FROM [SS III] : Good Kernels and Approx. to Identity

$\{K_\delta\}_{\delta>0}$ $\quad K_\delta \in L^1(\mathbb{R}^d)$

Good Kernels

(i) $\displaystyle\int_{\mathbb{R}^d} K_\delta(x)\,dx = 1$

(ii) $\displaystyle\int_{\mathbb{R}^d} |K_\delta(x)|\,dx \leq A$ indep. of δ

(iii) For every $\eta>0$ $\displaystyle\int_{|x|\geq\eta} |K_\delta(x)|\,dx \xrightarrow[\delta\to 0^+]{} 0$

Let $f\in L^1(\mathbb{R}^d)$

Q: Does $K_\delta * f(x) \xrightarrow[\delta\to 0^+]{} f(x)$ a.e. x ?

Remark: If f is bounded then at every point x_0 where f is continuous (at x_0)

$$K_\delta * f(x_0) \xrightarrow[\delta\to 0^+]{} f(x_0)$$

Note: (1) f cnt at $x_0 \Rightarrow x_0 \in \mathcal{L}_f$.

(2) How about if $x_0\in\mathcal{L}_f$ (= all $x_0\in\mathbb{R}^d$ for which $f(x_0)$ is finite and $\displaystyle\lim_{\substack{m(B)\to 0\\ B\ni x_0}} \frac{1}{m(B)}\int_B |f(y)-f(x_0)|\,dy = 0$)?

Gives also $\displaystyle\lim_{\substack{m(B)\to 0\\ B\ni x_0}} \frac{1}{m(B)}\int_B f(y)\,dy = f(x_0)$

To have an affirmative answer to our Q. we need to restrict our class of good kernels to one where:

$K_\delta \in L^1(\mathbb{R}^d)$ for all $\delta > 0$ and

(i) $\int_{\mathbb{R}^d} K_\delta(x)\,dx = 1$

STRONGER CONDITIONS MORE Restrictive than (ii) (iii)

(ii)' $|K_\delta(x)| \le A\delta^{-d}$ for all $\delta > 0$

(iii)' $|K_\delta(x)| \le \dfrac{A\delta}{|x|^{d+1}}$ for all $\delta > 0$, $x \in \mathbb{R}^d$, $x \neq 0$

This family is called an approximation to the identity because the answer to the Q. above for THIS FAMILY is now Yes.

Remark: Conditions above imply (ii) and (iii)

To see this first recall that $\displaystyle\int_{|x| \ge \delta} \frac{dx}{|x|^{d+1}} \le \frac{c}{\delta}$

for all $\delta > 0$ and some $c > 0$.

Then,

$$\int_{\mathbb{R}^d} |K_\delta(x)|\,dx = \int_{|x| \le \delta} |K_\delta(x)|\,dx + \int_{|x| > \delta} |K_\delta(x)|\,dx$$

$$\leq A \int_{|x| \leq \delta} \frac{dx}{\delta^d} + A\delta \int_{|x| > \delta} \frac{1}{|x|^{d+1}}\, dx$$

$$\underbrace{\leq C_1 \frac{\cancel{\delta^d} A}{\cancel{\delta^d}} + C_2 A}_{\text{independent of } \delta} < \infty .$$

Moreover we can check (iii) also since

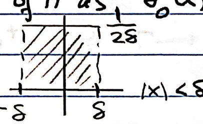
$$\int_{|x| \geq \eta} |K_\delta(x)|\, dx \leq A\delta \int_{|x| \geq \eta} \frac{dx}{|x|^{d+1}}$$

$$\leq \frac{A\delta}{\eta} \longrightarrow 0 \text{ as } \delta \to 0^+$$

for any $\eta > 0$ fixed.

