M623 HOMEWORK – Fall 2024

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SET 1: Due Date 09/19/2024

Problem 1 Give an example of a decreasing sequence of nonempty closed sets in $\mathbb{R}^n$ whose intersection is empty.

Problem 2 Give an example of two **closed** sets $F_1, F_2 \subset \mathbb{R}^2$ such that $F_1 \cap F_2 = \emptyset$ and $\text{dist}(F_1, F_2) = 0$.

Problem 3 a) Given an interval $[a, b] \subset \mathbb{R}$, construct a sequence of continuous functions $\phi_k(x)$ such that for every fixed $x \in \mathbb{R}$ we have

$$\lim_{k\to\infty} \phi_k(x) = \begin{cases} 1 & \text{if} \quad x \in [a, b] \\ 0 & \text{if} \quad x \notin [a, b] \end{cases}$$

b) Can one construct such a sequence ϕ_k so that it also converges uniformly as $k \to \infty$? Explain and justify your answer.

Problem 4 A continuous function $\phi : \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y), \quad ; \forall\, x, y \in \mathbb{R}, \forall\, \lambda \in [0, 1]$$

Show that if ϕ is convex, then if $x_1, \ldots , x_n$ are points in $\mathbb{R}$ then

$$\phi\left(\frac{x_1 + \ldots + x_n}{n}\right) \leq \frac{\phi(x_1) + \ldots \phi(x_n)}{n}$$

More generally, show that if $\alpha_1, \ldots , \alpha_n$ is a sequence of nonnegative numbers with

$$\sum_{i=1}^{n} \alpha_i = 1$$

Then, for any n points $x_1, \ldots , x_n$ in $\mathbb{R}$ we have

$$\phi\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \phi(x_i)$$

This last inequality is known as *Jensen's inequality.*

Problem 5 Let $x_1, \ldots , x_n$ be all nonnegative numbers. Prove the *arithmetic-geometric mean inequality*

$$\big(x_1 x_2 \ldots x_n\big)^{1/n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}$$

Hint Apply Jensen's inequality with a conveniently chosen convex function.

Problem 6 Compute the following Riemann integrals:

$$\int_0^1 x^k\,dx, \quad k>0; \quad \int_0^1 x^{-k}\,dx, \ \ k\in(0,1); \quad \int_1^\infty x^{-k}\,dx, \ \ k\in(1,\infty)$$

$$\int_0^\infty e^{-ax^2}x\,dx, \quad a>0; \quad \int_0^\infty e^{-ax^2}x^2\,dx, \quad a>0 \ \ \text{(use that } \int_{-\infty}^\infty e^{-x^2/2}\,dx=\sqrt{2\pi}\text{)}$$

$$\int_a^b \cos(mx)\,dx, \quad m\in\mathbb{N}.$$

For the last one, fix a and b and investigate the limit $m\to\infty$. Does the result depend on a, b?

SET 2: Due Date 09/26/2024

From Chapter 1 (pp 37-42): 1, 2, 3.

Additional Problems

A.I. Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage, each interval removed has length $\delta 3^{-k}$, for some $0<\delta<1$. Show that the resulting set is perfect, has measure $1-\delta$, and it is totally disconnected (in particular contains no intervals).

A.II. For $x\in[0,1]$, let

$$x=\sum_{n=1}^\infty \frac{a_n}{2^n}, \quad a_n\in\{0,1\},$$

be the binary expansion of x. Let A be the set of points x which admit a binary expansion with zero in all even positions (i.e., $a_{2n}=0$ for all $n\geq 1$). Show that A is a set of Lebesgue measure 0.

Hint: Write the set A has $A=\cap_{n=0}^\infty A_n$ where $A_0=[0,1]$, $A_{n+1}\subset A_n$ and A_{n+1} is obtained from A_n by removing some of the dyadic intervals in A_n.

A.III. The following problem is a special case of Problem 4 in [SS, Ch1] dealing with what we call *Fat Cantor Sets*.

Construct a closed set $\mathcal{C}$ analogous to the Cantor $\frac{1}{3}$-set by removing instead at the stage k^{th} stage 2^{k-1} centrally situated open intervals each of length $\ell_k=\frac{1}{4^k}$. The set $\mathcal{C}$ is again defined as the (countably) infinity intersection of the closed sets C_k appearing at stage. k.

a) Show that $\mathcal{C}$ is compact, totally disconnected and has no isolated points (this is similar to problem 1).

b) Show that $m_*(\mathcal{C}) = \frac{1}{2}$ and conclude (with justification) that $\mathcal{C}$ is uncountable.

