It follows that that there is no countable local base for the weak* topology on V^* at 0 in this situation.

10 Second duals

Let V be a real or complex vector space with a norm $\|v\|$, and let V^* be the corresponding dual space, with the dual norm $\|\lambda\|_*$. Similarly, let V^{**} be the dual space of V^*, consisting of all continuous linear functionals L on V^* with respect to the topology on V^* defined by $\|\lambda\|_*$. There is also a dual norm $\|L\|_{**}$ on V^{**}, which is the dual norm associated to the norm $\|\lambda\|_*$ on V^*. If $v \in V$, then $L_v(\lambda) = \lambda(v)$ defines an element of V^*, with

$$\|L_v\|_{**} \leq \|v\|, \tag{10.1}$$

by (5.2). Remember that for each $v \in V$ with $v \neq 0$ there is a $\lambda \in V^*$ such that $\|\lambda\|_* = 1$ and $L_v(\lambda) = \lambda(v) = \|v\|$, as in Section 3. This implies that

$$\|L_v\|_{**} = \|v\| \tag{10.2}$$

for every $v \in V$.

Of course, $v \mapsto L_v$ defines a linear mapping from V into V^{**}. It is easy to see that this mapping is also a homeomorphism from V equipped with the weak topology onto its image in V^{**} with the topology induced by the weak* topology on V^{**}, as the dual of V^*. If V is complete with respect to the norm $\|v\|$, then the image of V in V^{**} is complete with respect to the norm $\|L\|_{**}$, because of (10.2). This implies that the image of V is a closed set in V^{**} with respect to the topology defined by the norm $\|L\|_{**}$, as in the previous section.

A real or complex Banach space V is said to be *reflexive* if every element of V^{**} is of the form L_v for some $v \in V$. If V has finite dimension, then V^* and V^{**} have the same finite dimension, which implies that V is reflexive. More precisely, $v \mapsto L_v$ is an injective linear mapping from V into V^{**} for any V, and it follows that this mapping is also surjective when V and V^{**} are finite-dimensional with the same dimension. It is well known that L^p spaces are reflexive when $1 < p < \infty$, and that Hilbert spaces are reflexive. If V is reflexive, then the weak and weak* topologies on V^* are the same.

Suppose that V is a real or complex vector space with a norm $\|v\|$ which is not necessarily complete. Note that V^{**} is automatically complete with respect to the norm $\|L\|_{**}$, by the discussion in the previous section applied to V^* instead of V. If W is the closure of the image of V in V^{**} under the mapping $v \mapsto L_v$ with respect to the topology defined by the norm $\|L\|_{**}$, then W is complete with respect to the restriction of $\|L\|_{**}$ to $L \in W$, because V^{**} is complete and W is a closed set in V^{**}. Thus W may be used as a completion of V. In particular, if every element of V^{**} is of the form L_v for some $v \in V$, then V is necessarily complete, since it is isometrically equivalent to V^{**}, which is complete. If V is not complete, then the dual of V is basically the same as the dual of the completion of V, as in the previous section. This implies that the dual of the dual of V is also basically the same as for the completion.

11 Separability

Remember that a metric space M is said to be *separable* if there is a dense set $E \subseteq M$ with only finitely or countably many elements. Let V be a real or complex vector space with a norm $\|v\|$. In order for V to be separable as a metric space, it suffices to have a set $A \subseteq V$ with only finitely or countable many elements whose linear span is dense in V. By definition, the linear span of A in V is the set of all $v \in V$ which can be expressed as a linear combination of finitely many elements of A, which is also the smallest linear subspace of V that contains A. If $v_1, \ldots, v_n$ are finitely many elements of V, then the collection of $v \in V$ that can be expressed as a linear combination of $v_1, \ldots, v_n$ with rational coefficients is countable. Similarly, if V is a complex vector space, then the set of $v \in V$ that can be expressed as a linear combination of $v_1, \ldots, v_n$ with coefficients whose real and imaginary parts are rational is also countable. If A has only finitely or countably many elements, then one can use this to show that the linear span of A has a countable dense subset. If the linear span of A is also dense in V, then it follows that V is separable, as desired.

Suppose that $B \subseteq V$ is bounded with respect to the norm $\|v\|$, and that A is a subset of the dual space V^* whose linear span is dense in V^* with respect to the dual norm. Under these conditions, one can show that the topology on B induced by the weak topology on V is the same as the topology on B induced by the weak topology on V determined by the elements of A. In particular, if A has only finitely or countably many elements, then it follows that the topology on B induced by the weak topology on V is metrizable.

Similarly, suppose that A is a subset of V whose linear span is dense in V, and that $B \subseteq V^*$ is bounded with respect to the dual norm on V^*. Under these conditions, one can show that the topology induced on B by the weak* topology on V^* is the same as the topology on B induced by the weak topology on V^* defined by the linear functionals $L_v(\lambda) = \lambda(v)$ with $v \in A$. If A has only finitely or countably many elements, then it follows that the topology on B induced by the weak* topology on V^* is metrizable.

12 Uniform boundedness

Let M be a metric space, and let E be a collection of real or complex-valued continuous functions on M. Suppose that E is bounded pointwise on M, in the sense that

$$E(x) = \{f(x) : f \in E\} \tag{12.1}$$

is a bounded set in $\mathbf{R}$ or $\mathbf{C}$ for every $x \in M$. Equivalently, if

$$A_n = \{x \in M : |f(x)| \le n \text{ for every } f \in E\} \tag{12.2}$$

for each positive integer n, then

$$\bigcup_{n=1}^{\infty} A_n = M. \tag{12.3}$$

Note that A_n is a closed set in M for each n, because every $f \in E$ is continuous. If M is complete, then the Baire category theorem implies that A_n contains a nonempty open set in M for some n.

Now let V be a real or complex vector space with a norm $\|v\|$, and let E be a collection of continuous linear functionals on V. If V is complete, and if E is pointwise bounded on M, then the previous argument implies that E is uniformly bounded on a nonempty open set in V. Using linearity, one can show that E is uniformly bounded on the unit ball in V. This is the same as saying that the dual norms of the elements of E are bounded, so that E is a bounded set in V^* with respect to the dual norm. This is part of the theorem of Banach and Steinhaus.

Suppose now that $A \subseteq V$ has the property that

$$A(\lambda) = \{\lambda(v) : v \in A\} \tag{12.4}$$

is a bounded set in $\mathbf{R}$ or $\mathbf{C}$ for every $\lambda \in V^*$. If $L_v(\lambda) = \lambda(v)$ is the linear functional on V^* corresponding to $v \in V$, then it follows that the collection of L_v with $v \in A$ is bounded pointwise on V^*. Applying the previous argument to V^* instead of V, we get that the set of L_v with $v \in A$ is bounded in V^{**}. Note that this uses the completeness of V^*, instead of V. This shows that A is a bounded set in V under these conditions, since $v \mapsto L_v$ is an isometric embedding of V into V^{**}.

