Lusztig conjecture for modular Lie algebras

with Bezrukavnikov, Rumynin, Riche, Sommers

A. Modular Representation Theory: k is a closed field, $\mathrm{char}(k)=p$

G is a reductive algebraic group over k, $\mathfrak{g}=\mathrm{Lie}(G)$.

(a) $\mathrm{Irr\,Alg\,Rep}(G)=?$, (b) $\mathrm{Irr\,Rep}(\mathfrak{g})=?$

(1) $p=0$, (2) $p\le h_G$?, (3) $p>h_G$: Lusztig Conjectures

(4) $p>>0$: proved AJS, BM

In general (a) is a special case of (b), I will consider (b).

Also, only $p>h_G$ (structural results for $U\mathfrak{g}$).

E1. $G=SL_2$: $k_p[x,y]\supseteq kx^p\oplus ky^p$

E2. $G=SL_n$: $\mathfrak{g}\supseteq Z(\mathfrak{g})=k[\ldots]$.

So: $p>0$ introduces complexity of coincidences,

Claim: new complexity comes from commutative world, modular case is "easier".

B. Key mechanism of "modular" settings ("Critical Quantization"):

Large Center — Azumaya algebras — Reduction to Alg. Geom.

Crystalline Differential operators $\mathcal{D}_X \overset{\text{def}}{=} U_{\mathcal{O}_X}(\mathcal{T}_X) \overset{\text{locally}}{=} \oplus\, \mathcal{O}_X\cdot\partial^I$

B1. Large Center: case of differential operators:

L1. a) $Z(\mathcal{D}_{\mathbb{A}^1}) = k[x^p,\partial^p]$

b) $\mathcal{D}_{\mathbb{A}^1}|_{x^p=0=\partial^p}$ has basis $x^i\partial^j$, $0\le i,j<p$

c) It has irreducible module $k[x]/x^p$.

Pf. $[\partial,x^p] = \cancel{\ \ } = \partial(x^p) = px^{p-1} = 0$.

②

B2. Azumaya algebras: ① An algebra vector bundle A on a smooth X is Azumaya if all fibers are matrix algebras.

② A splitting of an Azumaya algebra A is a vector bundle V with $A \xrightarrow{\cong} End(V)$.

②' It gives: $Coh(X) \xrightarrow{\cong} Coh(A)$ by $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} V$.

So, where it splits representation theory becomes alg. geometry.

L2. (a) $Z(\mathcal{D}_X) = \mathcal{O}(T^* X^{(1)})$, for $Y^{(1)} = Spec[\mathcal{O}(Y)^p]$.

(b) $\mathcal{D}_X$ is an Azumaya algebra on $T^* X^{(1)}$.

(c) It splits on Frobenius twists $L^{(1)}$ of conormals $T^*_Y X = L$

. Pf. $\mathcal{D}_X |_{X^{(1)}}$ has splitting $\mathcal{O}_X$ (a vector bundle on $X^{(1)}$).

B3. Enveloping algebra $\mathcal{U} = U\mathfrak{g}$ is **"singular Azumaya"**:

T3. $Z(U\mathfrak{g}) = \mathcal{O}(\mathfrak{Z})$, $\mathfrak{Z} = \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^*//W^{(1)}} \mathfrak{h}^*//W$

"Pf." $U\mathfrak{g} = D(G)^{G \times 1}$. However, historically $\{$the result for $\mathcal{U}$ is classical.

Specializations for compatible $\lambda \in \mathfrak{h}^*$, $x \in \mathfrak{g}^*(1)$:

U^λ, U_x, U^λ_x.

0° $Irr(\mathfrak{g}) = \amalg\ Irr(U^\lambda_x)$

1° $\dim(U_x) = p^{\dim \mathfrak{g}}$

2° irreducibles are finite dimensional!

3° Compatibility for integral λ is that x is nilpotent,

More: • Quantum groups at roots of 1 (analogous theory) Backelin – Kremnitzer.

• Affine algebras at critical level: analogous phenomena.

