REPRESENTATION THEORY

Contents

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Part 1. Representations of finite groups

Section 1 is an elementary introduction into representations of finite groups. The emphasis is on algebraic structure of representations and less on computational aspect such as characters. The example of symmetric groups is sketched in section 2 from algebraic point of view, and then again in 3 from the geometric point of view ("Springer construction").

0.0.1. *Notation.* $\Bbbk$ will denote a field of characteristic p and $\mathcal{V}ec(\Bbbk)\supseteq\mathcal{V}ec^{\mathrm{fd}}(\Bbbk)$ the categories of (finite dimensional) vector spaces.

1. **Representations of finite groups**

1.0. **Summary.** For a group G of interest our goal is to classify irreducible representations, understand them in detail and then decompose interesting representations of G into their irreducible constituents.

1.0.1. *The approach.* There are many ways to introduce these ideas. We start by emphasizing the linear algebra ideas and from this we later derive the harmonic analysis.

We define representations in 1.1 as actions on vector spaces. Then in 1.2 we notice some obvious properties of the Category $Rep_G(\Bbbk)$ of representations of a group G over a field $\Bbbk$ (and we call those by their categorical names).

In 1.3 we define *irreducible* representations and we consider some characterizations of *semisimple* representations (the ones that are sums of irreducibles).

In the next two sections 1.4 and we consider first examples: irreducible representations of abelian groups and irreducible representations of products of groups.

In 1.8 we introduce the coinduction construction of representations of G from representations of subgroups. The *Frobenius reciprocity* provides a powerful tool for understanding coinduced representations.

In 1.7 we consider a general tool: the relation of functions $\mathcal{O}(G)$, the group algebra $\Bbbk[G]$ and of irreducible representations $Irr(G)$.

1.1. **Actions and representations of groups.**

1.1.1. *Structured sets.* The language of structures is a simple version of the idea of the idea of a categories.

We will only consider examples of what we mean by structures $\mathcal{S}$ on a set X.

Example. A structured group $(G,\mathcal{T})$ is a group G with an additional structure compatible with operations in G. For instance if $\mathcal{T}$ is a topology on G, the compatibility with a group structure means that the maps $G{\times}G \xrightarrow{\cdot} G \xrightarrow{-1} G \ni e$ are continuous.

1.1.2. *Structured actions.* An action of a group G on a structured set $(X,\mathcal{S})$ is an action $\circ : G{\times}X \to X$ on the set X that preserves structure $\mathcal{S}$, i.e., for each $g \in G$ the action $g{\circ}- :\ X \to X$ preserves $\mathcal{S}$. *representation of a group G on a vector space V over a field* $\Bbbk$ is an action of G on a the vector space V. This means an action $G{\times}V \xrightarrow{\circ} V$ of G on the set V which preserves the structure of a vector space on V, in other words the action for any $g \in G$ its action $g{\circ}- : V \to V$ is a linear operator.

Lemma. An action $\circ$ of G on a vector space V over $\Bbbk$ is the same as a homomorphism of groups $\pi : G \to GL(V)$. The transition is by $g{\circ}v = \pi(g)v$ for $v \in V$ and $g \in G$. $\square$

1.1.3. *Linearization of actions.* If G acts on a set X then

Notice the principle that

- If G acts on structured sets $(X_i, \mathcal{S}_i)$, $i \in I$, then it acts on every structured set $(V, \mathcal{S})$ that is naturally produced from these.

In particular we have the "linearization" mechanisms that produce vector spaces from structured sets.

Lemma. Each vector space V produced naturally from $(X, \mathcal{S})$ is a representation of X, i.e., the natural action of G on V is by linear operators. $\square$

Example. Structure $\mathcal{S}$ gives a notion $\mathcal{O} = \mathcal{O}_{\mathcal{S}}$ of a natural class of functions on X, then G acts on functions $\mathcal{O}(X)$ by $(g\circ f)(x) \stackrel{\text{def}}{=} f(g^{-1}\circ x)$. Then $\mathcal{O}_{\mathcal{S}}(X)$ is a representation of X. If $\mathcal{S}$ is a structure of a smooth or holomorphic manifold then the differential forms and vector fields on $(X, \mathcal{S})$ are representations of G.

Example. Group $SL_(\mathbb{C})$ acts on the complex projective line $\mathbb{P}^1(\mathbb{C})$. The subgroup $SL_2(\mathbb{R})$ preserves the upper half plane $\mathbb{H}$ so the holomorphic function $\mathcal{H}(\mathbb{H})$ on $\mathbb{H}$ are a representation of $SL_2(\mathbb{R})$.

1.2. **Category $Rep_G(\Bbbk)$ of representations of G on vector spaces over the field $\Bbbk$.** We will see that representations form a category. Moreover, the properties of the wonderful notion of a group will be reflected in the richness of properties of the category of its representations.

We sometimes write just Rep_G if $\Bbbk$ is obvious.

Lemma. (a) Rep_G is naturally a category enriched over the category.

(b) Category Rep_G is naturally an abelian category.

(c) Category Rep_G is naturally a monoidal abelian category for the operation of tensoring $U\otimes_{\Bbbk} V$.

(d) Monoidal category $(Rep_G, \otimes)$ is naturally a closed monoidal abelian category where the inner Hom functor is given by linear maps $\underline{\mathrm{Hom}}(U, V) = \mathrm{Hom}_k(U, V)$.

Proof. *Proof.* (a) For two representations U, V one defines the Hom set $\mathrm{Hom}_{Rep_G(\Bbbk)}(U, V)$ as the set of all liner maps $\phi : U \to V$ that are compatible with the action of G, i.e., for any $g \in G$

$$g\circ\phi = \phi\circ g.$$

More precisely this means that $g_V\circ\phi = \phi\circ g_U$ where g_V denotes the action of g on V, i.e., for any $u \in U$ one has $g(\phi u) = \phi(gu)$. The composition of Hom sets is just given by the composition of linear operators.

(b) A *subrepresentation* of a representation V of G is a vector subspace which is invariant under G.

(c) Any vector space V can be considered as a trivial representation of G by $gv = v$. What we call *the trivial* representation of $\mathbb{G}$ is the one dimensional representation on $\mathbb{k}$ with trivial action. $\square$

Remark. We will see later that there are more structures on $Rep_G(\mathbb{k})$. For instance it comes with the "forgetful" functor $\mathcal{F} : Rep_G(\mathbb{k}) \to \mathcal{V}ec(\mathbb{k})$ which just forgets the action of G. This is a closed monoidal functor, i.e., it preserves tensoring and the inner Hom.

1.2.1. *Invariants.* A vector $v \in V$ is G-invariant if all elements act on it trivially, i.e., $gv = v$ for $g \in G$. For any representation V we denote by $V^G \subseteq V$ the subspace of G-invariants, this is a subrepresentation of V.

Lemma. (a) $V^G \cong \mathrm{Hom}_G(\mathbb{k}, V)$.