As an application, suppose that $\{\lambda_j\}_{j=1}^{\infty}$ is a sequence of continuous linear functionals on V that converges with respect to the weak* topology on V^*. Thus $\{\lambda_j(v)\}_{j=1}^{\infty}$ is a convergent sequence of real or complex numbers for every $v \in V$, which implies that $\{\lambda_j(v)\}_{j=1}^{\infty}$ is a bounded sequence in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$. If V is complete, then the Banach–Steinhaus theorem implies that $\{\lambda_j\}_{j=1}^{\infty}$ is a bounded sequence in V^* with respect to the dual norm. Similarly, if a sequence $\{v_j\}_{j=1}^{\infty}$ of elements of V converges with respect to the weak topology on V, then $\{\lambda(v_j)\}_{j=1}^{\infty}$ is a bounded sequence in $\mathbf{R}$ or $\mathbf{C}$ for every $\lambda \in V^*$. This implies that $\{v_j\}_{j=1}^{\infty}$ is bounded with respect to the norm on V, as in the preceding paragraph.

13 ℓ^p Spaces

If f is a nonnegative real-valued function on a nonempty set X, then the sum

$$\sum_{x \in X} f(x) \tag{13.1}$$

is defined to be the supremum of the sums

$$\sum_{x \in A} f(x) \tag{13.2}$$

over all nonempty finite subsets A of X. Of course, this is finite if and only if the finite subsums (13.2) are bounded, in which case f is said to be *summable* on X. Similarly, a real or complex-valued function f on X is said to be summable if $|f(x)|$ is summable on X. The sum (13.1) of a summable function on X may be defined by expressing f as a linear combination of nonnegative real-valued summable functions on X, or by reducing to the case of absolutely convergent infinite series.

A real or complex-valued function f on X is said to be *p-summable* for some p, $1 \le p < \infty$, if $|f(x)|^p$ is a summable function on X. The space of p-summable functions on X is denoted $\ell^p(X)$, and is a vector space with respect to pointwise addition and scalar multiplication. It is well known that $\ell^p(X)$ is a Banach space with the norm

$$\|f\|_p = \Big(\sum_{x \in X} |f(x)|^p\Big)^{1/p}. \tag{13.3}$$

Similarly, the space $\ell^\infty(X)$ of bounded real or complex-valued functions on X is a Banach space with the supremum norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|. \tag{13.4}$$

Equivalently, X may be considered as a measure space with respect to counting measure, in which all subsets of X are measurable, and $\ell^p(X)$ is the same as the corresponding L^p space for each p, $1 \le p \le \infty$.

Suppose that $1 \le p, q \le \infty$ are conjugate exponents, so that $1/p + 1/q = 1$. If $g \in \ell^q(X)$, then

$$\lambda_g(f) = \sum_{x \in X} f(x)\, g(x) \tag{13.5}$$

defines a countinuous linear functional on $\ell^p(X)$, by Hölder's inequality. It is well known that every continuous linear functional on $\ell^p(X)$ is of this form when $1 \le p < \infty$. Of course, this is easier to show than the analogous statement for arbitrary measure spaces.

Let $\{f_j\}_{j=1}^\infty$ be a sequence of elements of $\ell^p(X)$ with bounded ℓ^p norm. If $\{f_j\}_{j=1}^\infty$ converges pointwise to a function f on X, then $f \in \ell^p(X)$ too. If $p > 1$ and $g \in \ell^q(X)$, then

$$\lim_{j \to \infty} \sum_{x \in X} f_j(x)\, g(x) = \sum_{x \in X} f(x)\, g(x). \tag{13.6}$$

This is a special case of the analogous statement for arbitrary measure spaces discussed in Section 4, although the proof is simpler in this situation. As before, this implies that $\{f_j\}_{j=1}^\infty$ converges to f with respect to the weak topology on $\ell^p(X)$ when $1 < p < \infty$, and with respect to the weak* topology on $\ell^\infty(X)$ as the dual of $\ell^1(X)$ when $p = \infty$.

Conversely, if $\{f_j\}_{j=1}^\infty$ is a sequence of elements of $\ell^p(X)$ that converges to $f \in \ell^p(X)$ with respect to the weak topology, then the ℓ^p norms of the f_j's are necessarily bounded, as in the previous section. Weak convergence of the

f_j's also implies pointwise convergence here, as one can see by taking g to be equal to 1 at a single point in X and zero elsewhere. Similarly, if $\{f_j\}_{j=1}^\infty$ is a sequence of bounded functions on X that converges to a bounded function f on X with respect to the weak* topology on $\ell^\infty(X)$ as the dual of $\ell^1(X)$, then the f_j's have bounded ℓ^∞ norms, and $\{f_j\}_{j=1}^\infty$ converges to f pointwise on X.

Let $c_0(X)$ be the space of real or complex-valued functions $f(x)$ on X that "vanish at infinity", in the sense that

$$\{x \in X : |f(x)| \ge \epsilon\} \tag{13.7}$$

has only finitely many elements for each $\epsilon > 0$. One can check that $c_0(X)$ is a closed linear subspace of $\ell^\infty(X)$ with respect to the supremum norm, and hence a Banach space. If $g \in \ell^1(X)$, then the restriction of (13.5) to $f \in c_0(X)$ is a continuous linear functional on $c_0(X)$, and it is well known that every continuous linear functional on $c_0(X)$ is of this form.

If $\{f_j\}_{j=1}^\infty$ is a sequence of summable functions on X with bounded ℓ^1 norms which converges pointwise to a function f on X, then f is a summable function on X too. One can also show that (13.6) holds for every $g \in c_0(X)$, which is analogous to the other convergence theorems mentioned earlier. This implies that $\{f_j\}_{j=1}^\infty$ converges to f with respect to the weak* topology on $\ell^1(X)$ as the dual of $c_0(X)$ under these conditions. Conversely, if $\{f_j\}_{j=1}^\infty$ is a sequence of summable functions on X that converges to a summable function f on X with respect to the weak* topology on $\ell^1(X)$ as the dual of $c_0(X)$, then the f_j's have bounded ℓ^1 norms and converge to f pointwise on X, for essentially the same reasons as before.

