Problem 1. Given the initial condition below for the differential equation
\[ \frac{dy}{dt} = (y - 2)(y - 3)y \]
What does the Existence and Uniqueness theorem say about the corresponding solution? Explain. Sketch a graph for \( y(t) \) indicating its initial condition on it.

Since \( y(y - 2)(y - 3) \) is continuous for all \( (y, t) \) the existence theorem guarantees that for the given initial conditions a solution must exist. Moreover, for all \( (y, t) \), \( y(y - 2)(y - 3) \) is also differentiable; hence we have that for each initial condition there must exist a unique solution.

By uniqueness, we then know that for different initial conditions the graphs of the solutions cannot intersect.

By inspecting \( y(y - 2)(y - 3) \) we conclude that there are three equilibrium solutions, namely \( y = 0 \), \( y = 2 \) and \( y = 3 \). By checking the signs we have that \( \frac{dy}{dt} > 0 \) for \( y > 3 \) and \( 0 < y < 2 \); while \( \frac{dy}{dt} < 0 \) for \( y < 0 \) and \( 2 < y < 3 \).

In case (a) for the initial condition \( y(0) = 1 \) we have that the solution is increasing and stays in between the equilibrium solutions \( y = 0 \) and \( y = 2 \) by uniqueness (it cannot ‘cross’ these lines). Moreover the \( \lim_{t \to \infty} y(t) = 2 \) and \( \lim_{t \to -\infty} y(t) = 0 \) asymptotically.

In case (b) for the initial condition \( y(0) = 3 \) by uniqueness we must have that the only solution with that initial condition must be the equilibrium solution \( y = 3 \).

Problem 2. Given the differential equation
\[ \frac{dy}{dt} = 3y(y - 1) \]
(a) Sketch its phase line and identify equilibrium points as sinks, sources or nodes.
(b) For the equation above and initial conditions \( y(0) = 1/2 \) on the one hand and \( y(0) = 2 \) on the other; sketch the graphs of the solutions satisfying these initial conditions. Put both graphs on one pair of axes.

Problem 3 Solve the following initial value problem for the given linear differential equation

\[
\frac{dy}{dt} = -2t y + 4e^{-t^2} \quad y(0) = 3
\]

First rewrite as

\[
\frac{dy}{dt} + 2ty = 4e^{-t^2} \implies \mu(t) = e^{\int 2t \, dt} = e^{t^2}
\]

Multiplying both sides by \( \mu(t) \) we get

\[
e^{t^2} \frac{dy}{dt} + e^{t^2} 2ty = 4 \implies (e^{t^2} y)' = 4
\]

Integrating both sides with respect to \( t \) we obtain,

\[
e^{t^2} y = 4t + C \implies y = 4te^{-t^2} + Ce^{-t^2}
\]

Since \( y(0) = 3 \), plugging in we have that \( C = 3 \). Hence

\[
y = 4te^{-t^2} + 3e^{-t^2}
\]