(b) $\mathrm{Hom}_{\mathbb{k}}(U, V)^G = \mathrm{Hom}_G(U, V)$.

Example. if G acts on a set X with finitely many orbits then $\mathcal{O}(X)^G$ has a basis by characteristic functions 1_α of orbits α in G.

1.3. **Irreducible representations.** Any representation V of G has two obvious (trivial) subrepresentations 0 and V. We say that V is *irreducible* if it has no non-trivial subrepresentations. These are the smallest representations of G and they are basic building blocks ("atoms") for building any representation of G.

Let $Irr(G)$ be the set of isomorphism classes of irreducible representations.

An imprecision. We will often assume that for each isomorphism class $\alpha \in Irr(G)$ we have made a choice U_α of a representative of the class α. Then we will think of $Irr(G)$ as a set $\{U_\alpha,\ \alpha \in Irr(G)\}$ of representations.

1.3.1. *Schurr lemma.* This is the basic property of irreducible representations.

Lemma. Let U, V be two irreducible finite dimensional representations of G.

(a) If U, V are not isomorphic then $\mathrm{Hom}_G(U, V) = 0$. (So, if $\mathrm{Hom}_G(U, V) \neq 0$ then $U \cong V$.)

(b) $\mathrm{End}_G(V)$ is a field which is a division algebra finite dimensional over $\mathbb{k}$.

(c) If the field $\mathbb{k}$ is closed then $\mathrm{End}_G(V)$ is a field which is a division algebra $\mathrm{End}_G(U) = \mathbb{k}1_U \cong \mathbb{k}$. $\square$

1.3.2. *1-dimensional representations.* The multiplicative group over $\Bbbk$ is $G_m = \Bbbk^* \stackrel{\text{def}}{=} \Bbbk - \{0\}$. We will call $\mathrm{Hom}(G, G_m) = Char(G)$ the *characters* of G.[1] As $G_m = GL_1(\Bbbk)$, each character $\chi : G \to G_m$ is a 1-dimensional representation of G on $\Bbbk$.

Lemma. (a) $Char(G)$ is the set of isomorphism classes of 1-dimensional representations of G.

(b) $Char(G)$ is a commutative group.

(c) $Char(G) = Char(G^{\mathrm{ab}})$ for the abelianization G^{ab} of G (the quotient of G by the subgroup generated by all commutators in G). □

Proof. (b) Characters are functions and the operation on $Char(G)$ is just the multiplication of functions. □

1.4. **Irreducible representations of abelian groups.** Let $\Bbbk$ be closed. We will see that all finite dimensional irreducible representations of any abelian group are characters (1.4.1). So, $Irr\mathfrak{d}(A) = Char(A) = \widehat{A}$ is the Pontryagin dual of A. If characteristic is also zero then we will see that the Pontryagin duality behaves like duality of vector spaces.

1.4.1. *Irreducible representations are characters.*

Theorem. If $\Bbbk$ is closed then any irreducible finite dimensional representation of an abelian group A is 1-dimensional. □

Proof. Let $(V, \pi) \in Irr\mathfrak{d}(A)$. Since $\Bbbk$ is closed for any $a \in A$, $\pi(a)$ has an eigenvector v with eigenvalue λ_a. Now, $\mathrm{Ker}(a - \lambda_a)$ is a nonzero subrepresentation (since $a - \lambda_a$ commutes with A), hence $\mathrm{Ker}(a - \lambda_a) = V$, i.e., $\pi(a) = \lambda_a$.

Since all $a \in A$ act by scalars all subspaces are A-invariant. Now, irreducibility of V implies that $\dim(V) = 1$. □

Example. It is necessary that $\Bbbk$ be closed. Over $\mathbb{R}$ the group with generator g of order 4 has an irreducible 2-dimensional representation $R = \mathbb{R}^2$ where g acts by $(\, 0\ 1//-1\ 0 \,)$.

1.4.2. *Pontryagin duality.* For an abelian group A we call $\widehat{A} \stackrel{\text{def}}{=} Char(A)$ the *Pontryagin dual* of A.

Example. (a) $\widehat{\mathbb{Z}}$ is G_m.

(b) $\widehat{\mathbb{Z}/n}$ is $\mu_n \stackrel{\text{def}}{=} \{\zeta \in G_m = \Bbbk^*;\ \zeta^n = 1\}$.

(c) $\widehat{G \times H} \cong \widehat{G} \times \widehat{H}$. □

[1] A confusion: there are several notions that are all called "character".

Corollary. If $\Bbbk$ is closed and of characteristic zero then $\widehat{\mathbb{Z}/n} = \mu_n \cong \mathbb{Z}/n$.

Proof. $\widehat{\mathbb{Z}/n} = \mu_n(\Bbbk)$ is the set of solutions of $X^n = 1$ in $\Bbbk$. If $p = 0$ this equation has n different solutions on $\overline{\Bbbk}$. So, $|\mu_n(\Bbbk) = n$.

Actually, $\mu_n(\Bbbk)$ is cyclic. This is obvious if $\Bbbk = \mathbb{C}$ hence also $\Bbbk$ is the closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. However, any field $\Bbbk$ of characteristic zero contains $\mathbb{Q}$ and if it is closed it contains $\overline{\mathbb{Q}}$. □

Theorem. [Pontryagin duality.] If $\Bbbk$ is closed and $p = 0$ then the Pontryagin duality preserves the category $\mathcal{A}b^f$ of finite abelian groups and here it is involutive, i.e., $\widehat{\widehat{A}}$ is canonically isomorphic to A. □

Remarks. (0) This is an analogue of involutivity of duality on the category $\mathcal{V}ec\mathfrak{d}_\Bbbk$ of finite dimensional vector spaces.

(b) The statement extends to the case of positive characteristic. However, this requires passing to a larger category of finite abelian *group schemes.*

1.5. **Semisimple representations.** A representation is said to be *semisimple* if it is a sum of irreducible representations $V \cong \oplus_{i\in I} V_i$ with V_i irreducible.

1.5.1. *Semisimplicity property.* By a *multiple* of a representation V we will mean representations $M \otimes V$ for vector spaces M (This is a representation when we think of M as a trivial representation of G.) If we choose a basis $m_i,\ i \in I$, of M we get that $M \otimes V \cong \oplus_{i\in I} M$ is a multiple in the usual sense.

Lemma. (a) Semisimple representations are closed under sums.

(b) A representation is semisimple iff it is isomorphic to a sum of multiples $\oplus_{U\in Irr(G)} M_U \otimes U$ of representations in $Irr(G)$.

(c) Representation V is semisimple iff

- $(*)$ for any subrepresentation $V' \subseteq V$ there exists a complementary subrepresentation V''.

