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<u>Convergence in L^1-norm</u> : Let $f \in L^1(\mathbb{R}^d)$.

In fact to prove just that $f * K_\delta \to f$ in L^1 is enough to require that $\{K_\delta\}_{\delta>0}$ be a good kernel family; ie:

(i) $\int_{\mathbb{R}^d} K_\delta(y)\, dy = 1 \quad \forall \delta > 0$

(ii) $\int_{\mathbb{R}^d} |K_\delta(y)|\, dy \le A$ $\forall \delta>0$ and for some $A>0$ indep. of δ.

(iii) For each $\eta > 0$ $\int_{|y| \ge \eta} |K_\delta(y)|\, dy \to 0$ as $\delta \to 0$.

We use (i) to write (for fixed $f \in L^1(\mathbb{R}^d)$):

$$f * K_\delta(x) - f(x) = \int_{\mathbb{R}^d} [f(x-y) - f(x)]\, K_\delta(y)\, dy$$

Thus $|f * K_\delta(x) - f(x)| \le \int_{\mathbb{R}^d} |f(x-y) - f(x)|\, |K_\delta(y)|\, dy$

(*) Next recall that since $f \in L^1(\mathbb{R}^d)$, both $f(x)$

and $f(x-y)$ regarded as functions on $\mathbb{R}^d \times \mathbb{R}^d$ are measurable.

(Recall Corollary 3.7 and Prop. 3.3 in Chapter 2)

(**) Furthermore by ~~Proposition~~ 2.5 Ch. 2, given $\varepsilon > 0$ $\exists\, \eta = \eta(\varepsilon)$ small such that $\| f_y - f \|_{L^1_x(\mathbb{R}^d)} < \varepsilon$ whenever $|y| < \eta$. Here $f_y(x) = f(x-y)$

From (*) and (**) we now invoke Fubini's theorem

$$\left\| \int_{\mathbb{R}^d} |f(x-y) - f(x)|\, |K_\delta(y)|\, dy \right\|_{L^1_x(\mathbb{R}^d)}$$

Used Fubini-Tonelli

$$= \int_{\mathbb{R}^d} \underbrace{\left[\int_{\mathbb{R}^d} |f(x-y) - f(x)|\, dx \right]}_{= \| f_y - f \|_{L^1_x(\mathbb{R}^d)}} |K_\delta(y)|\, dy$$

$(f_y(x) = f(x-y))$

$$\leq \int_{|y| < \eta} \| f_y - f \|_{L^1(\mathbb{R}^d)} |K_\delta(y)|\, dy \; +$$

$$+ \int_{|y| \geq \eta} \| f_y - f \|_{L^1_x(\mathbb{R}^d)} |K_\delta(y)|\, dy$$

③

$$\leq A.\varepsilon \qquad + \quad 2\,\|f\|_{L^1(\mathbb{R}^d)} \cdot \int_{|y|\geq \eta} |K_\delta(y)|\,dy$$

(by ⊛⊛ and (ii))

(using transl. invariance and triangle inequality)

$\|f_y - f\| \leq 2\|f\|_{L^1}$

By property (iii) $\exists\, \delta = \delta(\varepsilon) \,/\, \int_{|y|\geq\eta} |K_\delta(y)|\,dy < \varepsilon$

Hence all in all we have

$$\|f * K_\delta - f\|_{L^1(\mathbb{R}^d)} \leq A.\varepsilon + 2\|f\|_{L^1}.\,\varepsilon$$

$$\leq \tilde{\varepsilon} \qquad \tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon, A, \|f\|_{L^1})$$

⊛.