Chapter 4 of Beals:

4A: 1, 2, 3. 4B: 1, 2, 5, 9, 10, 12, 25. Additional ones (pp 59-60): 1, 2.

Additional Problem: Prove Theorem 4.8 (2^m test) in Chapter 4. Hint Prove first that
\[ 2^{m-1}a_{2m} \leq \sum_{j=2^{m-1}+1}^{2^m} a_j \leq 2^{m-1}a_{2m-1} \]
and use the first inequality when assuming that \( \sum_j a_j \) converges and the second inequality when assuming that \( \sum_{m=1}^{\infty} 2^m a_{2m} \) converges.

Chapter 5 of Beals:

5A: 1, 2, 6, 9, 10. 5B: 1, 2, 7.

Hints. For 5A. 1. Consider all cases for \( a \in \mathbb{R} \): positive, negative, zero.

For 5A. 9. Consider \( \lambda := \limsup_{j \to \infty} |\frac{a_{j+1}}{a_j}| < 1 \) and choose a real number \( \mu \) such that \( \lambda < \mu < 1 \). Note then that by the definition of lim sup there is an \( N \) large enough so that if \( j > N \) the \( |\frac{a_{j+1}}{a_j}| < \mu \). This in turn gives for any \( k \geq 0 \)
\[ |a_{N+k}| \leq \mu |a_{N+k-1}| \leq \mu a_{N+k-2} \cdots \leq \mu^k |a_N| \]
for any \( k \geq 0 \), which can be written (change variables \( n = N + k \)) as \( |a_n| \leq \mu^{N-n} |a_N| \). Use the root test.

Additional problem. In light of Theorem 5.9 (p. 71) we have the following

Definition: A series \( \sum_n a_n \) is said to be Abel summable (to \( L \)) if:

a) The power series \( \sum_n a_n x^n \) converges for all \( |x| < 1 \), and

b) \( f(x) := \sum_n a_n x^n \to L \) as \( x \to 1^- \).

Consider
\[ g(x) = \sum_{n \geq 1} (-1)^{n+1} n x^n. \]
i) Find the radius and interval $I \subset \mathbb{R}$ of convergence (i.e. domain of $g$).

ii) Prove that if $h(x) = \sum_{n \geq 0} (-1)^{n+1} x^n$ then $g(x) = xh'(x)$ on $-1 < x < 1$. Justify your answer.

iii) Show that $\lim_{x \to 1^-} g(x) = \frac{1}{4}$

iv) Deduce that the divergent series $\sum_n (-1)^{n+1} n$ is Abel summable to $\frac{1}{4}$.

Note As this example shows a series that is Abel summable is not necessarily convergent. So the converse of Th. 5.9 is false. However if one add an extra condition to $a_n$ a ‘partial’ result can be obtained (this is due to Tauber). Namely:

**Theorem** (Tauber, circa 1897) Suppose that the series $\sum_n a_n$ is Abel summable and that $\lim_{n \to \infty} a_n = 0$. Then $\sum_n a_n$ converges.

**SET 2 - DUE 02/11/16**

Chapter 6 of Beals:

**6A**: 1.  
**6B**: 1, 2, 4, 10 (see hint).

Hint. Before proving exercise 10 in 6B, show the Additional Problem 1 below.

In what follows, $A$ is a subset of a metric space $(S, d)$.

**Additional Problem 1.** Show that $\bar{A} = A^\circ \cup \partial A$. That is, show that if $p \in \bar{A}$ then $p \in A^\circ \cup \partial A$ and that if $p \in A^\circ \cup \partial A$ then $p \in \bar{A}$.

As a consequence note that one also has that $\bar{A} = A \cup \partial A$.

**Additional Problem 2.** Let $p \in A$. Prove that either $p$ is a limit point of $A^c$ or $p$ is an interior point of $A$, but not both.

**SET 3 - DUE 02/18/16**

Chapter 6 of Beals: (cont.).

**6B**: 3, 13 (first part only for $[0, 1]$).  
**6C**: 1, 2.  
**6D**: 1, 2, 3, 5.

Note An alternative definition of connected is: A subset $B$ of a metric space $(S, d)$ is
said to be *connected* if whenever $U$ and $V$ are *disjoint open* subsets of $S$

$$B \subseteq U \cup V \implies B \subseteq U \text{ or } B \subseteq V$$

The empty set is connected.