Proof. (a) holds by definition. In (b) if $V = \oplus_{i\in I} V_i$ with V_i irreducible then for $U \in Irr(G)$ let $I_U = \{i \in I;\ V_i \cong U\}$. This is a partition $I = \sqcup_{U\in Irr(G)} I_U$ of I. Therefore,

$$V = \oplus_{U\in Irr(G)} \oplus_{i\in I_U} V_i \cong \oplus_{U\in Irr(G)} \oplus_{i\in I_U} U = \oplus_{U\in Irr(G)} \Bbbk[I_U] \otimes V_i.$$

(c') Any non-zero finite dimensional representation V contains an irreducible subrepresentation $V' \subseteq V$. If V satisfies $*$ then $V = V' \oplus V''$ for some subrepresentation V''.

Then V'' again satisfies $(*)$ because for any subrepresentation W of V'' the sum $V' \oplus W$ has a complement C in V and then the image of C under the projection $V = V' \oplus V'' \to V''$ is a complement of U in V''.

By induction in dimension of V we know that V'' is semisimple and hence so is V.

(c") Now let V be semisimple, i.e., $V = \oplus_1^n V_i$ with irreducible V_i. We want to show that any subrepresentation V' has a complement.

(1) First we will prove this when V' is irreducible that it has a complement W which is semisimple.

The point is that since the inclusion $\iota : V' \subseteq V$ is a non-zero vector in $\mathrm{Hom}_G(V', V) = \oplus \mathrm{Hom}_G(V', V_i)$ then for some $k \in I$ the component $\iota_k : V' \to V_k$ of ι is not zero. Since V', V_k are irreducible this implies that ι_k is an isomorphism.

We will see that this actually implies a decomposition $V = V' \oplus C$ for $C = \oplus_{i \neq k} V_i$. By dimension count it suffices to notice that $V' \cap C = 0$. However, $V' \cap C$ is the kernel of ι_k. Since $\iota_k \neq 0$, V' does not lie in C, hence $V' \cap C \subseteq V'$ is a proper subrepresentation. Because V' is irreducible this gives $V' \cap C = 0$. So, we have proved that if V' is irreducible then it has a semisimple complement C.

(2) Now consider arbitrary $V' \subseteq V$. If $V' = 0$ the $V'' = V$ works. If $V' \neq 0$ then it contains some irreducible subrepresentation U.

By (1) we know that $U \subseteq V$ has a semisimple complement W. By induction in dimension $W \cap V' \subseteq W$ has a complement C. But then C is also a complement of V' in V. $\square$

1.5.2. *Multiplicity spaces for a semisimple representation.*

Corollary. (a') The multiplicity spaces M_U in a decomposition $V = \oplus_{U \in Irr(G)} M_U \otimes U$ of a semisimple representation V appear in

$$\mathrm{Hom}_G(U, V) \;=\; M_U \otimes \mathrm{End}_G(U).$$

(a') If $\Bbbk$ is closed then the multiplicity space M_Y for irreducible Y in V is $\mathrm{Hom}_G(Y, V)$.

(b') If V is semisimple so are its subrepresentations and quotients.

(b") Actually if $V = \oplus_{U \in Irr(G)} M_U \otimes U$ and $\Bbbk$ is closed then all subrepresentations V' of V are exactly the subspaces $V' = \oplus_{U \in Irr(G)} M'_U \otimes U$ for some systems of subspaces $M'_U \subseteq M_U$, $U \in Irr(G)$. Similarly for quotients.

Proof. (a) follows from Schurr lemma.

In (b') consider a subrepresentation S of a semisimple V. For any subrepresentation S' of S we know that S' a complement C in V. Then $C \cap S$ is a complement of S' in S. So, S is semisimple.

A quotient of semisimple V is of the form V/S for some subrepresentation S of V. We know that S has a complement C in V. But then V/S is isomorphic to a subrepresentation C, so it is semisimple.

(b") If $V'\subseteq V$ then V' is also semisimple so we have decompositions $V' = \oplus_{U\in Irr(G)} \operatorname{Hom}_G(U,V')\otimes U$ and $V = \oplus_{U\in Irr(G)} \operatorname{Hom}_G(U,V)\otimes U$ and the multiplicity spaces have obvious inclusions $\operatorname{Hom}_G(U,V')\subseteq\operatorname{Hom}_G(U,V)$. □

1.5.3. *Sums of semisimple subrepresentations.*

Lemma. (a) If W_i are semisimple subrepresentations of V then the sum of subspaces $\sum W_i$ inside V is semisimple.

(b) Let V_i be different irreducible representations of G over a closed $\Bbbk$. Suppose we have embeddings $\alpha_i : M_i\otimes V_i\hookrightarrow V$ of multiples of different irreducible representations V_i of G into the same representation V. Then V contains $\oplus_i\ M_i\otimes V_i$, i.e., the sum of vector subspaces $\sum_i\ \alpha_i(M_i\otimes V_i)$ inside V is direct.

Proof. (a) $\sum W_i$ is a quotient of the semisimple representation $\oplus_i W_i$ hence it is semisimple.

(b) If the sum were not direct then for some i $C = \alpha_i(M_i\otimes V_i)$ would meet $C' = \sum_{j\neq i}\ \alpha_i(M_j\otimes V_j)$. We will see that $C\cap C'\neq 0$ would produce a nonzero Hom between V_i and some V_j with $j\neq i$ which is impossible by Schurr lemma. The point is that

- $C\cap C'$ is a sub of $C\cong M_i\otimes V_i$ so it is isomorphic to $M_i'\otimes V_i$ for some subspace $M_i'\subseteq M_i$.
- C' is a quotient of the direct sum $\oplus_{j\neq i}\ \alpha_i(M_j\otimes V_j)$ hence it is isomorphic to $\oplus_{j\neq i}\ \alpha_i(M_j''\otimes V_j)$ for some quotients M_j'' of M_j.

Then the inclusion $i: C'\cap C\subseteq C'$ is a vector in

$$\operatorname{Hom}(M_i'\otimes V_i,\ \oplus_{j\neq i}M_j'\otimes V_j))\cong\ \oplus_{j\neq i}\ \operatorname{Hom}(M_i', M_j')\otimes\operatorname{Hom}_G(V_i,V_j)).$$

The RHS is zero by Schur lemma, hence $C\cap C' = 0$. □

Corollary. □

1.5.4. *Semisimplicity over $\mathbb{C}$.*

Lemma. Let $\Bbbk=\mathbb{C}$. Then for any $V\in Rep_G^{\mathrm{fd}}(\Bbbk)$:

(a) There exists a G-invariant hermitian inner product on V.

(b) For any subrepresentation $V'\subseteq V$ there exists a complementary subrepresentation V''.

(c) V is semisimple. □

Remark. Later we will see that all representations V of G over $\Bbbk$ are semisimple iff the characteristic p does not divide the order $|G|$ of the group.

1.6. **Representations of products of groups.** This section is an application of semisimple representations. It describes irreducible representations of products $G{\times}H$ (lemma 1.6.1). This has a consequence for any irreducible representations V of any group G: $V^*{\otimes}V$ embeds into functions on G.