If a sequence $\{f_j\}_{j=1}^\infty$ of summable functions on X converges to a summable function f on X with respect to the weak topology on $\ell^1(X)$, then it is well known that $\{f_j\}_{j=1}^\infty$ actually converges to f with respect to the ℓ^1 norm. Of course, this is trivial when X has only finitely many elements. Otherwise, one may as well suppose that X is countably infinite, since summable functions on X are supported on sets with only finitely or countably many elements. In fact, it is helpful to simply take X to be the set $\mathbf{Z}_+$ of positive integers. One may also suppose that $f = 0$ here, since otherwise one can replace f_j with $f_j - f$ for each j. As before, $\{f_j\}_{j=1}^\infty$ converges to 0 pointwise on X when it converges to 0 with respect to the weak topology on $\ell^1(X)$. If $\{f_j\}_{j=1}^\infty$ does not converge to 0 with respect to the ℓ^1 norm, then there is an $\epsilon > 0$ such that

$$\|f_j\|_1 \ge \epsilon \tag{13.8}$$

for infinitely many j. To get a contradiction, one can try to find a subsequence $\{f_{j_l}\}_{l=1}^\infty$ of $\{f_j\}_{j=1}^\infty$ and a $g \in \ell^\infty(X)$ such that

$$\Big|\sum_{x \in X} f_{j_l}(x)\, g(x)\Big| \tag{13.9}$$

has a positive lower bound.

14 Convergence and norms

Let V be a real or complex vector space with a norm $\|v\|$, and let V^* be the corresponding dual space, with the dual norm $\|\lambda\|_*$. Let us check that the closed ball

$$B_r = \{v \in V : \|v\| \le r\} \tag{14.1}$$

in V with center 0 and radius $r \ge 0$ is a closed set in V with respect to the weak topology. Observe first that

$$B_r = \{v \in V : |\lambda(v)| \le r \text{ for every } \lambda \in V^* \text{ with } \|\lambda\|_* \le 1\}, \tag{14.2}$$

by the definition of the dual norm and the Hahn–Banach theorem. Of course,

$$\{v \in V : |\lambda(v)| \le r\} \tag{14.3}$$

is a closed set in V with respect to the weak topology for every $\lambda \in V^*$ and $r \ge 0$. This implies that B_r is the intersection of a family of closed subsets of V with respect to the weak topology, as desired.

Similarly, the closed ball

$$B_r^* = \{\lambda \in V^* : \|\lambda\|_* \le r\} \tag{14.4}$$

in V^* with center 0 and radius $r \ge 0$ is a closed set with respect to the weak* topology on V^*. In this case,

$$B_r^* = \{\lambda \in V^* : |\lambda(v)| \le r \text{ for every } v \in V \text{ with } \|v\| \le 1\}, \tag{14.5}$$

simply by the definition of the dual norm $\|\lambda\|_*$. As before,

$$\{\lambda \in V^* : |\lambda(v)| \le r\} \tag{14.6}$$

is a closed set in V^* with respect to the weak* topology for every $v \in V$ and $r \ge 0$. Thus B_r^* can be expressed as the intersection of a family of closed subsets of V^* with respect to the weak* topology.

If $\{v_j\}_{j=1}^\infty$ is a sequence of vectors in V that converges to $v \in V$ with respect to the weak topology, and if $\|v_j\| \le r$ for some $r \ge 0$ and every j, then it follows that $\|v\| \le r$ too. This can be refined a bit to get that

$$\|v\| \le \liminf_{j \to \infty} \|v_j\|. \tag{14.7}$$

Similarly, if $\{\lambda_j\}_{j=1}^\infty$ is a sequence of continuous linear functionals on V that converges to $\lambda \in V^*$ with respect to the weak* topology, and if $\|\lambda_j\|_* \le r$ for some $r \ge 0$ and each j, then $\|\lambda\|_* \le r$. This can also be refined a bit to get that

$$\|\lambda\|_* \le \liminf_{j \to \infty} \|\lambda_j\|_*. \tag{14.8}$$

In both cases, if the sequence converges with respect to the topology defined by the norm, then the sequence of norms converges to the norm of the limit.

15 Continuous functions

Let X be a locally compact Hausdorff topological space, and let $C_0(X)$ be the space of continuous real or complex-valued functions on X that vanish at infinity, in the sense that for each $\epsilon > 0$ there is a compact set $K \subseteq X$ such that

$$|f(x)| < \epsilon \tag{15.1}$$

for every $x \in X \setminus K$. If X is compact, then this reduces to the space $C(X)$ of all continuous real or complex-valued functions on X. If X is equipped with the discrete topology, then $C_0(X)$ is the same as the space $c_0(X)$ discussed in Section 13. In any case, it is well known that $C_0(X)$ is a closed linear subspace of the space $C_b(X)$ of bounded continuous functions on X, with respect to the supremum norm. In particular, $C_0(X)$ is a Banach space with respect to the supremum norm.

A version of the Riesz representation theorem implies that the bounded linear functionals on $C_0(X)$ correspond exactly to real or complex measures on X with certain regularity properties. The dual norm of such a linear functional λ is equal to the total variation of the corresponding measure μ on X. A bounded linear functional λ on $C_0(X)$ is said to be *nonnegative* if $\lambda(f)$ is a nonnegative real number for every nonnegative real-valued function f on X. Nonnegative bounded linear functionals on $C_0(X)$ correspond exactly to finite nonnegative regular Borel measures on X.

Suppose that $\{f_j\}_{j=1}^\infty$ is a sequence of elements of $C_0(X)$ with uniformly bounded supremum norms that converges to $f \in C_0(X)$ pointwise everywhere on X. Using the representation of bounded linear functionals on $C_0(X)$ by Borel measures on X mentioned in the previous paragraph and the dominated convergence theorem, it follows that $\{f_j\}_{j=1}^\infty$ converges to f with respect to the weak topology on $C_0(X)$ under these conditions. Conversely, if $\{f_j\}_{j=1}^\infty$ converges to f with respect to the weak topology on $C_0(X)$, then the supremum norms of the f_j's must be uniformly bounded, as in Section 12. Weak convergence also implies pointwise convergence in this case, because $f \mapsto f(x)$ is a continuous linear functional on $C_0(X)$ for each $x \in X$.

16 The Banach–Alaoglu theorem

Let V be a real or complex vector space with a norm $\|v\|$, and let V^* be its dual space, with the dual norm $\|\lambda\|_*$. The *Banach–Alaoglu theorem* states that the closed unit ball

$$B^* = \{\lambda \in V^* : \|\lambda\|_* \le 1\} \tag{16.1}$$

in V^* is compact with respect to the weak* topology on V^*. To prove this, one can show that B^* is homeomorphic with respect to the topology induced on B^* by the weak* topology on V^* to a closed set in a product of closed intervals in the real line or disks in the complex plane, and use Tychonoff's theorem on the compactness of products of compact sets.