Remark : If $\{K_\delta\}_{\delta>0}$ is an approximation to the identity then " $K_\delta \longrightarrow \delta_0(x)$ " as $\delta \to 0^+$ where $\delta_0(\cdot)$ is the Dirac Delta "function" (distribution) at 0. We can think of it as " $\delta_0(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$ "

and $\int_{\mathbb{R}^d} \delta_0(x)\, dx = 1$



Then $(f * \delta_0)(x)$ can be interpreted as

$$\int_{\mathbb{R}^d} \underbrace{f(x-y)\,\delta_0(y)}_{=\begin{cases} 0 & y \neq 0 \\ f(x) & y = 0 \end{cases}}\,dy \overset{\text{"}\,=\,\text{"}}{} f(x)$$

$\leftarrow$ suitably interpreted

Some examples of approximations to the identity:

a) The heat kernel in $\mathbb{R}^d$

$$K_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}, \quad t > 0$$

$$\boxed{\delta = \sqrt{t}}$$

b) The Poisson kernel on the upper half plane $\mathbb{R} \times \mathbb{R}_+$

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2} \quad x \in \mathbb{R}, \ y > 0$$

$$\boxed{\delta = y.}$$

c) Poisson kernel for the unit disk D

$$\frac{1}{2\pi} P_r(x) = \begin{cases} \dfrac{1}{2\pi} \dfrac{1-r^2}{1-2r\cos x + r^2} & \text{if } |x| \leq \pi \\ 0 & \text{if } |x| > \pi \end{cases}$$

$0 < r < 1$

$$\boxed{\delta = 1 - r} \qquad \delta \to 0^+ \iff r \to 1^-$$

d) The Féjer kernel on $[-\pi,\pi]$

$$\frac{1}{2\pi} F_N(x) = \begin{cases} \frac{1}{2\pi N} \sin^2\left(\frac{Nx}{2}\right) & \text{if } |x| \le \pi \\ 0 & \text{if } |x| > \pi \end{cases}$$

$$\boxed{\delta = \frac{1}{N}}$$

In general we can easily construct approximations to the identity on $\mathbb{R}^d$ as follows

Consider $\varphi : \mathbb{R}^d \to [0,\infty)$ bounded such that support $\varphi \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$

Then, define

$$K_\delta := \frac{1}{\delta^d} \varphi\left(\frac{x}{\delta}\right), \quad \delta > 0$$

$\{K_\delta\}_{\delta>0}$ is an approximation to the identity family.

<u>Theorem 2.1</u> : If $\{K_\delta\}_{\delta>0}$ is an approximation to the identity and $f \in L^1(\mathbb{R}^d)$ then

$$(*) \quad \lim_{\delta \to 0^+} f * K_\delta(x) \longrightarrow f(x) \quad \forall x \in \mathcal{L}_f \text{ ; in}$$

particular a.e. x

Remark: Note that

$$f * K_\delta(x) - f(x) = \int_{\mathbb{R}^d} \left[f(x-y) - f(x) \right] K_\delta(y)\, dy$$

$$\Rightarrow \left| f * K_\delta(x) - f(x) \right| \leq \underbrace{\int_{\mathbb{R}^d} \left| f(x-y) - f(x) \right| \left| K_\delta(y) \right| dy}_{(\dagger)}$$

Hence to prove (*) is enough to show is enough to show that $(\dagger) \longrightarrow 0$ as $\delta \to 0^+$ $\forall x \in \mathcal{L}_f$

Lemma 2.2: Let $f \in L^1(\mathbb{R}^d)$ and $x \in \mathcal{L}_f$ (fixed)

Let $$A^x(r) = \frac{1}{r^d} \int_{|y| \leq r} \left| f(x-y) - f(x) \right| dy \geq 0 \quad (r > 0)$$

Then:

(i) A is continuous in r

(ii) $A(r) \longrightarrow 0$ as $r \to 0^+$

(iii) $A(r)$ is bounded; i.e. $\exists M > 0 \,/\, A(r) \leq M$ $\forall r > 0$

Proof: (i) follows from Prop. 1.12 ii) Chap. 2

(ii) $m(B(0,r)) = v_d r^d$ $\quad v_d = m(B(0,1))$

So, since $x \in \mathcal{L}_f$ (fixed) $A(r) \longrightarrow 0$ as $r \to 0$

(iii) By i) ii) $A(r)$ is bounded $\forall 0 < r < 1$; so prove $A(r)$ is bounded $\forall\, r>1$ we note that

$$A(r) \leq \frac{1}{r^d} \int_{|y|\leq r} |f(x-y)|\, dy + \underbrace{\frac{1}{r^d} \int_{|y|\leq r} |f(x)|\, dy}_{= |f(x)|}$$

So

$$A(r) \leq \frac{1}{r^d} \|f\|_{L^1} + \nu_d |f(x)| \qquad (x \text{ is fixed})$$

\#.