1.6.1. *The outer tensor product of representations.* This is the functor (i.e., a construction)

$$Rep_G(\Bbbk){\times}Rep_H(\Bbbk) \xrightarrow{\boxtimes} Rep_{G\times H}(\Bbbk)$$

where the representation $U{\boxtimes}V$ of $G{\times}H$ is $U{\otimes}V$ as a vector space and the product of groups acts placewise, i.e.,

$$(g,h)(u{\otimes}v) \overset{\text{def}}{=} gu{\otimes}hv.$$

Theorem. For any closed field $\Bbbk$ the outer tensor product gives a bijection

$$Irr_G(\Bbbk){\times}Irr_H(\Bbbk) \xrightarrow{\cong} Irr_{G\times H}(\Bbbk).$$

1.6.2. *Irreducibles come from* $\boxtimes$*.*

1.6.3. *Sublemma.* For any representations W,V of $G{\times}H$ and H, $\mathrm{Hom}_G(V,W)$ is naturally a representation of G and the following evaluation map is a $G{\times}H$-map:

$$ev: \ \mathrm{Hom}_G(V,W){\otimes}V \to W, \quad ev(\phi{\otimes}u) \overset{\text{def}}{=} \phi u.$$

Proof. □

Lemma. Any irreducible representation W of a product $G{\times}H$ is of the form $U{\boxtimes}V$ for some irreducible representations U,V of G and H.

Proof. Since $W \neq 0$, when we consider W as a representation of the subgroup H of $G{\times}H$, it contains an irreducible subrepresentation V of H. Let If $\iota : V \to W$ be the inclusion map. Then the evaluation map $ev: \ \mathrm{Hom}_H(V,W){\otimes}V$ is not zero since $ev(\iota{\otimes}v) = \iota v = v$ for $v \in V$.

Since W is irreducible for $G{\times}H$, the map ev must be surjective (since its image is not zero).

Now notice that when we view the LHS $\mathrm{Hom}_H(V,W){\otimes}V$ just as a representation of H this is multiple of V, so it is a semisimple representation of H. Therefore, the subrepresentation $\mathrm{Ker}(ev)$ is of the form $K{\otimes}V$ for some subspace K of $\mathrm{Hom}_G(V,W)$ (see corollary 1.5.b", this is where we use the assumption that $\Bbbk$ is closed).

Then K consists of all $k \in \mathrm{Hom}_G(V,W)$ such that $ev(k{\otimes}-)$ is zero on V. This description implies that K is G-invariant.

Therefore, as a $G{\times}H$-module, $W \cong U{\boxtimes}V$ for the G-module $U = \mathrm{Hom}_H(V,W)/K$.

Finally, if U were not a irreducible then it would contain some irreducible $U' \neq U$ and then $W = U \boxtimes V$ would properly contain $U' \boxtimes V \neq 0$. So, U is irreducible. $\square$

1.6.4. $\boxtimes$ *preserves irreducibility.*

Lemma. If U, V are irreducible representations of G and H then $U \boxtimes V$ is irreducible for $G \times H$.

Proof. Let W be an irreducible $G \times H$-submodule of $U \boxtimes V$. By lemma 1.6.2 we know that $W \cong U' \boxtimes V'$ for some irreducible representations U', V' of G and H. Now $U' \boxtimes V' \cong W \subseteq U \boxtimes V$.

This embedding will easily identify G-representations U' and U. The point is that

$$0 \neq \mathrm{Hom}_G(U' \boxtimes V', U \boxtimes V) \; = \; \mathrm{Hom}_G(U' \otimes V', U \otimes V) \; = \; \mathrm{Hom}_G(U', U) \otimes \mathrm{Hom}_{\Bbbk}(V', V)$$

implies that $\mathrm{Hom}_G(U', U) \neq 0$. However, since U, U' are irreducible this implies that $U' \cong U$.

Similarly, $V' \cong V$ as H-representations. So, the inclusion $U' \boxtimes V' = W \hookrightarrow U \boxtimes V$ is an isomorphism since the dimensions are the same. So, $U \otimes V = W$ is irreducible. $\square$

1.6.5. *Proof of the theorem 1.6.1.* Lemma 1.6.4 says that that the map $\boxtimes : \; Irr_G(\Bbbk) \times Irr_H(\Bbbk) \to Irr_{G \times H}(\Bbbk)$ is defined. Then 1.6.2 says that this map is surjective. So, it remains to prove that $U_1 \otimes V_1 \cong U_2 \otimes V_2$ for irreducible U_i, V_i implies that $U_1 \cong U_2$ and $V_2 \cong V_2$. However, this and more has already been proved at the end of the proof of lemma 1.6.4. $\square$

1.7. **Matrix coefficients.** For a group G, the set G has symmetry G^2 where $(g, h)u \overset{\text{def}}{=} guh^{-1}$. This induces a representation of G^2 on $\mathcal{O}(G)$ by $[(g, h)f](u) = f(g^{-1}uh)$.

Any representation V of G, produces functions on G called *matrix coefficients* of V. For $v \in V$ and $v^* \in V^*$

$$c^V_{v^*, v}(g) \overset{\text{def}}{=} \langle v^*, gv \rangle.$$

Lemma. If $\Bbbk$ is closed and V is irreducible then the matrix coefficient map is an injective morphism of representations of G^2

$$c : \; V^* \boxtimes V \hookrightarrow \mathcal{O}(G), \quad c(v^* \otimes v) \overset{\text{def}}{=} c_{v^*, v}.$$

Proof. c is a G^2-map since $c_{(g,h)(v^* \otimes v)}(x) = c_{gv^* \otimes hv}(x) = \langle xgv^*, hv) \rangle$ and this will be the same as

$$[(g, h)c_{v^* \otimes v}](x) = c_{v^* \otimes v}(g^{-1}xh) = \langle g^{-1}xhv^*, v \rangle.$$

Now the G^2-map c is injective since $V^* \boxtimes V$ is an irreducible G^2-module and $c \neq 0$ (since $c_{v^*, v}(e) = \langle v^*, v \rangle$). $\square$

Corollary. If $\Bbbk$ is closed then the representation $\mathcal{O}(G)$ of G^2 contains a subrepresentation $\oplus_{V\in Irr(G)}\ V^*\boxtimes V$.

Proof. The lemma gives for each $V \in Irr(G)$ an embedding $i_V :\ V^*\boxtimes V \hookrightarrow \mathcal{O}(G)$. Since all these $V^*\boxtimes V$ are different irreducible representations for G^2 we know that the sum $\sum_{V\in Irr_G}\ i_V(V^*\boxtimes V)$ is direct, i.e., isomorphic to $\oplus_{V\in Irr(G)}\ V^*\boxtimes V$. □

Remarks. (a) If G is finite then $Irr(G)$ is too.

(b) If $G = S_2$ and characteristic of $\Bbbk$ is 2 then we know the only irreducible subrepresentation of $\mathcal{O}(G)$ is the trivial representation $\Bbbk$. This now implies that $Irr(G) = \{\Bbbk\}$. Moreover, $\oplus_{V\in Irr(S_2)}\ V^*\boxtimes V$ is then just $\Bbbk^*\boxtimes k$ which the trivial representation of $(S_2)^2$. So, in this case the inclusion $\oplus_{V\in Irr(G)}\ V^*\boxtimes V \subseteq \mathcal{O}(G)$ is proper.

