M523Honors: Introduction to Modern Analysis

Homework

Fall 2018

Note: **Do** but don't turn in yet the problems followed by an asterisk $*$.

Assignment 1. Due Thursday 09/27/2018.

From Section 1.1: 1, 3, 4, 5, 6a), 7, 10, 11, 12$*$

From Section 1.2: 1, 2, 3, 5a), 5d), 6a), 6b), 6c), 7a), 7b), 7d), 9b), 10.

Additional Problem 1*: Show that $(\mathbb{C}, +, \cdot)$ as defined in class is a field.

From Section 1.3: 1, 3, 5, 7, 8, 9.

Assignment 2. Due Thursday 10/04/2018.

From Section 1.4: 1, 4, 7, 8, 9, 10a)c) 11a)

(In 10c) f^2 means $[f(x)]^2$ and **not** $f \circ f$)

Assignment 3. Due Thursday 10/11/2018.

From Section 2.1: 1a)c), 2a)c)d), 3b)c)d), 4, 5, 6, 7

Assignment 4. Due Thursday 10/25/2018.

From Section 2.2: 1, 2, 3, 4, 5(modified), 7, 8, 9.

Problem 5**(modified)**. Prove theorem 2.26 in the special case when $a_n = 1$. That is prove $\frac{1}{b_n} \to \frac{1}{b}$ under the same hypothesis.

(Note: the general case is posted as a Handout.).

From Section 2.4: 1, 2, 3, 5, 6, 7, 9, 13.

Special Project I:* Do Project #1 at the end of Chapter 2.

Assignment 5. Due Thursday 11/08/18.

From Section 2.5: 1, 3(modified), 4, 5, 7, 8.

Problem 3(modified). *Suppose a set S of real numbers is bounded and let η be a lower bound for S. Show that η is the greatest lower bound of S if and only if for every $\varepsilon > 0$ there is an element of S in the interval $[\eta, \eta + \varepsilon]$*

Extra Problem 2* for Section 2.5 Prove the existence of *greatest lower bounds* just as we proved the existence of *least upper bound* in Theorem 2.5.1

From Section 2.6: 1, 2, 3, 4, 6, 8, 9, 10, 11, 13.

Special Project II:* Do Problem 14 of Section 2.4.

Special Project III:* Do Project #5 at the end of Chapter 2 (page 70)

Assignment 6. Due Thursday 11/29/2018.

From Section 3.1 : 2, 4, 5, 7, 8a)b), 9.

Assignment 7. Due Thursday 12/06/2018.

From Section 3.2 : 1, 3, 4, 5, 7, 10, 11.

From Section 3.3: 1, 2, 3, 4, 5, 7, 12, 14, 15.

Special Project IV:* Do Problem 13 of Section 3.2

Assignment 8. Due Friday 12/14/2018 (no later than 12:30pm).

From Section 5.1: 2, 7, 8, 12.

From Section 5.2: 1, 2, 6, 9.

Hints For 5.1 #7: for each $n \geq 1$ choose an x in $[0, 1]$ such that $n\,x = 1$. Call that x, x_n and compute $f_n(x_n)$.

For 5.1 #8: for each $n \geq 1$ choose an x in $[0, 1]$ such that $\dfrac{x}{n} = 1$. Call that x, x_n and compute $f_n(x_n)$.

For 5.1 #12: Given $\varepsilon > 0$, write $|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|$ and find $N = N(\varepsilon)$ so that (a) the first term on the r.h.s of the inequality is less than $\varepsilon/2$ thanks to the *uniform convergence* of f_n to f; (b) the second term on the r.h.s of the inequality is less than $\varepsilon/2$ thanks to the *continuity* of f

For 5.2 #2a): first prove that the sequence of functions $f_n(x) = (x + \frac{1}{n})^2$ converges uniformly to the function $f(x) = x$ on $[0, 1]$ as n goes to infinity. Then use Theorem 5.2.2 to compute.

For 5.2 #6: Denote by f the limiting function and write $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)|$.
First note that since the convergence is uniform on $[0, 1]$, f must be continuous (why?) and hence bounded (why?). Second, prove that there exists N (think of $\varepsilon = 1$) such that for all $n \geq N$ the first term on the right hand side is less than 1. Third, note that each of the remaining functions f_n, $1 \leq n \leq N - 1$ is continuous and bounded on $[0, 1]$ (and there are only a finite $N - 1$, a finite number of them).
Finally, put all the ingredients together to conclude!

Special Project V:*

Let f be a continuous function on $\mathbb{R}$ such that the improper integral $\int_{-\infty}^{\infty} f(x)\,dx < \infty$.

Let f_n be a sequence of continuous functions defined on $\mathbb{R}$ such that f_n converge uniformly to f on every finite, closed interval $[a, b]$ of $\mathbb{R}$.

Suppose that there exists a **continuous** function $g : \mathbb{R} \to \mathbb{R}$ such that:
(i) $g(x) \geq 0$
(ii) the improper integral $\int_{-\infty}^{\infty} g(x)\,dx < \infty$,
(iii) for all $n \geq 1$ and all $x \in \mathbb{R}$ we have that $|f_n(x)| \leq g(x)$ and also $|f(x)| \leq g(x)$.

(a) Prove that each of the improper integrals $\int_{-\infty}^{\infty} f_n(x)\,dx < \infty$

(b) Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx \;=\; \int_{-\infty}^{\infty} f(x)\,dx.$$

Hint for b) The improper integral $\int_{-\infty}^{\infty} g(x)\,dx = \lim_{M\to\infty} \int_{-M}^{M} g(x)\,dx$ and since $\int_{-\infty}^{\infty} g(x)\,dx < \infty$ then the limit in M of the sequence $\int_{-M}^{M} g(x)\,dx$ of real numbers (why are each of these finite?) exists.

Hence given $\varepsilon > 0$ there exists an $M_0 = M_0(\varepsilon) > 0$ such that

$$\left| \int_{-\infty}^{\infty} g(x)\,dx - \int_{-M}^{M} g(x)\,dx \right| \;=\; \left| \int_{|x|>M} g(x)\,dx \right| \leq \varepsilon \qquad M \geq M_0.$$

To prove part for b), you need to consider

$$\left| \int_{-\infty}^{\infty} f_n(x)\,dx - \int_{-\infty}^{\infty} f(x)\,dx \right| \;=\; \left| \int_{-\infty}^{\infty} \Big(f_n(x) \;-\; f(x) \Big)\,dx \right| \qquad (\dagger)$$

Next, rewrite the r.h.s in (†) as the sum of two integrals, one over the set $|x| \leq M$ and the other over the set $|x| > M$ and use triangle inequality to bound (†) by

$$\left| \int_{|x| \leq M} \Big(f_n(x) \, - \, f(x) \Big) \, dx \right| + \left| \int_{|x| > M} \Big(f_n(x) \, - \, f(x) \Big) \, dx \right|.$$

Use uniform convergence over the set $|x| \leq M$. For the integral over the set, $|x| > M$, use triangle inequality, the hypothesis (iii) and part (b) to bound each term by integrals over $|x| > M$ of $g(x)$.

Put all the pieces together to conclude!.

Assignment 9: Do but do not to turn in.

First read/review on your own the results in 6.1 & 6.2. Then do.

From Section 6.3 1, 2, 6, 7, 9.