Hint for Exercise 1 (Ch. 3.5 pg 145)

For part a) use the dilation property of the Lebesgue integral for the first two properties of good kernels. For the last one you may use a change of variables and Prop. 1.12 i) in a suitable fashion.

For part b) Use that φ vanishes outside a (big) ball centered at 0 and - say - radius $R > 0$, and φ is bounded - say - $|\varphi(x)| \leq B$ $\forall x$ (for some $B > 0$).

Then note $\varphi(x/\delta) = 0$ if $|x/\delta| > R$ and that

$$\frac{|x|^{d+1}}{\delta} |K_\delta(x)| = \frac{|x|^{d+1}}{\delta^{d+1}} |\varphi(x/\delta)| \leq R^{d+1} \cdot B$$

and conclude from here the desired properties.

For part c): One way to prove it is to denote by

$$\mathcal{L} = \{ f \in L^1(\mathbb{R}^d) : \|(f * K_\delta) - f\|_{L^1} \to 0 \text{ as } \delta \to 0 \}$$

and show that $\mathcal{L} = L^1(\mathbb{R}^d)$ by proving:

(a) $\mathcal{L}$ is closed in $L^1(\mathbb{R}^d)$ (i.e. show that if $f \in \overline{\mathcal{L}}$ (closure of $\mathcal{L}$) then $f \in \mathcal{L}$.

(b) Show that $C_c(\mathbb{R}^d) \subset \mathcal{L}$ and use that $C_c(\mathbb{R}^d)$ are dense in $L^1(\mathbb{R}^d)$ (plus (a) to conclude)

Another way to prove it is directly by following these steps:

Write $\| f * K_\delta - f \| = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \big(f(x-y) - f(x)\big) K_\delta(y)\, dy \right| dx$

WHY

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y) - f(x)|\,|K_\delta(y)|\, dx\, dy$$

$$= \underbrace{\int_{|y| \leq \eta} \int_{\mathbb{R}^d} |f(x-y) - f(x)|\,|K_\delta(y)|\, dx\, dy}_{(I)} +$$

$$+ \underbrace{\int_{|y| > \eta} \int_{\mathbb{R}^d} |f(x-y) - f(x)|\,|K_\delta(y)|\, dx\, dy}_{(II)}$$

for any $\eta > 0$.

For (I) note $= \int_{|y| \leq \eta} |K_\delta(y)|\, \| f(x-y) - f(x) \|_{\substack{L^1(\mathbb{R}^d) \\ (\text{in } x)}}\, dy$

and use the L^1 continuity of translations to control this term (argue carefully).

For (II) note that

$$(II) = \int_{|y| > \eta} |K_\delta(y)| \int_{\mathbb{R}^d} |f(x-y) - f(x)|\, dx\, dy$$

$$\leq \int_{|y|>\eta} |K_\delta(y)| \int_{\mathbb{R}^d} \left(|f(x-y)| + |f(y)| \right) dx\,dy$$

$$\underset{\text{WHY?}}{=} \int_{|y|>\eta} |K_\delta(y)|\; 2\, \|f\|_{L^1(\mathbb{R}^d)}\, dy$$

Argue from here to conclude #.