1.8. **Restriction and Coinduction.** These are two operations that relate representations of a group and of its subgroup. This relation is a strong tool with a very nice property.

Let H be a subgroup of G. The *restriction* of a representation V of G to a representation of H will be denoted by $Res_H^G\ V$ (the vector space does not change but now we consider only the action of elements of H).

In the more interesting direction we will see that a representation (σ, U) of H gives a representation $(\pi, Coind_H^G\ U)$ of G, called the *coinduced representation.*[(2)] The underlying vector space is the space of functions on G with values in U and a certain compatibility with the representation of H on U :

$$Coind_H^G\ U \stackrel{\text{def}}{=} \{\phi : G \longrightarrow U,\ \phi(hx) = \sigma(h)\cdot\phi(x),\ h \in H,\ x \in G\}.$$

The action of G is by the right translations of functions $(\pi(g)\phi)(x) \stackrel{\text{def}}{=} \phi(xg)$.

Lemma. $Coind_H^G\ U$ is a well defined representation, i.e., the operators $\pi(g)$ do act on $Coind_H^G\ U$ and then π is a representation of G.

1.8.1. *Frobenius reciprocity.* Show that one can identify vector spaces

$$\mathrm{Hom}_H(Res_H^G\ V, U)\ \cong\ \mathrm{Hom}_G(V, Coind_H^G\ U),$$

using maps α and β, where for $v \in V$

(1) $\alpha : \mathrm{Hom}_G(V, Coind_H^G\ U) \to \mathrm{Hom}_H(Res_H^G\ V, U)$ by

$$\alpha(A)\ v\ \stackrel{\text{def}}{=}\ (Av)(1) \quad \text{for} \quad A \in \mathrm{Hom}_G(V, Coind_H^G\ U),$$

(2) $\beta : \mathrm{Hom}_H(Res_H^G\ V, U) \to \mathrm{Hom}_G(V, Coind_H^G\ U)$ by

$$(\beta B)v\ (g)\ \stackrel{\text{def}}{=}\ B(g\cdot v) \quad \text{for} \quad B \in \mathrm{Hom}_H(Res_H^G\ V, U).$$

[2] Actually there are two versions of this construction called *induction* and *coinduction*. For finite groups these are the same (meaning isomorphic) operations.

Remark. In the language of categories such identification is called an *adjunction* of the pair of functors (i.e., constructions) $(Coind_H^G, Res_H^G)$, or that Res_H^G is a left adjoint of $Coind_H^G$, while $Coind_H^G$ is a right adjoint of Res_H^G.

Adjunction is a strong relation between functors which generalizes inverses of functions and adjunction of linear operators on a vector sp-ace with an inner product. In particular. each functor uniquely determines the other. This means that a trivial forgetful construction Res_H^G determines an interesting construction $Coind_H^G$ as its right adjoint.

1.9. **The G^2-decomposition of functions on G.**

Lemma. $\mathcal{O}(G)$ as a representation of $G \cong\ 1{\times}G \subseteq G^2$ is the coinduced representation $Coind_1^G\ \Bbbk$.

Theorem. If $\Bbbk = \mathbb{C}$ then for any finite group G there is a canonical G^2-decomposition

$$c:\ \oplus_{V\in Irr(G)}\ V^*{\otimes}V \xrightarrow{\cong} \mathcal{O}(G)$$

which is on each summand $V^*{\otimes}V$ given by the matrix coefficient map c^V for V.

Proof. First recall that over $\mathbb{C}$ all representations of finite groups are semisimple (for this we used the hermitian inner products!).

Next for any semisimple representation M over a closed field (such as $\mathbb{C}$) the evaluation map

$$ev_M:\ \oplus_{V\in Irr_G}\ \mathrm{Hom}_G(V,M){\otimes}V\ \to M, \quad ev_M(A{\otimes}v) = Av$$

is an isomorphism.

Now, for $M = \mathcal{O}(G)$ viewed as a G-module via the right translation action $(g\phi)(x) = \phi(xg)$ we can calculate $\mathrm{Hom}_G(V,\mathcal{O}(m))$ via Frobenius reciprocity.

$$\mathrm{Hom}_G(V,\mathcal{O}(m)) =\ \mathrm{Hom}_G(V, Coind_1^G\ k) \xrightarrow[\cong]{Fr} \mathrm{Hom}_1(V,k) = V^*$$

where for $A\in \mathrm{Hom}_G(V,\mathcal{O}(m))$, by $FrA:\ V\to\Bbbk$ is given by $(FrA)v\ =\ (Av)(e)$, the evaluation of $Av\in\mathcal{O}(G)$ at $e\in G$. $\square$

Corollary. $\sum_{V\in Irr(G)}\ \dim(V)^2$ is the number $|G|$ of elements in G.

1.9.1. *Invariant functions.* Consider the action of G^2 on G so that (g,h) acts by L_gR_h, i.e., $(g,h)u = guh^{-1}$. Then G^2 acts on $\mathcal{O}(G)$ so that $[(g,h)\phi](u) = \phi(g^{-1}uh)$. The *conjugation action* of G on G is then the restriction of the action of G^2 to the diagonal $\Delta_G{\subseteq}G^2$, so the action on G is ${}^gu \overset{\text{def}}{=} gug^{-1}$ and on $\mathcal{O}(G)$ it is ${}^g\phi)(u) = \phi(g^{-1}ug)$.

By *class functions on G* one means the functions that are constant on conjugacy classes, i.e., that are invariant under the conjugation action of G on $\mathcal{O}(G)$. This means that $\phi(g^{-1}ug) = \phi(u)$ or equivalently that $\phi(vg) = \phi(gv)$. These form the subspace $\mathcal{O}(G)^G$.

Lemma. (a) For any representation V of G, the character function $\chi_V(g) \stackrel{\text{def}}{=} Tr_V(g)$ is an invariant function on G. and it lies in the image of the matrix coefficient map c^V : $V^*\boxtimes V \to \mathcal{O}(G)$.

(b) For $V \in Irr(G)$, $\dim[(V\otimes V^*)^G] = 1$.

Corollary. (a) The characters of irreducible representations form a basis of invariant functions.

(b) The number of irreducible representations of G is the same as the number of conjugacy classes in G.

1.10. **Hopf algebras [to be developed].**

1.11. **Functions and distributions.** For a finite set X the functions $\mathcal{O}(X, \Bbbk)$ have a $\Bbbk$-basis of *characteristic functions of points* $1_,\ a \in X$, where $1_a(x) = \delta_{a,x}$ for $a \in X$. The space of *distributions* on X is defined as the dual of functions $\mathcal{D}(X) \stackrel{\text{def}}{=} \mathcal{O}(X)^*$. It has a basis of "point distributions" $\delta_a,\ a \in X$ where $\langle \delta_a, f\rangle \stackrel{\text{def}}{=} f(a)$.