If V is separable, then the topology on B^* induced by the weak* topology on V^* is metrizable, as in Section 11. In this case, compactness of B^* is equivalent to sequential compactness, which can be shown more directly, as follows. Let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of elements of B^*, and let $v_1, v_2, v_3, \ldots$ be a sequence of vectors in V whose linear span is dense in V. Thus $\{\lambda_j(v_k)\}_{j=1}^\infty$ is a bounded sequence of real or complex numbers for each k, which implies that for each k there is a subsequence of $\{\lambda_j(v_k)\}_{j=1}^\infty$ that converges in $\mathbf{R}$ or $\mathbf{C}$, since closed and bounded subsets of the real line and the complex plane are compact. Using standard diagonalization arguments, one can check that there is a subsequence $\{\lambda_{j_l}\}_{l=1}^\infty$ of $\{\lambda_j\}_{j=1}^\infty$ such that $\{\lambda_{j_l}(v_k)\}_{l=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every k.

It follows that $\{\lambda_{j_l}(v)\}_{l=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$ in the linear span of $v_1, v_2, v_3, \ldots$, because of linearity. This implies that $\{\lambda_{j_l}(v)\}_{l=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$, because the linear span of $v_1, v_2, v_3, \ldots$ is dense in V, and $\|\lambda_{j_l}\|_* \le 1$ for each l. Thus $\{\lambda_{j_l}(v)\}_{l=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$, by completeness. It is easy to see that

$$\lambda(v) = \lim_{l \to \infty} \lambda_{j_l}(v) \tag{16.2}$$

defines a continuous linear functional on V with $\|\lambda\|_* \le 1$, because of the corresponding properties of λ_{j_l} for each l. This says exactly that $\lambda \in B^*$ and that $\{\lambda_{j_l}\}_{l=1}^\infty$ converges to λ with respect to the weak* topology on V^*, as desired.

17 The dual of L^∞

Let $(X, \mathcal{A}, \mu)$ be a measure space, and consider the corresponding space $L^\infty(X)$ of bounded real or complex-valued measurable functions on X, where two such functions are identified when they are equal almost everywhere. It is well known that $L^\infty(X)$ is a Banach space with respect to the essential supremum norm $\|f\|_\infty$. Suppose that λ is a continuous linear functional on $L^\infty(X)$. Let $\mathbf{1}_A(x)$ be the indicator or characteristic function of a set $A \subseteq X$, which is equal to 1 when $x \in A$ and equal to 0 when $x \in X \backslash A$. Thus $\mathbf{1}_A \in L^\infty(X)$ when $A \subseteq X$ is measurable, in which case we put

$$\nu(A) = \lambda(\mathbf{1}_A). \tag{17.1}$$

If $A, B \subseteq X$ are measurable and $A \cap B = \emptyset$, then

$$\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B, \tag{17.2}$$

and hence

$$\nu(A \cup B) = \nu(A) + \nu(B). \tag{17.3}$$

This shows that ν is a finitely additive measure on $\mathcal{A}$, and we also have that

$$\nu(A) = 0 \quad \text{when} \quad \mu(A) = 0, \tag{17.4}$$

because $\mathbf{1}_A$ is identified with 0 in $L^\infty(X)$ when $\mu(A) = 0$.

Suppose that $A_1, \ldots, A_n$ are finitely many pairwise-disjoint measurable subsets of X. If $t_1, \ldots, t_n$ are real or complex numbers, as appropriate, such that $|t_j| \le 1$ for each j, then

$$f(x) = \sum_{j=1}^{n} t_j \, \mathbf{1}_{A_j}(x) \tag{17.5}$$

is in $L^\infty(X)$, and satisfies $\|f\|_\infty \le 1$. This implies that

$$|\lambda(f)| \le \|\lambda\|_*, \tag{17.6}$$

where $\|\lambda\|_*$ denotes the dual norm of λ with respect to the L^∞ norm. Of course,

$$\lambda(f) = \sum_{j=1}^{n} t_j \, \lambda(\mathbf{1}_{A_j}(x)) = \sum_{j=1}^{n} t_j \, \nu(A_j), \tag{17.7}$$

so that

$$\biggl|\sum_{j=1}^{n} t_j \, \nu(A_j)\biggr| \le \|\lambda\|_*, \tag{17.8}$$

and hence

$$\sum_{j=1}^{n} |\nu(A_j)| \le \|\lambda\|_*, \tag{17.9}$$

using suitable choices of $t_1, \ldots, t_n$.

Conversely, suppose that $\nu(A)$ is a real or complex-valued finitely-additive measure defined for $A \in \mathcal{A}$. This means that ν is a real or complex-valued function on $\mathcal{A}$ that satisfies (17.3) when A, B are disjoint measurable subsets of X. It is easy to see that there is a linear functional λ on the vector space of real or complex-valued measurable simple functions on X such that (17.1) holds for every measurable set $A \subseteq X$. If $\nu(A) = 0$ for every measurable set $A \subseteq X$ with $\mu(A) = 0$, then $\lambda(f) = 0$ when f is a measurable simple function on X that is equal to 0 almost everywhere with respect to μ.

Suppose in addition that ν has finite total variation on X. This means that there is a nonnegative real number C such that

$$\sum_{j=1}^{n} |\nu(A_j)| \le C \tag{17.10}$$

for any collection $A_1, \ldots, A_n$ of finitely many pairwise-disjoint measurable subsets of X. This implies that

$$|\lambda(f)| \le C \, \|f\|_\infty \tag{17.11}$$

for every measurable simple function f on X, where λ is as in the preceding paragraph. It follows that there is a unique extension of λ to a bounded linear functional on $L^\infty(X)$, as in Section 9, because simple functions are dense in

$L^\infty(X)$. Alternatively, $\lambda(f)$ may be considered as a type of integral of f with respect to ν when $f \in L^\infty(X)$, which amounts to essentially the same thing.

Let ν be a real or complex-valued finitely additive measure on $\mathcal{A}$ with finite total variation again. The corresponding total variation measure is defined by

$$|\nu|(A) = \sup \sum_{j=1}^{n} |\nu(A_j)| \tag{17.12}$$

for each measurable set $A \subseteq X$, where the supremum is taken over all finite collections $A_1, \ldots, A_n$ of pairwise-disjoint measurable subsets of X such that

$$A = \bigcup_{j=1}^{n} A_j. \tag{17.13}$$

It is easy to see that this is a nonnegative real-valued finitely additive measure on $\mathcal{A}$, and that $|\nu|(X)$ is the same as the smallest constant C for which (17.10) holds. If $\nu(A) = 0$ for every measurable set $A \subseteq X$ such that $\mu(A) = 0$, then $|\nu|$ has the same property. Note that any nonnegative real-valued finitely-additive measure on $\mathcal{A}$ has finite total variation on X.