A.IV. a) Let $A = \cup_{n=1}^{\infty} A_n$ with $m_*(A_n) = 0$. Use the definition of exterior measure to prove that $m_*(A) = 0$.

b) Use a) to prove that any countable set in $\mathbb{R}^d$ is measurable and has measure zero.

SET 3: Due Date 10/10/2024

A.I.) Prove that a set E in $\mathbb{R}^d$ is measurable if and only if for every set A in $\mathbb{R}^d$,

$$m_*(A) = m_*(A \cap E) + m_*(A - E) \tag{1}$$

Hint: First assume E is measurable and prove (1). Then to prove the converse, to prove that (1) implies that E is measurable, assume first that $m_*(E) < \infty$. Then do the case of $m_*(E) = \infty$. For the later write $E = \bigcup_{k=1}^{\infty}[E \cap B(0,k)]$ where $B(0,k)$ is the ball centered at the origin of radius k. This characterization of measurability is called the *Carathéodory condition.*

Remark: Note that $A - E = A \cap E^c$ so the *Carathéodory condition* could be rephrased as: A set E in $\mathbb{R}^d$ is measurable if and only if for every set A in $\mathbb{R}^d$,

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c) \tag{1'}$$

From Chapter 1 (pp 37-42): 5, 6, 7, 10, 11, 16, 25.

SET 4: Due Date 10/17/2024

From Chapter 1 (pp 37-42): 28, 29.

A.I. Let $\{E_n\}_{n\geq 1}$ be a countable collection of measurable sets in $\mathbb{R}^d$. Define

$$\limsup_{n\to\infty} E_n := \{x \in \mathbb{R}^d \,:\, x \in E_n, \text{ for infinitely many } n\}$$

$$\liminf_{n\to\infty} E_n := \{x \in \mathbb{R}^d \,:\, x \in E_n, \text{ for all but finitely many } n\}$$

a) Show that

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k \qquad \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty} E_j$$

b) Show that

$$m(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} m(E_n)$$

$$m(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} m(E_n) \quad \text{provided that } m(\bigcup_{n=1}^{\infty} E_n) < \infty$$

c) Find $\limsup E_k$ and $\liminf E_k$ for the sequence $\{E_k\}$ defined as follows:

$$E_k := \begin{cases} [-1/k, 1] & \text{for } k \text{ odd} \\ [-1, 1/k] & \text{for } k \text{ even} \end{cases}$$

AII. Do problem 13a) in Chapter 1 page 41.

Hints for 13a) First show that for each $n \in \mathbb{N}$, the set $\mathcal{O}_n := \{x : d(x, F) < \frac{1}{n}\}$ is open.
Then show that if $x \notin F$ then since F is closed, $d(x, F) > \delta$ for some $\delta > 0$.
Finally prove that if F is closed then $F = \bigcap_{n=1}^{\infty} \mathcal{O}_n$.
Conclude.

Additional Problem (do but do not turn in): Read, and then write/type, explaining key points and expanding/filling gaps where necessary the construction of a *nonmeasurable set* in Stein-Shakarchi's book in pages 24-25-26. Note in the course of the proof you might need to use the Invariance properties of the Lebesgue measure in the bottom half of page 22.

AIII. First do 32 (pp 44-45). (Hint for part a) consider the sets $E_k = E + r_k \subset \mathcal{N}_k$, where $\{r_k\}_{k\geq 1}$ is an enumeration of the rationals.)

Note that part b) should read "...prove that *there exists* a subset of G which is....".

Furthermore, show:

c) $\mathcal{N}^c = I \setminus \mathcal{N}$ satisfies $m_*(\mathcal{N}^c) = 1$. (Hint: argue by contradiction and use a))

d) Conclude that

$$m_*(\mathcal{N}) + m_*(\mathcal{N}^c) \neq m_*(\mathcal{N} \cup \mathcal{N}^c)$$

Additional Problem (do but do not turn in): From Chapter 1 (pp 37-42) do problem 10

SET 5: DUE DATE 10/24/2024

From Chapter 1 (pp 37-42): 17, 22

Hint for 17: Note that

$$\{x : |f_n(x)| = \infty\} = \bigcap_{j=1}^{\infty}\{x : |f_n(x)| > \frac{j}{n}\}.$$

Hence the hypothesis implies that for each n,

$$m(\bigcap_{j=1}^{\infty}\{x : |f_n(x)| > \frac{j}{n}\}) = 0.$$

But then $\lim_{j\to\infty} m(\{x : |f_n(x)| > \frac{j}{n}\}) = 0$ (Why? Justify.). Next, follow the hint in the book from here.