These bases give an isomorphism of vector spaces $\mathcal{O}(X) \xrightarrow{\cong} \mathcal{D}(X),\ f \mapsto \widetilde{f}$ by $\widetilde{1_a} = \delta_a$. This can be useful but it is in some sense unnatural: functions and distributions are different objects.

1.11.1. *Distributions.* For a finite set X, besides $\Bbbk$-valued functions $\mathcal{O}(X, \Bbbk)$ there is also the dual vector space $\mathcal{D}(X, k) \stackrel{\text{def}}{=} \mathcal{O}(X, \Bbbk)^*$ of *distributions* on X (We omit k when it is understood.). There are natural dual bases $1_a,\ \delta_a,\ a \in A$; of functions and distributions with $1_a(x) = \delta_{a,x}$ for $x \in X$ and $\delta_a(f) = f(a)$ for $f \in \mathcal{O}(X)$. Since the distribution δ_a comes from a point a it is natural to denote δ_a by a and $\mathcal{D}(X)$ by $\Bbbk X$ (or $\Bbbk[X]$) with a basis X.

Remark. When X has more structure (an algebraic variety or a manifold) this is reflected in the natural functions $\mathcal{O}(X)$ on X and it makes the definition of distributions more subtle.

1.11.2. *Group algebra.* Can define algebra $\Bbbk G$.

Identify it with $\mathcal{D}(G)$ as a vector space in a natural and unnatural way.

Prove that over $\mathbb{C}$ $\Bbbk G$ is

Lemma. For a finite G, $Rep_G(\Bbbk) = Mod[\Bbbk G] = \ Comod[\mathcal{O}(G)]$.

Remark. (Use the identification $\mathcal{O}(G)\xrightarrow{\cong}\mathcal{D}(G)$ by $1_g\mapsto\delta_g$ to define the convolution operation on $\mathcal{O}(G)$ then $1_g * 1_h = 1_{gh}$.

This gives an action of $(\mathcal{O}(G), *)$ on any $V \in Rep_G(\Bbbk)$. $1_g v = \delta_g v = gv$.

$\Bbbk G = \oplus V\boxtimes V^* \cong \oplus\mathrm{End}(V)$.

(a) $\mathrm{End}(V)\subseteq\Bbbk G$ acts on V as usual and on U by zero. By G^2-irreducibles: $\mathrm{End}(V) \to \mathrm{End}(U)$.

Corollary. This is an algebra isomorphism. $\square$

We know that the composition of algebra maps $\mathrm{End}(V)\hookrightarrow\Bbbk G \to \mathrm{End}(V)$ is identity.

1.11.3. *Hopf algebras.* Algebra, coalgebra, Hopf algebra Modules, comodules.

$(\mathcal{O}(G), m, u, \Delta, \varepsilon)$ is a Hopf algebra

$(\mathbb{D}(G), m, u, \Delta, \varepsilon$ is a Hopf algebra $\delta_g{\cdot}\Delta_h = \delta_{gh}$ ...

1.12. **Hopf algebra structures on $\mathcal{O}(G)$ and $\mathcal{D}(G) = \Bbbk G$.** The following is the basic principle of algebraic geometry (spaces are related to commutative rings), stated in the case of finite sets.

Lemma. A map of finite sets $\pi : X \to Y$ is equivalent to a map of algebras $\pi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. $\square$

Lemma. (a) For a finite set G the following are equivalent

(1) A monoid structure $G\times G \xrightarrow{m} G \xleftarrow{1}$ pt on G;
(2) a commutative bialgebra structure on $\mathcal{O}(G)$

$$\mathcal{O}(G)\otimes\mathcal{O}(G) \xrightarrow{\mu} \mathcal{O}(G) \leftarrow \mathrm{pt} \quad \text{and} \quad \mathcal{O}(G)\otimes\mathcal{O}(G) \xleftarrow{\mu} \mathcal{O}(G) \to \mathrm{pt}.$$

(3) a cocommutative bialgebra structure on $\mathcal{D}(G)$

$$\Bbbk G\otimes\Bbbk G) \xrightarrow{\mu} \Bbbk G \leftarrow \mathrm{pt} \quad \text{and} \quad \Bbbk G\otimes\Bbbk G) \xleftarrow{\mu} \Bbbk G \to \mathrm{pt}.$$

(b) Similarly, for a finite monoid G the following is equivalent:

(1) G is a group;
(2) bialgebra $\mathcal{O}(G)$ is a Hopf algebra;
(3) bialgebra $\Bbbk G$ is a Hopf algebra;

$\square$

Lemma. For a finite group G and a finite dimensional vector space V the following data are equivalent

(1) A representation $\pi : G \to \mathrm{End}_{\Bbbk}(V)$ of G on V;
(2) An action $\Pi : \Bbbk G \to \mathrm{End}(V)$ of the Hopf algebra $\Bbbk G$ on V;
(3) A matrix coefficient map $c : V^*\otimes V \to \mathcal{O}(G)$;
(4) A coaction $V \to \mathcal{O}(G)\otimes V$ of the Hopf algebra $\mathcal{O}(G)$ on V;
(5) An action $\Bbbk G\otimes V \to V$ of the Hopf algebra $\Bbbk G$ on V;

Proof. (1)$\Leftrightarrow$ (2) because G is a basis of $\Bbbk G$: one extends π to Π by linearity and one restricts Π to π.

The rest of the equivalences come from passing form a map to its dual or via the identity $\mathrm{Hom}(A\otimes B, C) \;\cong\; \mathrm{Hom}[A, \mathrm{Hom}(B, C)]$. $\square$

1.13. **Orthogonality relation.** $Tr(\chi_V, \mathcal{O}(G)) = Tr[dim(V)e, \mathcal{O}(G)] = \dim(V)|G|$.

In $\Bbbk G$ $\widetilde{\chi_V} = \frac{|G|}{\dim(V)} 1_V$ $I_V \overset{\text{def}}{=} \frac{\dim(V)}{|G|}\chi_V$ is a basis of $\Bbbk G$ by orthogonal idempotents.

Lemma. (a) For $U, V \in Irr(G)$

$$c^V_{v^*,v} * c^U_{u^*,u} = \delta_{U,V} c_{v^*,u}\langle u^*, v\rangle.$$

(b) $e_V = \dim(V)\chi_V$, $V \in Irr_G$, is a basis of invariant functions with the property that

$$e_U * e_V = \delta_{U,V} e_U.$$

In particular this is an orthonormal basis of invariant functions for $(-,-)$ or $\langle -,-\rangle$.

2. **Representations of symmetric groups S_n**

2.1. **Combinatorics.**

2.1.1. Partitions Π_n. The set of partitions Π_n of n consists of all sequences integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_q > 0)$. such that $|\lambda| \overset{\text{def}}{=} \sum \lambda_i$ equals n.

