1. (a) \[ \frac{0.025}{10^3} = \frac{25}{10^3} + \frac{25}{10^6} + \frac{25}{10^9} + \cdots = \sum_{n=1}^{\infty} \frac{25}{10^{3n}} \] or \[ \frac{0.025}{10^3} = \frac{25}{1000} + \frac{25}{10000} + \frac{25}{100000} + \cdots = \sum_{n=1}^{\infty} \frac{25}{10000^n} \]

Note: Another, correct, way to write the answer is \( \sum_{n=1}^{\infty} \frac{25}{1000} \left( \frac{1}{1000} \right)^{n-1} \).

However, \( \sum_{n=0}^{\infty} \frac{25}{1000^n} \) and \( \sum_{n=1}^{\infty} \frac{25}{1000^n} \left( \frac{1}{1000} \right)^n \) are very wrong!

(b) [8%] This is a geometric series having initial term \( a = \frac{25}{1000} \) and ratio \( r = \frac{1}{1000} \). Hence:

\[ \sum_{n=1}^{\infty} \frac{25}{1000^n} = \frac{a}{1-r} = \frac{25/1000}{1-1/1000} = \frac{25}{1000-1} = \frac{25}{999} \]

2. (a) [8%] By definition, the sum of the series is the limit of its sequence of partial sums:

\[ \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{3^n - 9}{3^{n+1}} \]

\[ = \lim_{n \to \infty} \left( \frac{3^n}{3^{n+1}} \right) - \lim_{n \to \infty} \left( \frac{9}{3^{n+1}} \right) = \frac{1}{3} - \frac{9}{3^{n+1}} = \frac{1}{3} - 0 = \frac{1}{3} \]

Notes: \( s_n \) is the \( n \)th partial sum and \( \text{not} \) the \( n \)th term \( a_n \) — so the series is \( \text{not} \sum_{n=1}^{\infty} \frac{3^n-9}{3^{n+1}} \) ! For (a) it’s pointless to find a formula for \( a_n \); even if you do (using \( a_n = s_n - s_{n-1} \)), the fact that then you find \( \lim_{n \to \infty} a_n = 0 \) does \( \text{not} \) tell you that \( \sum_{n=1}^{\infty} a_n = 0 \) converges!

(b) [8%] For each \( n = 2, 3, \ldots \),

\[ a_n = s_n - s_{n-1} = \frac{3^n - 9}{3^{n+1}} - \frac{3^{n-1} - 9}{3^{n-1+1}} = \frac{3^n - 3^{n-1} - 9}{3^n} - \frac{3^{n-1} - 9}{3^{n-1}} = \frac{3^n - 9 - 3^{n-1} - 9}{3^n} = \frac{3^n - 9 - 3^{n-1} + 27}{3^n} = \frac{18}{3^n} \]

(OK, or simplify more:)

\[ = \frac{2}{3^n} \cdot \frac{2}{3^{n-1}} = \frac{2}{3^n} \cdot \frac{3^n}{3^{n-1}} = \frac{2}{3} \]

And: \( a_1 = s_1 = \frac{3^1 - 9}{3^{1+1}} = -6 \cdot \frac{1}{3^2} = -\frac{2}{3} \)

3. (a) [8%]

\[ \lim_{n \to \infty} \left| \frac{(x-3)^n (n+1)}{(x-3)^n / n} \right| = \lim_{n \to \infty} \left| \frac{(x-3)^n + 1}{(x-3)^n} \cdot \frac{n}{n+1} \right| = \lim_{n \to \infty} \left| (x-3) \cdot \frac{n}{n+1} \right| = |x-3| \lim_{n \to \infty} \frac{n}{n+1} = |x-3| \cdot 1 = |x-3| \]

According to the Ratio Test, the power series converges when \( |x-3| < 1 \) and diverges when \( |x-3| > 1 \). Hence its radius of convergence is \( R = 1 \).

(b) [8%] It remains to test the endpoints—where \( |x-3| = 1 \), in other words, where \( x = 3 \pm 1 \).

[Note: From (a), we already know that the power series converges when \( |x-3| < 1 \), that is, when \( -1 < x-3 < 1 \), that is, when \( 2 < x < 4 \); and that it diverges when \( x < 2 \) or \( x > 4 \).]

Test \( x-3 = 1 \) (that is, \( x = 4 \)). The series is \( \sum_{n=1}^{\infty} \frac{1}{n} \), the \textit{divergent harmonic series}.

Test \( x-3 = -1 \) (that is, \( x = 2 \)). The series becomes \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \), the series whose terms are the \textit{negatives} of the corresponding terms of the alternating harmonic series. Since the \textit{alternating harmonic series} converges, this series \textit{converges}, too.