Let λ be a continuous linear functional on $L^\infty(X)$, and let $A_1, \ldots, A_n$ be finitely many pairwise-disjoint measurable subsets of X. Also let $f_1, \ldots, f_n$ be elements of $L^\infty(X)$ such that $\|f_j\|_\infty \le 1$ for each j, and $f_j(x) \ne 0$ only when $x \in A_j$. If $t_1, \ldots, t_n$ are real or complex numbers, as appropriate, such that $|t_j| \le 1$ for each j, then

$$f(x) = \sum_{j=1}^{n} t_j\, f_j(x) \tag{17.14}$$

satisfies $\|f\|_\infty \le 1$. Thus

$$\lambda(f) = \sum_{j=1}^{n} t_j\, \lambda(f_j) \tag{17.15}$$

satisfies $|\lambda(f)| \le \|\lambda\|_*$, which implies that

$$\sum_{j=1}^{n} |\lambda(f_j)| \le \|\lambda\|_*, \tag{17.16}$$

using suitable choices of $t_1, \ldots, t_n$.

Suppose now that $A_1, A_2, A_3, \ldots$ is an infinite sequence of pairwise disjoint measurable subsets of X, and that $f_1, f_2, f_3, \ldots$ is a sequence of elements of $L^\infty(X)$ such that $\|f_j\|_\infty \le 1$ for each j, and $f_j(x) \ne 0$ only when $x \in A_j$. The previous discussion implies that

$$\sum_{j=1}^{\infty} |\lambda(f_j)| \le \|\lambda\|_*, \tag{17.17}$$

and hence that $\lambda(f_j) \to 0$ as $j \to \infty$. It follows that $\{f_j\}_{j=1}^\infty$ converges to 0 with respect to the weak topology on $L^\infty(X)$ under these conditions. In particular, one can apply this to $f_j = \mathbf{1}_{A_j}$, which has L^∞ norm equal to 1 when $\mu(A_j) > 0$.

18 Bounded continuous functions

Let us begin with a remark about weak topologies and subspaces. Suppose that V is a real or complex vector space with a norm $\|v\|$, and that W is a linear subspace of V. If λ is a continuous linear functional on V, then the restriction of λ to W is a continuous linear functional on W. Conversely, every continuous linear functional on W is of this form, by the Hahn–Banach theorem. This implies that the weak topology on W is the same as the topology induced on W by the weak topology on V.

Now let X be a topological space, and let $C_b(X)$ be the space of bounded continuous real or complex-valued functions on X. It is well known that $C_b(X)$ is a Banach space with respect to the supremum norm. Suppose that μ is a nonnegative Borel measure on X such that $\mu(X) > 0$ for every nonempty open set $U \subseteq X$. This implies that the supremum norm of $f \in C_b(X)$ is equal to the L^∞ norm of f with respect to μ, so that $C_b(X)$ may be identified with a closed linear subspace of $L^\infty(X)$ with respect to μ under these conditions. In particular, the weak topology on $C_b(X)$ is the same as the topology induced on $C_b(X)$ by the weak topology on $L^\infty(X)$, as in the previous paragraph. If X is equipped with the discrete topology, then every function on X is continuous, and $C_b(X)$ is the same as $\ell^\infty(X)$. In this case, one can take μ to be counting measure on X, where every subset of X is measurable, and $\ell^p(X)$ is the same as $L^p(X)$ for each p, including $p = \infty$.

Let X be any topological space again, and let us take $C_b(X)$ to be the algebra of bounded continuous complex-valued functions on X. This is a commutative C^*-algebra, using complex conjugation as the involution. It is well known that $C_b(X)$ is isomorphic as a commutative C^*-algebra to the algebra $C(Y)$ of all continuous complex-valued functions on a compact Hausdorff topological space Y, known as the Stone–Čech compactification of X. Of course, one can simply take $Y = X$ when X is a compact Hausdorff space. If X is a locally compact Hausdorff space which is not compact, then X is homeomorphic to a dense open set in Y, and every bounded continuous complex-valued function on X can be extended to a continuous function on Y.

Now let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) > 0$, and let $L^\infty(X)$ be the usual space of complex-valued bounded measurable functions on X, where two such functions are identified when they are equal almost everywhere on X. This is also a commutative C^*-algebra with respect to pointwise multiplication of functions, and using complex conjugation as the involution. It follows that $L^\infty(X)$ is isomorphic as a C^*-algebra to the space of continuous complex-valued functions on a compact Hausdorff topological space Z. In particular, the dual of $L^\infty(X)$ can be identified with the dual of $C(Z)$, which can be characterized in terms of regular Borel measures on Z. If X is equipped with counting measure, then this is the same as the Stone–Čech compatification of X as a topological space with the discrete topology.

19 Limits at infinity

Let X be a locally compact Hausdorff topological space which is not compact, and let f be a continuous real or complex-valued function on X. We say that $f(x)$ tends to a real or complex number a as x tends to infinity in X if for each $\epsilon > 0$ there is a compact set $K \subseteq X$ such that

$$|f(x) - a| < \epsilon \tag{19.1}$$

for every $x \in X \backslash K$. This reduces to the condition that $f(x)$ vanish at infinity on X when $a = 0$, and is equivalent to asking that $f(x) - a$ vanish at infinity on X for any a. Note that the limit a of $f(x)$ as $x \to \infty$ on X is unique when it exists, because X is not compact.

It is well known that $C_0(X)$ is a closed linear subspace of $C_b(X)$ with respect to the supremum norm. Similarly, the space of continuous real or complex-valued functions on X that have a limit at infinity is a closed linear subspace of $C_b(X)$, and in fact a closed subalgebra of $C_b(X)$ with respect to pointwise multiplication. This is the same as the collection of continuous functions on X that have a continuous extension to the one-point compactification of X. In particular, if X is an infinite set with the discrete topology, then $C_b(X) = \ell^\infty(X)$, $C_0(X) = c_0(X)$, and the space of functions on X with a limit at infinity may be denoted $c(X)$. If $X = \mathbf{Z}_+$, then $c = c(\mathbf{Z}_+)$ consists of the real or complex-valued functions $f(x)$ on $\mathbf{Z}_+$ that have a limit as $x \to \infty$ in the usual sense, and $c_0 = c_0(\mathbf{Z}_+)$ is the subspace of these functions for which the limit is equal to 0.