• etc.

C. Lusztig Conjecture:

C1. Irreducibles live in Springer fibers: (suggested by Humphreys)

$\mathcal{P}$ is a partial flag variety (a G-orbit of parabolic subgroups)

For $P \in \mathcal{P}$, $\mathfrak{p} = \mathrm{Lie}(P) \supseteq \mathfrak{U} =$ nilpotent radical

$$\mathcal{P} \times \mathfrak{g} \supseteq \mathfrak{g}_{\mathcal{P}} = \{(\mathfrak{p}, x),\ x \in \mathfrak{p}\} \supseteq \{(\mathfrak{p}, x);\ x \in \mathfrak{U}\} = T^*\mathcal{P}$$

$$x \in \mathfrak{g} \xleftarrow{\mu} \mathfrak{g}_{\mathcal{P}} \longrightarrow \mathfrak{z}_{\mathcal{P}} \cong \mathfrak{p}^{ab} \ni \lambda,\quad P \in \mathcal{P}.$$

Springer fibers: $(\mathfrak{g}_{\mathcal{P}})^{\lambda}_{x}$, main case: $\lambda = 0$, x nilpotent, $\mathcal{P} = \mathcal{B}$

$$(\mathfrak{g}_{\mathcal{B}})^{\lambda}_{x} = \mathcal{B}_x = \{\mathfrak{b} \in \mathcal{B},\ \mathfrak{b} \ni x\}.$$

C2. ~~[illegible]~~ **Projectives live in Slodowy** varieties (Lusztig):

- Any nilpotent $e \in \mathfrak{g}$ extends to an $\mathfrak{sl}_2$-subalgebra e, h, f
- $S = Z_{\mathfrak{f}}(\mathfrak{g}) + e$ is a normal slice to $\mathcal{O} = {}^{G}e$ in $\mathfrak{g}$.
- For $\mathcal{P} = \mathcal{B}$, $T^*\mathcal{B} \xrightarrow{\mu} \mathcal{N}$ is a resolution of the nilpotent cone and the restriction to $S' \stackrel{\text{def}}{=} S \cap \mathcal{N}$ is a resolution $\widetilde{S}' \to S'$.

C3. Lusztig conjectures: $\mathbb{G}_m$ acts on $\mathfrak{g}$ by multiplication.

- There is $\mathbb{G}_m \to G^{\times \mathbb{G}_m}$ which preserves S & contracts it to e. (So it acts on $\widetilde{S}' \supseteq \mathcal{B}_e$ and contracts it to $\mathcal{B}_e$.)
- $K^{\mathbb{G}_m}(\mathcal{B}_e) \longleftrightarrow K^{\mathbb{G}_m}(\widetilde{S}')$.
- On $K^{\mathbb{G}_m}(\widetilde{S}')$ there are involutions β_e, β_S and a pairing $(-||-)$ with values in $\mathbb{Z}((v^{-1}))$ a completion of $K^{\mathbb{G}_m} = \mathbb{Z}[v^{\pm 1}]$.

Conj 1. $\mathbb{B}^{\pm}_{e}$ = all $b \in K^{\mathbb{G}_m}(\mathcal{B}_e)$ fixed by β_e and with $(b||b) \in 1 + v^{-1}\mathbb{Z}[[v^{-1}]]$;

$\mathbb{B}^{\pm}_{S}$ = the same for β_S and $K^{\mathbb{G}_m}(\widetilde{S}')$;

are signed basis of $K^{\mathbb{G}_m}$-modules $K^{\mathbb{G}_m}(\mathcal{B}_e)$ & $K^{\mathbb{G}_m}(\widetilde{S}')$.

Conj 2. $K(\mathcal{B}_e) \cong K^0[\mathrm{mod}_{fg}(U^0_e)]$ so that

$\mathbb{B}^{\pm}_{e}/\{\pm 1\} \longleftrightarrow$ irreducibles,

$\mathbb{B}^{\pm}_{S}/\{\pm 1\} \longleftrightarrow$ projectives.