Hint for 22: Argue by contradiction and use the continuity of such an f at $x = 1$ to show that $m(\{x : f(x) \neq \chi_{[0,1]}(x)\})$ contains an interval of small but positive measure and hence it can't be zero (which gives you the contradiction).

From Chapter 1 (pp 37-42): Prove problem 18 in this case only "Every measurable function $f : [a, b] \to \mathbb{R}$ (finite-valued) is the limit a.e. of a sequence of continuous functions on $[a, b]$."

Hint This is Lusin's Theorem. Note the Handout and fully rewrite on your own thinking the proof in this case.

Additional Problem AI. The following relates to the proof of Theorem 4.1 page 31. Prove that the sequence of nonnegative simple functions $\{\phi_k\}_k$ that approximate pointwise f is indeed increasing, ie. $\phi_k \leq \phi_{k+1}$.

From Chapter 2 (pp 89-97): 1

Hint One way to proceed is as follows. For $j = 1, \ldots, N$ $(N = 2^n - 1)$ write each j as an n-digit binary number $j_1 j_2 \ldots j_n$ For example, $2 = 000\ldots10$ in binary representation. Next define the set A_k to be F_k if $j_k = 1$ and F_k^c if $j_k = 0$ and let F_k^* be the intersection from $k = 1$ up to$n$ of A_k. You'll see that such F_k^* is intersection of n sets each of which could be F_j or F_j^c depending on the binary representation of k. Now:

i) Prove that the collection of F_k^* thus defined is pairwise disjoint and that

$$F_k = \bigcup_{F_j^* \subset F_k} F_j^*$$

.

ii) Argue from i) to deduce form here that $\bigcup_{\ell=1}^{n} F_\ell \subset \bigcup_{j=1}^{N} F_j^*$.

iii) Finally show that the reverse inclusion easily holds.

SET 6: DUE DATE 11/14/2024

From Chapter 2 (pp 89-93): 6, 8, 9, 10, 11.

Hints. For 6a) consider the positive real x axis and select intervals of the form $[k, k + \frac{1}{2^{2k}}], k \geq 0$ integer. Now think of a continuous (piecewise linear) function f whose graph looks like series of triangles that get higher and higher over each of these intervals so that the area under each one is 2^{-k} and f is zero elsewhere. You don't need to attempt to write the function analytically, but graph it identifying the height of the triangles and argue why this function gives you the desired conclusion.

For 6b) Use the $\varepsilon - \delta$ definition of uniform convergence to choose a suitable countable family of disjoint intervals of small fixed length (say $C\delta$) on which $|f| \geq c\varepsilon$ (for some suitable fixed constants $C, c > 0$. Tchebychev might then be useful to draw the conclusion.

Additional Problems:

A.I. If a function f is integrable then we proved in Proposition 1.12 (Chapter 2) that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any set A with $m(A) \leq \delta$, we have that $\int_A |f(x)|\, dm \leq \varepsilon$ (*absolute continuity of the Lebesgue integral*).

We say that a sequence of functions $\{f_n\}_{n\geq 1}$ is **equi-integrable** if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. for any set A with $m(A) \leq \delta$, we have that $\int_A |f_n(x)|\, dm \leq \varepsilon$ for all $n \geq 1$.

Now prove the following.

Let E be a set of finite measure, $m(E) < 1$, and let $\{f_n\} : E \to R$ be a sequence of functions which is equi-integrable. Show that if $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. x, then

$$\lim_{n\to\infty} \int_E |f_n(x) - f(x)|\, dm = 0.$$

Hint. Use Egorov's Theorem as in the bounded convergence theorem.

A.II. We say that a sequence of measurable functions $\{f_n\}_{n\geq 1}$ *converges in measure* to another measurable function f is for every $\varepsilon > 0$,

$$m(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0, \qquad \text{as} \qquad n \to \infty$$

Prove that if a sequence of measurable function f_n converges in measure to another measurable function f then there exists a subsequence $\{f_{n_j}\}_{j\geq 1}$ which converges almost everywhere to f, that is $f_{n_j}(x) \to f(x)$ a.e. x as $j \to \infty$.

Hint. First show that for $\varepsilon = 2^{-j}$ one can choose n_j such that for all $n \geq n_j$

$$m(\{x : |f_n(x) - f(x)| > 2^{-j}\}) \leq 2^{-j}.$$

Next note that for each $j \geq 1$ one may choose $n_{j+1} \geq n_j$ (note this is needed to satisfy the definition of subsequence) and define $A_j := \{x : |f_{n_j}(x) - f(x)| > 2^{-j}\}$.