Conclusion: The series converges only for \( -1 \leq x-3 < 1 \), that is, \( 2 \leq x < 4 \). In other words, the series has \textit{interval of convergence} \( [2, 4) \).
4. (a) [8%] The first term is \(1/(-1) < 0\), but all other terms are positive because, for \(n \geq 2\), \(n^3 + 7n^2 - 9 \geq 0\). For \(n \geq 2\),

\[
\frac{1}{n^3 + 7n^2 - 9} \leq \frac{1}{n^3} \iff n^3 + 7n^2 - 9 \geq n^3 \iff 7n^2 \geq 9 \iff n^2 \geq \frac{9}{7},
\]

and certainly \(n^2 \geq \frac{9}{7}\) when \(n \geq 2\). Hence indeed \(\frac{1}{n^3 + 7n^2 - 9} \leq \frac{1}{n^3}\) for all \(n \geq 2\).

Now \(\sum_{n=1}^{\infty} \frac{1}{n^3}\) converges since it is a \(p\)-series with \(p = 3 > 1\). Hence also \(\sum_{n=2}^{\infty} \frac{1}{n^3}\) converges.

By the **Comparison Test**, \(\sum_{n=2}^{\infty} \frac{1}{n^3 + 7n^2 - 9}\) also converges. So the given series converges.

(b) [8%]

\[
\lim_{n \to \infty} \frac{\sqrt{n^2 + 7}}{2n + 3} = \lim_{n \to \infty} \frac{(1/n) \sqrt{n^2 + 7}}{(1/n)(2n + 3)} = \lim_{n \to \infty} \frac{\sqrt{\frac{n^2 + 7}{n^2}}}{2 + 3/n} = \lim_{n \to \infty} \frac{\sqrt{1 + \frac{7}{n^2}}}{2 + \frac{3}{n}} = \frac{1}{3} \neq 0
\]

By the **Test for Divergence** (a.k.a. the **nth Term Test**), the given series diverges.

(c) [8%] The series is alternating, and the sequence of absolute values \(b_n = \frac{1}{\ln n}\) of absolute values of its terms clearly has properties:

- \((b_n)_{n=1}^{\infty}\) is decreasing; and
- \(\lim_{n \to \infty} b_n = 0\).

By the **Alternating Series Test** the given series converges.
5. (a) [8%] Method 1: Use summation notation. From \( \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \) get

\[
\frac{1}{1 - 4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}.
\]

Then

\[
f(x) = \frac{x}{1 - 4x^2} = x \cdot \frac{1}{1 - 4x^2} = x \sum_{n=0}^{\infty} 4^n x^{2n} = \sum_{n=0}^{\infty} x \cdot 4^n x^{2n} = \sum_{n=0}^{\infty} 4^n x^{2n+1}
\]

Note: An answer like \( \sum_{n=1}^{\infty} 4^n x^{2n+1} \) is wrong: it’s missing the constant term.

Method 2: Avoid summation notation. From \( \frac{1}{1-r} = 1 + r + r^2 + r^3 + r^4 + \cdots \) get

\[
\frac{1}{1 - 4x^2} = 1 + (4x^2) + (4x^2)^2 + (4x^2)^3 + \cdots = 1 + 4x^2 + 16x^4 + 64x^6 + 256x^8 + \cdots.
\]

Then:

\[
f(x) = x \cdot \frac{1}{1 - 4x^2} = x \left( 1 + 4x^2 + 16x^4 + 64x^6 + 256x^8 + \cdots \right) = x + 4x^3 + 16x^5 + 64x^7 + 256x^9 + \cdots
\]

(b) [8%] The series \( \frac{1}{1 - 4x^2} \) converges if and only if \(|4x^2| < 1\). Now

\[
|4x^2| < 1 \iff |x|^2 < \frac{1}{4} \iff |x| < \frac{1}{2}
\]

Since multiplying a series be a given number does not change whether it converges, the power series found in (a) also converges if and only if \(|x| < \frac{1}{2}\), in other words, when \(-\frac{1}{2} < x < \frac{1}{2}\).
6. (a) The fifth partial sum $S_5$ is:

$$S_5 = \sum_{n=1}^{5} (-1)^{n-1} \frac{1}{n^2 + 1} = \frac{1}{1^2 + 1} - \frac{1}{2^2 + 1} + \frac{1}{3^2 + 1} - \frac{1}{4^2 + 1} + \frac{1}{5^2 + 1}$$

$$= \frac{1}{2} - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} + \frac{1}{26} = \frac{839}{2210} \approx 0.379638$$

(b) This is an alternating series. The sequence of absolute values $b_n = \frac{1}{n^2 + 1}$ of its terms satisfies the conditions:

- $(b_n)_{n=1}^{\infty}$ is decreasing; and
- $\lim_{n \to \infty} b_n = 0$.

Hence the theory related to the Alternating Series Test applies: the error $R_5$ of the approximation satisfies

$$|R_5| \leq \frac{|(-1)^{(5+1)+1}|}{(5+1)^2 + 1} = \frac{1}{37} = 0.027 \leq 0.027028$$

Note: It would be wrong to conclude $|R_5| \leq 0.027027$, since the upper bound obtained is the repeating decimal 0.027027027... , which is larger than 0.027027.