Geometric Representation Theory: ④
reformulate RT into a
setting with strong machinery

D. Strategy of proof:

Rep. Theory	D-modules	Alg. Geometry	Hodge Topology
$\mathrm{mod}(U\mathfrak{g})$	$\mathrm{mod}(\mathcal{D}_{\mathcal{B}})$	$\mathrm{Coh}(T^*\mathcal{B})$	$\mathrm{Perv}(\mathcal{F})$ affine flag variety of G

① Translation between settings: equivalences of derived categories.

② Known for
$p>h$ or $p=0$ ⟷ $p>h$ ⟷ $p=0$ ⟷

③ Transition between $p>0$ & $p=0$: in Alg. Geometry.

④ Following your heart:
action of the affine braid group $\mathcal{B}_{aff}$
on derived categories: compatible with equivalences.

⑤ Resolution: use Hodge theory.

⑥ Beilinson-Bernstein equivalence ⟷ Splitting Azumayas ⟷ Langlands Duality of Bezrukavnikov ⟷

Extensions:

Travkin: Geometric q-Langlands

- use of Azumaya algebras of diff. operators: various
- the whole package: expect in quantum groups: Backelin-Kremnitzer
  and quiver varieties of type A. Critical Level?
- Bezrukavnikov - Okunkov program for geometry of
  symplectic resolutions

Ex1. Action of $\mathcal{B}_{aff}$ on $K(X)$ for X a symplectic
resolution like $T^*\mathcal{B}$, $\tilde{S}'$, Hilb^n(sympl. surface), ...
is the monodromy of the quantum cohomology connection.

Ex2. Dimension polynomials $\dim[T_{\lambda_0\to\lambda}(M)]$ for $p>0$
should provide change for a stability condition in $p=0$.

⑦ Mystery: 2 transition equivalences do not respect abelian
categories but the composition does !!!

E. Beilinson – Bernstein localization: $\left\{\begin{array}{l}\mathfrak{g}\text{-modules to}\\ \mathcal{D}\text{-modules}\end{array}\right.$

E1. Rings of "differential operators" on partial flags:

$$G \to Aut(\mathcal{P}),\quad \mathfrak{g} \to Vect(\mathcal{P}),\quad U\mathfrak{g} \longrightarrow \mathcal{D}_{\mathcal{P}} \ (\leftrightarrow T^*\mathcal{P})$$
$$U\mathfrak{g} \dashrightarrow \widetilde{\mathcal{D}}_{\mathcal{P}} \ (\leftrightarrow \mathfrak{g}_{\mathcal{P}})$$

$$\widetilde{\mathcal{D}}_{\mathcal{P}} = U_{\mathcal{O}_{\mathcal{P}}}(G \times_P \mathfrak{g}/\mathfrak{u}) \twoheadrightarrow U_{\mathcal{O}_{\mathcal{P}}}(G \times_P \mathfrak{g}/\mathfrak{p}) = \mathcal{D}_{\mathcal{P}}$$

- $(\widetilde{\mathcal{D}}_{\mathcal{P}})_P = U\mathfrak{g}/\mathfrak{u}\cdot U\mathfrak{g}$
- $\widetilde{\mathcal{D}}_{pt} = U\mathfrak{g} \twoheadrightarrow \mathcal{D}_{pt} = k.$
- $\widetilde{\mathcal{D}}_{\mathcal{P}} \otimes_{\mathcal{O}(\mathfrak{h}^*_{\mathcal{P}})} k_\lambda \overset{\text{def}}{=} \mathcal{D}^\lambda_{\mathcal{P}}$ for $\lambda \in \mathfrak{h}^*_{\mathcal{P}} \approx (\mathfrak{p}^{ab})^*.$
- For $\mathcal{P} = \mathcal{B}$: $\mathcal{D}^0_{\mathcal{B}} = \mathcal{D}_{\mathcal{B}} \underset{\text{locally}}{\approx} \mathcal{D}^\lambda_{\mathcal{B}}$ for $\lambda \in \mathfrak{h}^*.$

