The Cycle Lemma and Euler's Theorem

Lemma 1 (The Cycle Lemma). *Let G be a graph in which each vertex has* even *degree. Let a be a vertex of G for which* $\deg(a) \neq 0$. *Then there is some cycle in G from a to a.*

The proof is essentially an induction (or a recursion, depending on how you look at it). The method of proof will provide, in effect, an algorithm for finding such a cycle.

Proof. If there is a loop at a or multiple edges at a, we are done. Otherwise, remove all loops in G and, from each set of multiple edges between two adjacent vertices, remove all but one of these multiple edges. Now G is a simple graph. Still, each vertex has even degree and $\deg(a) \neq 0$.

Let $v_1 = a$. Since $\deg(v_1) \neq 0$, there is some edge e_1 from v_1 to a vertex v_2 with $v_2 \neq v_1$.

Since $\deg(v_2)$ is even, there is an edge e_2 from v_2 to a vertex v_3 with $e_2 \neq e_1$. And since there is no loop or multiple edges at v_1, necessarily $v_3 \neq v_1, v_2$. So far we have a trail v_1, e_1, v_2, e_2, v_3.

Since $\deg(v_3)$ is even, there is an edge e_3 from v_3 to a vertex v_4 with $e_3 \neq e_2$. Then the edges e_1, e_2, e_3 are all distinct (why?).

If $v_4 = v_1$ we are done, because then $v_1, e_1, v_2, e_2, v_3, e_3, v_4$ is a cycle at $v_1 = a$.

Suppose, however, that $v_4 \neq v_1$. Then v_1, v_2, v_3, v_4 are all distinct (why?). So far we have a trail $v_1, e_1, v_2, e_2, v_3, e_3, v_4$.

Since $\deg(v_4)$ is even, there is an edge e_4 from v_4 to a vertex v_5 with $e_4 \neq e_3$. Also $v_5 \neq v_4, v_3$ (why?)—but it is possible that $v_5 = v_1$ or $v_5 = v_2$. Then the edges e_1, e_2, e_3, e_4 are all distinct (why?). So far we have a trail $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5$.

If $v_5 = v_1$ we are done, because then $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5$ is a cycle at $v_1 = a$.

Suppose that $v_5 \neq v_1$ but $v_5 = v_2$. Remove the cycle $v_2, e_2, v_3, e_3, v_4, e_4, v_5$ from G. Still, each vertex has even degree and $\deg(a) \neq 0$. Now start the entire process again!

Suppose, though, that $v_5 \neq v_1$ and $v_5 \neq v_2$. Then the vertices v_1, v_2, v_3, v_4, v_5 are all distinct, and the edges e_1, e_2, e_3, e_4 are all distinct. And we have the trail $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5$. Now continue as before, getting a new edge from v_5, etc.
□

Theorem 1 (Euler's Theorem). *A necessary and sufficient condition that a graph G be Eulerian is that G be connected and each vertex of G have even degree.*

Proof. **Necessity.** Already done.

Sufficiency. Assume that G is connected and that each vertex of G has even degree.

The following algorithm shows how to construct an Eulerian trail in G.

(0) Temporarily remove all loops from G. (We shall put them all back at the end.)

(1) (1.1) Select an arbitrary vertex v_0 of G;

(1.2) form some cycle C in G from v_0 to v_0 {use Cycle Lemma method}; and

(1.3) remove all edges in C, leaving a subgraph H of G.

(2) Now repeat the following while H still has any edges remaining...

(2.1) Choose some vertex v of H that is an end of some edge of C;

(2.2) form some cycle S **in** H from v to v {use Cycle Lemma method};

(2.3) form the new walk obtained by inserting S into C at vertex v {so that C is a trail in G}, and call the resulting trail C again; and

(2.4) remove all the edges and all the *isolated* (that is, degree 0) vertices of H, and call the resulting graph H again.

(3) Finally, reinsert into C all the loops of G at their original ends. □

Then C is the desired Eulerian trail. □