Let $f_j(x)$ be the function on $X = \mathbf{Z}_+$ defined by $f_j(x) = 1$ when $x \le j$ and $f_j(x) = 0$ when $x > j$ for each positive integer j, and put $f(x) = 1$ for every $x \in \mathbf{Z}_+$. Thus $f_j \in c_0$ for each j, $f \in c$, and $\{f_j(x)\}_{j=1}^\infty$ converges to $f(x)$ for every $x \in \mathbf{Z}_+$. Remember that the dual of c_0 may be identified with $\ell^1 = \ell^1(\mathbf{Z}_+)$, and that the dual of ℓ^1 may be identified with $\ell^\infty = \ell^\infty(\mathbf{Z}_+)$. The obvious inclusion of c_0 in ℓ^∞ corresponds exactly to the standard inclusion of c_0 in its second dual with respect to these identifications. As in Section 13, the sequence $\{f_j\}_{j=1}^\infty$ converges to f with respect to the weak* topology on ℓ^∞ as the dual of ℓ^1. In this case, this follows from the definition of the sum of an infinite series as the limit of the sequence of partial sums. However, $\{f_j\}_{j=1}^\infty$ does not converge to f with respect to the weak topology on c, because the mapping from an element of c to its limit at infinity is a continuous linear functional on c. It follows that $\{f_j\}_{j=1}^\infty$ does not converge to f with respect to the weak topology on ℓ^∞, by the remark at the beginning of the previous section.

20 Cauchy sequences

Let V be a real or complex vector space, and let $\mathcal{N}$ be a nice collection of seminorms on V. A sequence $\{v_j\}_{j=1}^\infty$ of elements of V is said to be a *Cauchy sequence* in V if

$$\lim_{j,l \to \infty} N(v_j - v_l) = 0 \tag{20.1}$$

for each $N \in \mathcal{N}$. It is easy to see that $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence in V when it converges to an element of V with respect to the topology determined by $\mathcal{N}$, as usual. If $\mathcal{N}$ consists of a single norm N, then this Cauchy condition is the same as the usual one for the metric associated to N.

Equivalently, $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence in V if for any open set U in V that contains 0, there is an $L \geq 1$ such that

$$v_j - v_l \in U \tag{20.2}$$

for every $j, l \geq L$. This definition makes sense for any topological vector space V. If the topology on V is determined by a translation-invariant metric $d(w, z)$, then this is the same as saying that $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $d(w, z)$. Let us say that V is *sequentially complete* if every Cauchy sequence of elements of V converges to an element of V. One might simply say that V is complete when V is sequentially complete and there is a countable local base for the topology of V at 0, but otherwise one should also consider Cauchy conditions for nets or filters on V.

Now let V be a vector space with a norm $\|v\|$, and let V^* be the corresponding dual space, with the dual norm $\|\lambda\|_*$. A sequence $\{\lambda_j\}_{j=1}^\infty$ of elements of V^* is a Cauchy sequence with respect to the weak* topology on V if and only if $\{\lambda_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for every $v \in V$. This implies that $\{\lambda_j(v)\}_{j=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$, since $\mathbf{R}$ and $\mathbf{C}$ are complete, so that

$$\lambda(v) = \lim_{j \to \infty} \lambda_j(v) \tag{20.3}$$

defines a linear functional on V. If V is a Banach space, then there is a $C \geq 0$ such that

$$\|\lambda_j\|_* \leq C \tag{20.4}$$

for each j, by the Banach–Steinhaus theorem. Equivalently,

$$|\lambda_j(v)| \leq C\,\|v\| \tag{20.5}$$

for every $v \in V$ and $j \geq 1$, which implies that

$$|\lambda(v)| \leq C\,\|v\| \tag{20.6}$$

for every $v \in V$. Thus $\lambda \in V^*$ under these conditions, and hence $\{\lambda_j\}_{j=1}^\infty$ converges to λ with respect to the weak* topology on V^*. This shows that V^* is sequentially complete with respect to the weak* topology on V^* when V is a Banach space.

Let V be a real or complex vector space with a norm $\|v\|$ again. Suppose that $\{v_j\}_{j=1}^\infty$ is a sequence of elements of V which is a Cauchy sequence with respect to the weak topology on V. This means that $\{\lambda(v_j)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$ for every $\lambda \in V^*$, which implies that $\{\lambda(v_j)\}_{j=1}^\infty$ converges for every $\lambda \in V^*$, because $\mathbf{R}$ and $\mathbf{C}$ are complete. If

$$L_j(\lambda) = L_{v_j}(\lambda) = \lambda(v_j) \tag{20.7}$$

for each j, then it follows that $\{L_j\}_{j=1}^\infty$ is a Cauchy sequence in V^{**}, with respect to the weak* topology on V^{**} as the dual of V^*. Because V^* is automatically complete with respect to the dual norm, the argument in the previous paragraph implies that $\{L_j\}_{j=1}^\infty$ converges to some $L \in V^{**}$ with respect to the weak* topology on V^{**}. If $L(\lambda) = L_v(\lambda) = \lambda(v)$ for some $v \in V$ and every $\lambda \in V^*$, then it follows that $\{v_j\}_{j=1}^\infty$ converges to $v \in V$ with respect to the weak topology on V. In particular, V is sequentially complete with respect to the weak topology when V is a reflexive Banach space.

21 Uniform convexity

Let V be a real or complex vector space. A norm $\|v\|$ on V is said to be *uniformly convex* if for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that for each $v, w \in V$ with $\|v\| = \|w\| = 1$ and

$$\Big\|\frac{v+w}{2}\Big\| > 1 - \delta, \tag{21.1}$$

we have that

$$\|v - w\| < \epsilon. \tag{21.2}$$

Of course,

$$\Big\|\frac{v+w}{2}\Big\| \le \frac{\|v\| + \|w\|}{2} \le 1 \tag{21.3}$$

when $\|v\| = \|w\| = 1$, so that the point of (21.1) is for the norm of $(v+w)/2$ to be close to the maximum. Equivalently, for each $v, w \in V$ with $\|v\| = \|w\| = 1$ and $\|v - w\| \ge \epsilon$, one should have that

$$\Big\|\frac{v+w}{2}\Big\| \le 1 - \delta. \tag{21.4}$$

If the norm $\|\cdot\|$ is determined by an inner product on V, then one can check that $\|\cdot\|$ is uniformly convex, using the parallelogram law. It is well known that L^p norms are uniformly convex when $1 < p < \infty$, as a consequence of inequalities due to Clarkson. It is easy to see that this does not work when $p = 1$ or ∞, even on $\mathbf{R}^2$, basically because of a lack of strict convexity.