E2. Localization of U^λ-modules, $\lambda \in \mathfrak{h}^*$:

T1. If $\lambda \in \mathfrak{h}^*$ is $\mathcal{P}$-regular
$$D^b \text{ mod } \mathcal{D}^\lambda_{\mathcal{P}} \xrightarrow[\approx]{R\Gamma} D^b \text{ mod } U^\lambda$$

Version:
$$D^b \text{ mod}(\mathcal{D}^\lambda_{\mathcal{P}} | \widehat{\mathcal{P}^\lambda_\chi}) \xrightarrow{\approx} D^b \text{ mod}(U^\lambda_{\hat{\chi}})$$

F. From D-modules to coherent sheaves:

T2. If χ is $\mathcal{P}$-unramified: $\left\{\begin{array}{l}\text{Azumaya algebra } \mathcal{D}^\lambda_{\mathcal{P}} \text{ splits}\\ \text{on the formal neighborhood}\end{array}\right\}$:
$$D^b \text{ mod}(\mathcal{D}^\lambda_{\mathcal{P}} | \widehat{\mathcal{P}^\lambda_\chi}) \xleftarrow{\approx} D^b\, Coh(\widehat{\mathcal{P}^\lambda_\chi}).$$

C3. Each $D^b \text{ mod } U^\lambda_{\hat{\chi}}$ has a realization as $D^b\, Coh(\widehat{\mathcal{P}^\eta_\chi})$ for some $\mathcal{P}$ & some $\eta \in W\cdot\lambda.$

6.

G. From $p > h$ to $p = 0$ in $D^b Coh(\tilde{T}^* \mathcal{B})$:

The tilting vector bundle $\mathcal{E}$

T1. Over $R = \mathbb{Z}[\frac{1}{h!}]$ there is a vector bundle $\mathcal{E}$ on $\tilde{\mathfrak{g}} = \mathfrak{g}_{\mathcal{B}} = \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g} ;\ x \in \mathfrak{b}\}$ which is

(1) tilting: for $A = End(\mathcal{E})$

$$D^b Coh(\tilde{T}^* \mathcal{B}) \xrightarrow{\cong} D^b Coh(A), \quad \mathcal{F} \mapsto RHom(\mathcal{E}, \mathcal{F}).$$

(2) Exotic sheaves on $\tilde{T}^* \mathcal{B}$ $\stackrel{def}{=}$ the ones that corr. to A-modules, when ~~considered~~ considered on $\widehat{\mathcal{B}^\lambda_{\chi,k}}$ for $char(k) = p > h_G$, correspond to

U^λ_χ-modules (rather than to complexes of such).

(3) There is more: (still specialize R to k & restrict to $\widehat{\mathcal{B}^\lambda_\chi}$)

- Let $\mathcal{E}_i$, $i \in \mathcal{I}_\chi$, be distinct indecomposable summands of $\mathcal{E} | \widehat{\mathcal{B}^\lambda_\chi}$.

  These correspond to projectives (indecomposable) in U^λ_χ-modules.
- There are unique $\mathcal{L}_i \in D^b Coh(\mathcal{B}_e)$ such that

  $$RHom(\mathcal{E}_j, \mathcal{L}_i) = \delta_{i,j} \cdot k$$

  These correspond to irreducibles.

(4) $\mathcal{E}$ can be constructed using the action of $\mathbb{B}_{aff}$ on $D^b Coh(\tilde{T}^* \mathcal{B})$: start with $\mathcal{O}_{\tilde{T}^* \mathcal{B}}$, apply sufficiently many reflection functors

The generators T_α of $\mathbb{B}_{aff}$ act by functors I_α.

Reflection functor U_α is obtained as a cone for a natural map between I_α and identity.