Use Borel-Cantelli to prove $m(\limsup_{j\to\infty} A_j) = 0$ and show this is equivalent to the desired conclusion.

A.III. Suppose that $\{f_n\}_{n\geq 1}$ is a sequence of non-negative measurable function, that is $f_n \geq 0$ for all n, such that f_n converges in measure to f . Show that then

$$\int f(x)\, dm \leq \liminf_n \int f_n(x)\, dm$$

Hint. Let us call denote by $I = \int f dx,\ a_n = \int f_n dx,$ and $A = \liminf \int f_n dx$.

We wish to prove that $I \leq A$. Note that since A is the smallest of all limit points, there must exist a subsequence of a_{n_k} of a_n that converges to A ; i.e. $\lim_{k\to\infty} a_{n_k} = A$. In particular note that

$$A = \lim_{k\to\infty} \int f_{n_k} dx \qquad \text{and that } f_{n_k} \to f \text{ in measure as well}$$

Then for every subsequence $a_{n_{k_j}}$ of a_{n_k} we also have that

$$A = \lim_{j\to\infty} a_{n_{k_j}} = \lim_{j\to\infty} \int f_{n_{k_j}} dx$$

and $f_{n_{k_j}} \to f$ in measure as well . Next obtain one (sub)subsequence for which we have a.e. convergence to f (you may use previous problem). Apply Fatou's Lemma and conclude.

SET 7: 11/21/2024.

From Chapter 2 (pp 89-93): 12, 15, 16.

Hint Recall the counter-examples we did in class.

From Chapter 2 (pp 95): 3.

Hint. Note $\{f_n\} \to f$ in L^1 as $n \to \infty$ means $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$. To demonstrate one direction suitably use Tchebychev's inequality. For the converse consider $f_n(x) = n\chi_{[0,\frac{1}{n})}$.

Additional Problems:

I. Let f and f_n, $n \geq 1$ be measurable functions on $\mathbb{R}^d$

a) Suppose that $\mu(E) < \infty$ and that f and f_n, $n \geq 1$ are all supported on E. Prove that $f_n \to f$ a.e implies $f_n \to f$ in measure.

b) Prove that the converse of (a) is false even under the hypothesis of (a) (ie. all functions supported on E a set of finite measure)

Hint. Let $E = [0, 1]$ and consider the (double) sequence $f_{m,k}(x) = 1_{E_{m,k}}(x)$ $(m, k \in \mathbb{N})$, where $E_{m,k} := [\frac{m-1}{k}, \frac{m}{k}]$.)

II. Consider the sequence of functions $f_n(x) := \frac{n}{1+(nx)^2}$. For $a \in \mathbb{R}$ be a fixed number consider the Lebesgue integral $I_a(f_n)(x) := \int_a^\infty f_n(x)\, dm$. Compute $\lim_{n\to\infty} I_a(f_n)(x)$ in each case: i) $a = 0$ ii) $a > 0$ and iii) $a < 0$. Carefully justify your calculations (recall the transformation of integrals under dilations).

III. In Chapter 2 we first prove the Bounded Convergence Theorem (using Egorov Theorem). Then, we proved Fatou's Lemma (using the BCT) and deduced from (a corollary of) it the Monotone Convergence Theorem. Finally we proved the Dominated Convergence Theorem (using both BCT and MCT). Here we would like to prove these sequence of results in a different order. Namely, prove:

a) Prove Fatou's Lemma *from* the MCT by showing that for any sequence of measurable functions $\{f_n\}_{n\geq 1}$,

$$\int \liminf_{n\to\infty} f_n\, dm \;\leq\; \liminf_{n\to\infty} \int f_n\, dm.$$

Hint. Note that $\inf_{n\geq k} f_n \leq f_j$ for any $j \geq k$, whence $\int \inf_{n\geq k} f_n\, dm \;\leq\; \inf_{j\geq k} \int f_j$.

b) Now prove the DCT from Fatou's Lemma.

Hint. Apply Fatou's Lemma to the nonnegative functions $g + f_n$ and $g - f_n$.

IV. Use the DCT to prove the following: let $\{f_n\}_{n\geq 1}$ be a sequence of integrable functions on $\mathbb{R}^d$ such that $\sum_{n=1}^\infty \int |f_n(x)|\, dm < \infty$. Show that $\sum_{n=1}^\infty f_n(x)$ *converges* a.e. $x \in \mathbb{R}^d$ to an *integrable function* and that $\sum_{n=1}^\infty \int f_n(x)\, dm \;=\; \int \sum_{n=1}^\infty f_n(x) dm$.