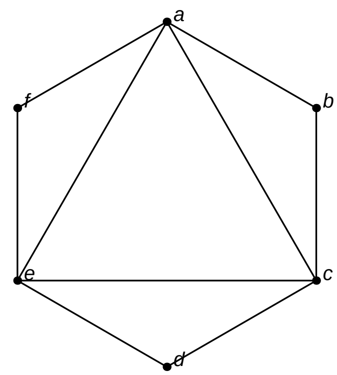
Math 455.1 **Homework Set 8** **Spring, 2009**

Due: Monday, May 4

Instructions: Work either individually or in a team.

1. Use our algorithmic **method of proof** (from class) of Euler's Theorem to construct an Eulerian trail from vertex a to a in the following graph:

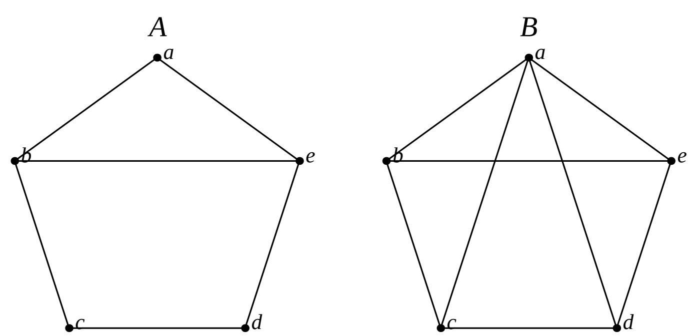


2. (a) Use the function `EulerianQ` from to determine for which $n = 2, 3, \ldots, 10$ the cube graph Q_n is Eulerian. Nicely format your results as a *Mathematica* table without directly entering the individual entries of the table. *Note:* In *Combinatorica* notation, the graph Q_n is `Hypercube[n]`.

   (b) In general, for which values of n is the graph Q_n is Eulerian? Formulate a conjecture based upon your results from (a).

   (c) Now prove that your conjecture is true in general.

3. (a) For each of the graphs A and B shown below, do the hypotheses of Ore's Theorem about Hamiltonian graphs hold?



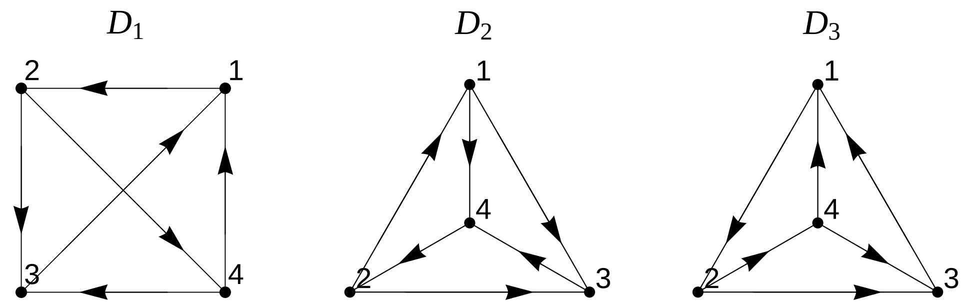
   (b) Determine whether each of the graphs in (a) is Hamiltonian and indicate why or why not.

4. In class you saw a method that constructs, from the reflective Gray code of n-bit binary words, the reflective Gray code of $(n+1)$-bit binary words. Apply that method to construct the reflective Gray code of 5-bit binary words starting with the following reflective Gray code of 4-bit binary words, shown here (horizontally, to save space).

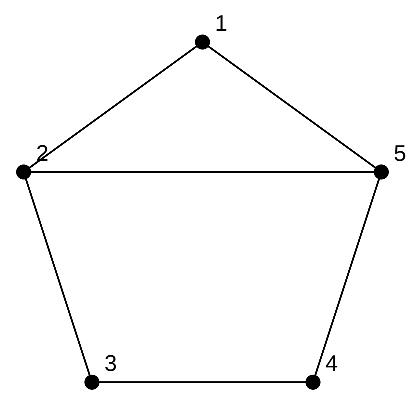
$$0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000$$

5. Repeat problem 2 but for "Hamiltonian" instead of "Eulerian". *Hint:* For (c), consider the relationship between Gray codes and (hyper)cubes.

6. For the digraphs D_1, D_2, and D_3 shown below, which are isomorphic to which others, and why?



7. (a) Write the adjacency matrix of the following graph, using the ordering 1,2,3,4,5 of the vertices:



(b) Draw a graph whose adjacency matrix is the following; number the vertices of your graph in the order corresponding to the order of entries in the matrix, of course.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

8. Recall the following result:

Theorem. *Let D be a digraph with n vertices $v_1, v_2, \ldots, v_n$ and let A be the adjacency matrix of D corresponding to that ordering of the vertices. Let*

$$B = A + A^2 + \cdots + A^{n-1}.$$

Then D is strongly *connected if and only if each* non-*diagonal entry is strictly positive.*

(a) State—but do *not* prove—an analog of this theorem for (undirected) graphs.

The theorem you state is to be about ordinary graphs, *not* digraphs! So its conclusion should be about when the graph is connected, *not* when it is strongly connected. And it really should be some sort of analog.

You might want to try your purported theorem on some more examples than those in (b), below, just to make sure you didn't rush to a conclusion too soon. But you do *not* need to prove your theorem.

(b) Here are the adjacency matrices A_1 and A_2 of two graphs G_1 and G_2. Use your theorem to determine whether each graph is connected.

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Note: Feel free to use *Mathematica* to calculate the matrix powers and sum of these powers that's involved. Just be sure to use `MatrixPower` to compute powers of matrices—and *not* mistakenly use the `^k` construction, which just raises all elements of a matrix to the power k.