H. Proof of Lusztig conjecture for $p \gg h$

Want: $\mathcal{L}_i, \mathcal{E}_i$ admit $\mathbb{G}_m$-equivariant structures $\tilde{\mathcal{L}}_i, \tilde{\mathcal{E}}_i$ with:

1. $\tilde{\mathcal{L}}_i$'s are representatives modulo ± 1 for Lusztig's $\mathbb{B}_e^{\pm} \subseteq K^{\mathbb{G}_m}(\mathcal{B}_e)$ i.e. $\begin{cases} \beta_e \tilde{\mathcal{L}}_i = \tilde{\mathcal{L}}_i, \\ (\tilde{\mathcal{L}}_i \| \tilde{\mathcal{L}}_j) \in \delta_{i,j} + v\mathbb{Z}[v] \end{cases}$.
2. The same for $\mathbb{B}_S^{\pm}$, $\mathcal{B}_S$ and extensions $\tilde{\mathcal{E}}_i^S$ of $\mathbb{G}_m$-equiv. bundles $\tilde{\mathcal{E}}_i$ from $\hat{\mathcal{B}}_e$ to $\tilde{\mathcal{S}}'$.

Reduction to $p=0$: (1) $\mathcal{L}_i^k, \mathcal{E}_i^k$ exists for any specializations $R \to k$ of R to a field, (2) For $p \gg 0$ they behave the same as for $k = \mathbb{Q}$!

So: only need to check for $p = 0$!!

Key: asymptotic orthogonality property, it amounts to

$$(\star)\ \begin{cases} \text{can choose eq. structures } \tilde{\mathcal{E}}_i \text{ so that} \\ \text{for all } s \leq 0: \text{ weight } s \text{ part of } \mathrm{Hom}(\tilde{\mathcal{E}}_i^S, \tilde{\mathcal{E}}_j^S) \\ \qquad = \delta_{i,j} \cdot \delta_{s,0} \cdot k \cdot \mathrm{id}_{\mathcal{E}_i}. \end{cases}$$

2° $\mathcal{E}_i$ has a canonical $\mathbb{G}_m$-structure

1° any other is just a shift by a character of $\mathbb{G}_m$

2° $(\tilde{\mathcal{E}}_i^S \| \tilde{\mathcal{E}}_j^S) = [\mathrm{Hom}(\tilde{\mathcal{E}}_i^S, \tilde{\mathcal{E}}_j^S)]$, so $(\star)$ = AsyOrth.

Proof of ($\star$); i.e. a choice of equivariant structures $\tilde{\mathcal{E}}_i$ of vector bundles $\mathcal{E}_i$ such that they satisfy ($\star$).

0° Reformulate the problem in terms of exotic irreducibles $\mathcal{L}_i$ instead of projectives $\mathcal{E}_i$.

1° [Arkhipov – Bezrukavnikov]

$$D^b \mathrm{Coh}(T^*\mathcal{B}) \approx \mathrm{Asph} \subseteq D^b(\mathrm{Perv}_I \mathcal{F})$$

for

$\mathcal{F}$ = affine flag variety for $\check{G}$, defined over a finite field $\mathbb{F}_q$

I = the corresponding Iwahori group.

2° Moreover [AB] says that The pull back under Frobenius on the RHS corresponds to pull back under multiplication by q on $T^*\mathcal{B}$ on LHS.

3° So, if $\mathcal{L}$ in LHS corresponds to L in RHS then a G_m-equivariance structure on $\mathcal{L}$ gives $q^{\#}$-equivariance, i.e., mixed structure on RHS.

4° A partial converse:

a mixed structure on L which is pure of weight 0 corresponds to a unique G_m-structure on $\mathcal{L}$.

5° Check that under AB equivalence exotic sheaves (LHS) correspond to perverse sheaves in RHS.

6° $\mathcal{L}_i$'s are irreducible exotic sheaves, so the counterparts L_i are irreducible perverse sheaves. Then

7° L_i's have canonical pure mixed structure of weight 0 and $\mathcal{L}_i$'s a canonical G_m-equivariant structure $\tilde{\mathcal{L}}_i$.

8° This one satisfies $\star$.