Let $v \in V$ with $\|v\| = 1$ be given, and let λ be a continuous linear functional on V such that $\|\lambda\|_* = 1$ and $\lambda(v) = 1$. Remember that such a linear functional exists, by the Hahn–Banach theorem. If $w \in V$ and $\|w\| = 1$ too, then

$$\Big\|\frac{v+w}{2}\Big\| \ge \Big|\lambda\Big(\frac{v+w}{2}\Big)\Big| = \frac{1}{2}\,|\lambda(v) + \lambda(w)| = \frac{1}{2}\,|1 + \lambda(w)|. \tag{21.5}$$

Using the triangle inequality, we get that

$$1 \le \frac{1}{2}\,|1 + \lambda(w)| + \frac{1}{2}\,|1 - \lambda(w)|, \tag{21.6}$$

so that

$$\Big\|\frac{v+w}{2}\Big\| \ge 1 - \frac{1}{2}\,|1 - \lambda(w)|. \tag{21.7}$$

If $\|\cdot\|$ is uniformly convex, then it follows that $\|v - w\| \to 0$ as $\lambda(w) \to 1$. More precisely, this also works when $\|\cdot\|$ is "locally uniformly convex" at v, in the sense that for each $\epsilon > 0$ there is a $\delta > 0$ such that (21.2) holds for every $w \in V$ that satisfies $\|w\| = 1$ and (21.1). This is the same as the uniform convexity condition in the previous paragraph, except that δ is allowed to depend on v as well as ϵ.

This shows that the topology on the unit sphere

$$\{v \in V : \|v\| = 1\} \tag{21.8}$$

induced by the weak topology on V is the same as the topology determined by the restriction of the metric associated to the norm $\|\cdot\|$ on (7.7) when $\|\cdot\|$ is locally uniformly convex at every point in (21.8). In particular, if $\{v_j\}_{j=1}^\infty$ is a sequence of vectors in V that converges with respect to the weak topology on V to $v \in V$, if $\|v_j\| = \|v\| = 1$ for each j, and if $\|\cdot\|$ is locally uniformly convex at v, then it follows that $\{v_j\}_{j=1}^\infty$ converges to v with respect to the norm on V. Similarly, if $\{v_j\}_{j=1}^\infty$ is a sequence of vectors in V that converges with respect to the weak topology on V to $v \in V$ and satisfies

$$\lim_{j \to \infty} \|v_j\| = \|v\|, \tag{21.9}$$

and if $\|\cdot\|$ is locally uniformly convex at each point in the unit sphere (21.8), then $\{v_j\}_{j=1}^\infty$ converges to v with respect to the norm on V. This is trivial when $v = 0$, and otherwise this can be derived from the previous statement by dividing the vectors by their norms. The case where V is an L^p space with $1 < p < \infty$ is known as the Radon–Riesz theorem.

As a counterexample for $p = 1$, one can take

$$f_j(x) = 1 + \cos(2\pi j x) \tag{21.10}$$

on $[0,1]$ for each positive integer j, which converges with respect to the weak topology on $L^1([0,1])$ to the constant function equal to 1 on $[0,1]$. It is easy to see that the L^1 norm of f_j is equal to 1 on $[0,1]$ for each j, using the fact that $f_j(x) \ge 0$ for every $x \in [0,1]$ and $j \ge 1$. However, one can check that $\{f_j\}_{j=1}^\infty$ does not converge with respect to the L^1 norm. To get a counterexample corresponding to $p = \infty$, let $\{f_j\}_{j=1}^\infty$ be the sequence of functions on $\mathbf{Z}_+$ such that $f_j(x) = 1$ when $x = 1$ and when $x = j$, and $f_j(x) = 0$ otherwise. This sequence converges pointwise on $\mathbf{Z}_+$ to the function $f(x)$ defined by $f(1) = 1$ and $f(x) = 0$ when $x \ge 2$. Clearly f_j is an element of $c_0(\mathbf{Z}_+)$ with supremum norm equal to 1 for each j, and f is an element of $c_0(\mathbf{Z}_+)$ with supremum norm equal to 1 as well. It follows that $\{f_j\}_{j=1}^\infty$ converges to f with respect to the weak topology on $c_0(\mathbf{Z}_+)$ under these conditions, although $\{f_j\}_{j=1}^\infty$ does not converge to f with respect to the supremum norm.

22 Weak compactness

Let V be a real or complex vector space with a norm $\|v\|$, and let V^* be the corresponding dual space, with the dual norm $\|\lambda\|_*$. Also let B^* be the closed unit ball in V^* with respect to $\|\lambda\|_*$. Thus B^* is compact with respect to the weak* topology on V^*, by the Banach–Alaoglu theorem. If V is separable, then the topology induced on B^* be the weak* topology on V^* is metrizable, as in Section 11. This implies that there is a countable subset of B^* which is dense in B^* with respect to the weak* topology, since compact metric spaces are separable.

If $V = \ell^1(\mathbf{Z}_+)$, for instance, then V is separable, but $V^* \cong \ell^\infty(\mathbf{Z}_+)$ is not separable with respect to the dual norm. Similarly, if $V = C([0,1])$, then V is separable with respect to the supremum norm, but V^* is not separable with respect to the corresponding dual norm.

Let V be any real or complex vector space with a norm $\|v\|$ again, and suppose that $A \subseteq V^*$ is a dense set in B^* with respect to the weak* topology. More precisely, it suffices to ask that the linear span of A be dense in V^* with respect to the weak* topology. Under these conditions, it is easy to see that A separates points in V. If A is countable, then it follows that the weak topology on V determined by A is metrizable, as in Section 8. If V is separable, then there is a countable set $A \subseteq V^*$ with this property, as before.

If $A \subseteq V^*$ separates points in V, then the weak topology on V associated to A is Hausdorff. Suppose that $E \subseteq V$ is compact with respect to the usual weak topology on V, associated to V^*. Because the weak topology on V associated to A is weaker than the usual weak topology on V, it follows that the topology induced on E by the weak topology on V is the same as the topology induced on E by the weak topology on V associated to A. This implies that the topology induced on E by the weak topology on V is metrizable when A is countable. In particular, if V is separable, then there is a countable set $A \subseteq V^*$ that separates points in V, and hence the topology induced on E be the weak topology on V is metrizable.

Let V be a real or complex vector space with a norm $\|v\|$ again, which is not necessarily separable. If W is a linear subspace of V which is closed with respect to the topology determined by the norm, then it is well known that W is also closed with respect to the weak topology on V, by the Hahn–Banach theorem. If $E \subseteq V$ is compact with respect to the weak topology on V, then it follows that $E \cap W$ is compact with respect to the weak topology on V too. This implies that $E \cap W$ is compact with respect to the weak topology on W as well, because the weak topology on W is the same as the topology induced on W by the weak topology on V, as mentioned previously. If W is separable, then it follows that the topology on $E \cap W$ induced by the weak topology on V is metrizable, by the remarks in the preceding paragraph.