2.1.2. Young diagrams $\mathcal{Y}_n$. Consider the part $\mathbb{R}^2_{\geq 0}$ of the x, y-plane as divided into squares (we will call them"boxes") of unit size and with corners at $\mathbb{N}^2$. We can reparametrize these boxes by the additive semigroup $\mathbb{N}^2_{>0}$ using the upper right corners of boxes. Let $\mathcal{I}$ be the set of all *cofinite ideals* in this semigroup, i.e., all $I \subseteq \mathbb{N}^2_{>0}$ such that the set theoretic difference $\mathbb{N}^2_{>0} - I$ is finite and $I + \mathbb{N}^2_{>0} \subseteq I$ (if we think of I as a collection of boxes then the second condition says that if Y contains a box b then it also contains the boxes to the right of b and the boxes above b).

The set $\mathcal{Y}$ of (2-dimensional) Young diagrams consists of all complements $\mathbb{N}^2_{>0} - I$ of elements I in $\mathcal{I}$. Let $\mathcal{Y}_n$ be all diagrams in $\mathcal{Y}$ of size n.

Proposition. The following sets are in canonical bijections

(1) Partitions Π_n;
(2) Young diagrams $\mathcal{Y}_n$;
(3) Conjugacy classes $\frac{S_n}{S_n}$ in the symmetric group S_n;
(4) Nilpotent conjugacy classes $\mathcal{N}/GL_n$ in matrices M_n;
(5) $(G_m)^2$-fixed points in the Hilbert scheme $\mathcal{H}^n_{\mathbb{A}^2}$ of points in the plane $\mathbb{A}^2$. (The Hilbert scheme $\mathcal{H}^n_X$ is the moduli of all subschemes D of X of length n. It is often denoted $X^{[n]}$.)

Proof. A partition $\lambda \in \Pi_n$ gives the Young diagram $Y_\lambda \in Y_n$ whose k^{th} row has length k. This is a bijection $\Pi_n \xrightarrow{\cong} \mathcal{Y}_n$. (We will pretty much think of Y_λ and λ as being the same.) □

2.1.3. Tableaux $\mathcal{T}_\lambda$. For $\lambda \in \Pi_n$, a *tableau* T of shape λ is a bijective coloring of boxes in Y_λ by $\{1, ..., n\}$. These form the set $\mathcal{T}_\lambda$ on which S_n acts simply transitively.

A tableau $T \in \mathcal{T}_\lambda$ defines two ordered partitions $\mathcal{C}_T = (C_1, ..., C_p)$ and $\mathcal{R}_T = (R_1, ..., R_q)$ of $\{1, ..., n\}$ where R_i and C_j are the sets of colors in the i^{th} row and $j\theta$ column of Y_λ.

The a tableau T defines two subgroups $P_T \subseteq S_n \supseteq Q_T$, the stabilizers of the row and column partitions $\mathcal{R}_T$ and $\mathcal{C}_T$ of T. Say, $\mathcal{R}_T$ consists of all $\sigma \in S_n$ such that $\mathcal{R}_{\sigma T} = \mathcal{R}_T$, i.e., σ permutes elements of each row of the tableaux T.

2.1.4. *Duality* $\lambda\mapsto\check{\lambda}$ *on partitions.* The dual $\check{Y}$ of a diagram $Y\mathcal{Y}$ is obtained by flipping Y across the line $x = y$. The same for tableaux. On partitions we define $\check{\lambda}$ so that $Y_{\check{\lambda}} = (Y_\lambda)^\vee$.

Notice that the row partition $\mathcal{R}_{\check{T}}$ of T is the column partition $\mathcal{C}_T$ of T. So,

$$P_\lambda \cong \prod_i S_{\lambda_i} \quad \text{and} \quad Q_\lambda \cong \prod_j S_{\check{\lambda}_j}.$$

We will be interested in two 1-dimensional representations, the trivial representation $\tau_\lambda = \boxtimes\, \tau_{\lambda_i}$ of P_λ, and the sign representation $\sigma_\lambda = \boxtimes\, \sigma_{\lambda_i}$ of Q_λ,

2.1.5. *The standard representations* M^λ *and* N^λ *of* S_n. These are the names for the coinduced representations

$$M^T \stackrel{\text{def}}{=} Coind_{P_\lambda}^{S_n}\tau_\lambda \quad \text{and} \quad N^T \stackrel{\text{def}}{=} Coind_{Q_\lambda}^{S_n}\sigma_\lambda.$$

We will only consider representations over $\Bbbk = \mathbb{C}$.

Lemma. For any two tableaux $T', T'' \in \mathcal{Y}_\lambda$, representations $M^{T'}$ and $M^{T''}$ are canonically isomorphic. The same for $N^{T'}$ and $N^{T''}$.

Remark. For any Young diagram Y_λ there is a canonical choice T_λ of a tableaux in $\mathcal{T}_\lambda$: one colors the first row by $1, ..., \lambda_1$, the second by $\lambda_1 + 1, ..., \lambda_1 + \lambda_2$ etc. So we can define M^λ, N^λ using this tableau T_λ. However, the choice of tableaux is not important since for each $T \in \mathcal{T}_\lambda$ we have $M^T \cong M^\lambda$ canonically.

2.2. **Classification of** $Irr(S_n)$**.** We will eventually prove that

Theorem. (a) For each partition λ, there exists a unique irreducible representation $\pi^\lambda \in Irr(S_n)$ that appears in both M^λ and N^λ. Moreover, its multiplicity is one in both representations.

(b) $\Pi_n \ni \ \mapsto \pi^\lambda \in Irr(S_n)$ is a bijection.

Lemma. Part (a) of the theorem is equivalent to the claim

$$\text{(A)} \qquad \dim\ \mathrm{Hom}_{S_n}(M^\lambda, N^\lambda) = 1.$$

2.2.1. *The dominance order* $\lambda \geq \mu$ *on partitions.* One says that $\lambda \geq \mu$ if one has

$$\lambda_1 \geq \mu_1, \quad \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2 \ \text{ etc.}$$

Lemma. (a) Under the standard bijection of Π_n to nilpotent conjugacy classes $\mathcal{O}_\lambda \in \mathcal{N}_n/GL_n$, the dominance order on partitions is the same as the closure order on nilpotent conjugacy classes.

(b) The orbit $\mathcal{O}_{(n)}$ corresponding to the one row partition (n) consists of all nilpotents with one Jordan block. It is open and dense in $\mathcal{N}_n$.

The orbit $\mathcal{O}_{(1^n)}$ corresponding to the one column partition (1^n) is the point $\{0\}$. consists of all nilpotents with one Jordan block. It is open and dense in $\mathcal{N}_n$.

Lemma. The part (b) of the theorem 2.2 follows from the following theorem.

Theorem. If the irreducible representation π_μ appears in M^λ then $\mu \leq \lambda$.

Proof. □

3. Springer construction of irreducible representations of S_n from nilpotent orbits

Since Springer construction is one of origins of the Geometric Representation Theory (GRT), the part A of this section consists of general remarks on goals and methods of Geometric Representation Theory. The the part B (3.1-) explains how evidence points out to a possible shape of a geometric construction of $Irr(S_n)$. Finally, part C explains the Springer construction using algebraic topology in the formalism of constructible sheaves. (Among all proofs this one is the most powerful, i.e., it leads to many consequences.)