Next let's prove Theorem 2.1

Step 1 : We write

$$\int_{\mathbb{R}^d} |f(x-y)-f(x)||K_\delta(y)|\, dy =$$

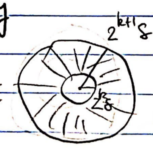
$$\underbrace{\int_{|y|\leq \delta} |f(x-y)-f(x)||K_\delta(y)|\, dy}_{(\mathrm{I})} + \underbrace{\int_{|y|>\delta} \text{.........}}_{(\mathrm{II})}$$

same integrand

Then we further decompose the second term as

$$(\mathrm{II})\ \int_{|y|>\delta} \text{.....} = \sum_{k=0}^{\infty} \int_{2^k\delta < |y| \leq 2^{k+1}\delta} |f(x-y)-f(x)||K_\delta(y)|\, dy$$

$\to$ annuli



Step 2: By (ii)' we have that

$$(I) = \int_{|y| \leq \delta} |f(x-y) - f(x)|\, |K_\delta(y)|\, dy$$

$$\leq \frac{c}{\delta^d} \int_{|y| \leq \delta} |f(x-y) - f(x)|\, dy$$

$$\leq c\, A(\delta)$$

Step 3: Consider each summand of (II):

$$\int_{2^k\delta < |y| \leq 2^{k+1}\delta} |f(x-y) - f(x)|\, |K_\delta(y)|\, dy \leq$$

by (ii)' and using $|y| > 2^k\delta$

$$\leq \frac{c\,\delta}{(2^k\delta)^{d+1}} \int_{|y| \leq 2^{k+1}\delta} |f(x-y) - f(x)|\, dy$$

$$\leq \frac{c'}{2^k (2^{k+1}\delta)^d} \int_{|y| \leq 2^{k+1}\delta} |f(x-y) - f(x)|\, dy$$

with $\boxed{c' = c\,2^d}$

$$\leq c'\, 2^{-k} A(2^{k+1}\delta)$$

Note: $\dfrac{c\,\delta}{2^k\delta\,(2^k\delta)^d} = \dfrac{c}{2^k (2^k\delta)^d} = \dfrac{c\,2^d}{2^k (2^{k+1}\delta)^d}$.

Then $(II) \leq c' \sum_{k=0}^{\infty} 2^{-k} A(2^{k+1}\delta)$

All in all from Step 1–3 we gather

$$|f * K_\delta(x) - f(x)| \leq c A(\delta) + c' \sum_{k=0}^{\infty} 2^{-k} A(2^{k+1}\delta)$$

$$= c A(\delta) + c' \Big[\sum_{k=0}^{N-1} 2^{-k} A(2^{k+1}\delta) + \underbrace{\sum_{k=N}^{\infty} 2^{-k} A(2^{k+1}\delta)}_{(*)(*)} \Big]$$

Let $\varepsilon > 0$ be given and choose $N = N(\varepsilon)$ / the tail of the geometric series $\sum_{k \geq N} 2^{-k} < \varepsilon$

$\Rightarrow (*)(*) \leq M_x \cdot \varepsilon$

$A(\cdot)$ is bounded $\quad$ $\hookrightarrow$ depends on x

Next since as $\delta \to 0$ $A(2^{k+1}\delta) \longrightarrow 0$ (for k fixed)

we choose $\delta = \delta(\varepsilon) > 0$ small enough so that

$$A(2^{k+1}\delta) < \frac{\varepsilon}{N} \quad \forall k = 0, \dots, N-1.$$

Hence, $\sum_{k=0}^{N-1} 2^{-k} A(2^{k+1}\delta) < \varepsilon$.

Similarly, by making $\delta > 0$ smaller if necessary we have that we can bound $(I) \leq C'\varepsilon$.

All in all we then have that

$$\left| f * K_\delta(x) - f(x) \right| \leq C\varepsilon \text{ for } \delta > 0 \text{ suff. small.}$$

#