**SET 4 - Due 02/25/16**

**Chapter 6 of Beals:** (cont.)

**6D:** 6, 8, 9, 10, 13.

**SET 5 - Due 03/10/16**

**Additional Problem (related to 6E*)** Consider the unit interval $[0, 1]$ and let $\xi$ be a fixed real number $0 < \xi < 1$. In stage 1 of the construction remove a centrally open interval of length $\xi$. In stage 2, remove two central open intervals each of *relative length* $\xi$, one in each of the remaining subintervals after stage 1. Note that each of the two subintervals has length $\frac{1-\xi}{2}$ (the total length that remains after stage 1 is $1 - \xi$) so what one removes in stage 2 is two intervals each of length $\xi(\frac{1-\xi}{2})$. So the total removed has length $\xi(1-\xi)$ and the total length left after stage 2 is $(1-\xi) - \xi(1-\xi) = (1-\xi)^2$. Continue in this manner. Let $C_\xi$ denote the set that remains after applying this procedure indefinitely and $C^k_\xi$ the set that remains after completing stage $k$. Prove that:

a) The complement of $C_\xi$ in $[0, 1]$ is the union of open intervals of total length 1 (this would be the set you have removed at the end).

b) Compute the length of the set $C^k_\xi$ and prove that the limit as $k \to \infty$ of the length of the set $C^k_\xi$ is zero.

(Note that when $\xi = \frac{1}{3}$ the above construction is the one that gives the Cantor 1/3 set.)

**Chapter 7 of Beals:**

**7A:** 1, 3, 4.

**SET 6 - Due 03/24/16**

**7C:** 1a) d)

*Hint.* Problem 1): for the uniform part it is useful to find $x_n$ the maximum of each $f_n$. For part a) a useful trick is to consider $\log[(f_n(x))]$. 
7D*: Graphing Problem: Consider $I = [0, 1]$ and the Bernstein polynomials $P_n(x)$, $x \in I$, $n \in \mathbb{N}$ defined by (9) (Beals p. 95). Consider the continuous function on $I$ defined by

$$f(x) = \frac{1}{1 + 10(x - \frac{1}{2})^2}.$$ 

Compute $P_6, P_{10}, P_{20}$ and plot all three together with $f$ on the same graph (scale suitably so the graphs are clearly visible).

Chapter 8 of Beals:

8A: 2, 3a), 5, 6 (use 5).

8B: 2 (use IVT and monotonicity).

SET 7 - DUE 04/07/16

10B: 1, 2, 3, 6.
10C: 1
11A: 1
11C: 1

SET 8 - DUE 04/14/16

12C: Additional Problem 1: Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and compactly supported (see Definition p.164) then $f$ is (Lebesgue) integrable.

12D: 1, 3, 4, 6. 7.

Additional Problem 2.

Let $f$ is a $2\pi$-periodic integrable function on any finite interval.

(a) Prove that for any $a, b \in \mathbb{R}$

$$\int_a^b f(x)dx = \int_{a+2\pi}^{b+2\pi} f(x)dx = \int_{a-2\pi}^{b-2\pi} f(x)dx$$

(b) Prove that for any $a \in \mathbb{R}$

$$\int_{-\pi}^{\pi} f(x + a)dx = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi+a}^{\pi+a} f(x)dx$$
SPECIAL PROJECTS (Due date: 04/29/16)

The projects below should be typed.
Show all your work and steps clearly, justifying where appropriate what are you using.
Check your work carefully.

SP I. Do Problem 2 Section 7C (p 94).
Hint. For 2b) Prove that $d(x_{n+m},x_n) = \sup_{x \in I} |x^{n+m} - x^n|$ does not converge to zero as $m \to \infty$ for each fixed $n$ (so that $f_n$ is not Cauchy). To do this consider the function $F(x) = x^{n+m} - x^n$. Note $F(x) \leq 0$ and $F(0) = 0 = F(1)$. Find the extremum $\bar{x}$ of $F$ (which depends on $n, m$) and show that as $m \to \infty$ the values of $|F(\bar{x})| \to 1$.

SP II. Prove that the series $\sum_{j=1}^{\infty} 2^{-j} \sin(2^j x)$, defines a continuous function that is nowhere differentiable.

Hints. First recall the M-test for series to first verify that the series is indeed absolutely convergent for each fixed $x$. You may then adapt a similar strategy to the one in the handwritten notes I distributed for the Weierestrass-type function to prove the continuity and the non-differentiability at each fixed $x$. 