Suppose that $E \subseteq V$ is compact with respect to the weak topology on V, and that $\{v_j\}_{j=1}^\infty$ is a sequence of elements of E. Let W be the closure of the linear span of the v_j's in V, with respect to the topology on V determined by the norm. Thus W is separable by construction, so that the topology on $E \cap W$ induced by the weak topology on V is metrizable, as in the previous paragraph. By construction, $\{v_j\}_{j=1}^\infty$ is a sequence of elements of $E \cap W$, and hence there is a subsequence $\{v_{j_l}\}_{l=1}^\infty$ of $\{v_j\}_{j=1}^\infty$ that converges to an element v of $E \cap W$ with respect to the weak topology on V, because $E \cap W$ is compact and metrizable with respect to the topology induced by the weak topology on V. This shows that $E \subseteq V$ is sequentially compact with respect to the weak topology on V when E is compact with respect to the weak topology on V.

Consider the case where $V = \ell^1(X)$ for some set X. If $E \subseteq \ell^1(X)$ is compact with respect to the weak topology on $\ell^1(X)$, then E is sequentially compact with respect to the weak topology on $\ell^1(X)$, as in the preceding paragraph. This implies that E is sequentially compact with respect to the topology on $\ell^1(X)$ determined by the norm, because every sequence in $\ell^1(X)$ that converges with respect to the weak topology also converges with respect to the norm, as in Section 13. It follows that E is also compact with respect to the topology on $\ell^1(X)$ determined by the norm, because sequentially compact subsets of any metric space are compact.

23 Closed sets

Let V be a real or complex vector space, and let Λ be a nonempty collection of linear functionals on V. Suppose that $v \in V$ is not in the closure of a set $E \subseteq V$ with respect to the weak topology on V determined by Λ. This means that there are finitely many elements $\lambda_1, \ldots, \lambda_n$ of Λ and a positive real number r such that

$$\max_{1 \le j \le n} |\lambda_j(v) - \lambda_j(w)| \ge r \tag{23.1}$$

for every $w \in E$. Let T be the linear mapping from V into $\mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate, such that the jth component of $T(u)$ is equal to $\lambda_j(u)$ for every $u \in V$. Note that $T(v)$ is not an element of the closure of $T(E)$ in $\mathbf{R}^n$ or $\mathbf{C}^n$ under these conditions.

If E is a linear subspace of V, then $T(E)$ is a linear subspace of $\mathbf{R}^n$ or $\mathbf{C}^n$, and hence $T(E)$ is automatically a closed set in $\mathbf{R}^n$ or $\mathbf{C}^n$. Because $T(v) \notin T(E)$, there is a linear functional μ on $\mathbf{R}^n$ or $\mathbf{C}^n$ such that $\mu \equiv 0$ on $T(E)$ and $\mu(T(v)) \ne 0$. Every linear functional on $\mathbf{R}^n$ or $\mathbf{C}^n$ is a linear combination of the coordinate functions, which implies that

$$\widetilde{\mu} = \mu \circ T \tag{23.2}$$

is a linear combination of $\lambda_1, \ldots, \lambda_n$ on V. It follows that $\widetilde{\mu}$ is a continuous linear functional on V with respect to the weak topology on V determined by Λ, $\widetilde{\mu} \equiv 0$ on E, and $\widetilde{\mu}(v) \ne 0$.

Similarly, if E is a convex set in V, then $T(E)$ is a convex set in $\mathbf{R}^n$ or $\mathbf{C}^n$. Because $T(v)$ is not in the closure of $T(E)$ in $\mathbf{R}^n$ or $\mathbf{C}^n$, classical finite-dimensional separation theorems can be applied to $T(v)$ and $T(E)$. This leads to analogous separation properties of v and E in V, using linear functionals on V that are continuous with respect to the weak topology associated to Λ.

Remember from Section 2 that a linear functional on V is continuous with respect to the weak topology on V determined by Λ if and only if it can be expressed as a linear combination of finitely many elements of Λ. The kernel of any continuous linear functional on V is automatically a closed linear subspace of V, and hence the intersection of the kernels of any family of continuous linear functionals on V is also a closed linear subspace of V. If E is a linear subspace of V, then the previous argument shows that the closure of E with respect to the weak topology on V determined by Λ can be expressed as the intersection of the kernels of the linear functionals on V that are continuous with respect to this topology and vanish on E. In particular, a linear subspace E of V is dense with respect to the weak topology on V corresponding to Λ if and only if the only linear functional on V that is continuous with respect to this topology and vanishes on E is identically 0 on V.

Suppose now that $\mathcal{N}$ is a nonempty collection of seminorms on V. If E is a closed linear subspace of V with respect to the topology on V determined by $\mathcal{N}$, and if $v \in V \setminus E$, then the Hahn–Banach theorem can be used to show that there is a continuous linear functional λ on V such that $\lambda \equiv 0$ on E and $\lambda(v) \neq 0$. This implies that E can be expressed as the intersection of the kernels of a collection of continuous linear functionals on V, and hence that E is a closed set with respect to the corresponding weak topology on V. Similarly, if E is a closed convex set in V with respect to the topology determined by $\mathcal{N}$, then it can be shown that E is also closed with respect to the weak topology on V.

Let V be a real or complex vector space with a norm $\|\cdot\|$, and let V^* be the corresponding dual space. If $E \subseteq V^*$ is dense in V^* with respect to the weak* topology, then it is easy to see that E separates points in V. Of course, this uses the fact that V^* separates points in V, by the Hahn–Banach theorem. Conversely, if E is a linear subspace of V^* that separates points in V, then the earlier argument implies that E is dense in V^* with respect to the weak* topology on V^*. Indeed, the condition that E separate points in V is equivalent to saying that if $u \in V$ satisfies $\lambda(u) = 0$ for every $\lambda \in E$, then $u = 0$.

Let V^{**} be the dual of V^*, and remember that $L_v(\lambda) = \lambda(v)$ defines an element of V^{**} for each $v \in V$. The collection E of elements of V^{**} of the form L_v for some $v \in V$ is a linear subspace of V^{**} that automatically separates points in V^*. It follows that E is dense in V^{**} with respect to the weak* topology on V^{**} as the dual of V^*, as in the preceding paragraph. As a refinement of this, let A be the collection of elements of V^{**} of the form L_v for some $v \in V$ with $\|v\| \leq 1$. One can show that the closure of A in V^{**} with respect to the weak* topology on V^{**} as the dual of V^* is the same as the closed unit ball in V^{**}.

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