3.A Geometric Representation Theory

(A) The Geometric Representation Theory (GRT) started as the following strategy. For a given representation theory $\mathcal{R}$ find an algebro geometric object X that in some way encodes $\mathcal{R}$. Then, from X one extracts information about $\mathcal{R}$ using powerful methods of algebraic geometry.

(B) With the growing experience of GRT a point of view arose where the geometric objects X were sometimes seen as "more real" (meaning more fundamental) than the related representation theories $\mathcal{R}$.

3.0.1. *Example: GRT and* Langlands program. The *Langlands program* arose (in 1968) as a conjectural representation theoretic encoding of (large parts of) Number Theory. Since the corresponding representation theoretic problems were hard, Drinfeld applied the above idea (A) and constructed an upgrade, the *geometric Langlands program* (GLP). This indeed led to dramatic successes in the Langlands program.

The phase (B) started with the Beilinson-Drinfeld understanding that GLP has deep content not only in the case studied by Number Theory (the curves over finite fields) but also in the case studied by Algebraic Geometry (curve over closed fields, in particular over $\mathbb{C}$). Moreover, they noticed that when working over $\mathbb{C}$ it is useful to use ideas from physics (vertex algebras and collisions). Phase (B) came into full bloom with Witten's discovery that the geometric Langlands program over $\mathbb{C}$ is a part of the Quantum Field Theory. This established a previously unknown bridge between Number Theory and physics which can (so far) only be seen on the geometric level.

3.0.2. *Example: Category $\mathcal{O}$.* This is category of representations of semisimple Lie algebras which is of a very wide importance. While irreducible objects in $\mathcal{O}$ were found by "elementary" algebra, the question of computing their characters has been out of reach for a long time. The Kazhdan-Lusztig breakthrough was a conjectural topological interpretation in terms of intersection cohomology sheaves of Schubert varieties. This was then proved by Beilinson-Bernstein and by Kashiwara-Brylinski. These ideas were imitated and developed further in a large number of representation theoretic settings. In particular a very fruitful intuitive principle was formulated: the character formulas in "non-trivial" settings can only be proved by geometric methods.

3.0.3. *Example: singularities.* The following principle has been found in both GRT and in QFT:

Information is hidden in singularities, i.e., the non-smooth points of algebraic varieties.

Two (related) ways to extract information from singularities of a space X is by studying

- (i) by fibers of resolutions $\widetilde{X} \to X$ of X;
- (ii) stalks of the intersection cohomology sheaf $IC(X)$of the space X and

We will consider both in the example of singularities of nilpotent cones $\mathcal{N}_n$. Here, $\mathcal{N}_n$ is the moduli of nilpotent operators in the vector space M_n of $n{\times}n$ matrices. The resolution will come from the *flag variety* $\mathcal{F}_n$.

3.B The idea

3.1. **Why this should be possible.**

3.1.1. *Combinatorics and irreducible representations.* The theorem 2.2 gives a bijection $\Pi_n \ni \lambda\ \pi_\lambda \in Irr(S_n)$. The construction is nontrivial: to each $\lambda \in \Pi_n$ we associate two subgroups P_λ, Q_λ of S_n and then two coinduced representations M^λ, N^λ. Then π_λ is defined as the unique irreducible representation that lies in both M_λ and N_λ.

3.1.2. *From partitions to nilpotent orbits.* Now, recall the bijection $\Pi_n \ni \lambda {\mapsto} \mathcal{O}_\lambda \in GL_n(\mathcal{N}_n)$ of partitions and nilpotent orbits. We construct a vector space V_λ with the basis given by all boxes in the Young diagram Y_λ of λ. So, any ordering of boxes (this is just a choice of tableau in $\mathcal{Y}_\lambda$), gives an identification $V_\lambda \cong \Bbbk^n$.

Then we define a nilpotent operator e_λ on V_λ which is given on boxes by the translation to the left (the boxes in the first column fall out of Y_λ which we take to mean that they are killed by e_λ). Then $\mathcal{O}_\lambda$ is defined as the conjugacy class of e_λ in $M_n = \mathrm{End}_{\mathbb{K}}(\Bbbk^n)$ (this is independent of the choice of the above ordering of boxes).

3.1.3. *Geometry and irreducible representations.* The question that arises is whether the composition of bijections

$$GL_n\backslash\mathcal{N}_n \xrightarrow{\cong} \Pi_n \xrightarrow{\cong} Irr(S_n)$$

has a natural interpretation, i.e., whether one can construct the irreducible representation π_λ from the nilpotent orbit $\mathcal{O}_\lambda$?

This has been achieved by Springer and this idea is among the origins of the Geometric Representation Theory.

3.1.4. *Appendix. S_n and GL_n.* Here is how one usually views the relation of a finite group S_n and the algebraic group GL_n.

Any permutation $\sigma \in S_n$ acts on the standard basis $e_1, ..., e_n$ of $\Bbbk^n$ and this gives an embedding of groups $i : S_n \hookrightarrow GL_n$ by $i(\sigma)e_k \stackrel{\text{def}}{=} e_{\sigma k}$.

The standard *Cartan* subgroup of $G = GL_n$ is the group $T = \{diag(z_1, ..., z_n);\ z_i \in \Bbbk^*\} \cong (G_m)^n$. This very simple subgroup controls the whole group GL_n in terms of certain discrete objects: the *system of roots* Δ and the Weyl group W defined as $N_G(T)/T$, the normalizer of T in G divided by T itself.

Lemma. $N_G(T) = T \ltimes S_n$, hence the Weyl group W of GL_n is S_n. $\square$

3.2. **Geometry associated to groups GL_n.** To $G = GL_n$ we will associate geometric objects: the nilpotent cone $\mathcal{N}$ in its Lie algebra $\mathfrak{g}$, its flag variety $\mathcal{F} = \mathcal{B}$, Grothendieck resolution $\widetilde{\mathfrak{g}}$, Springer resolution $\widetilde{\mathcal{N}}$ and Springer fibers $\mathcal{B}_x$.

3.2.1. By $G = GL_n$ we mean certain *algebraic group*, i.e., a factory that produces groups $GL_n(\Bbbk)$, one for each commutative ring $\Bbbk$.

However, here we will use $\Bbbk = \mathbb{C}$ and often think of GL_n as just denoting the single group $GL_n(\Bbbk)$. Since our G is obtained from the vector space $V = \Bbbk^n$, as the group of $GL(V)$ of its automorphisms, it may be more natural to construct geometric objects from V, i.e., in linear algebra, but it will turn out that they can all be reconstructed from the group G itself. (This opens the construction of the "same" geometry for all reductive groups.)

3.2.2. *Lie algebra $\mathfrak{g}$ and nilpotent orbits.* The Lie algebra of G is the tangent space $\mathfrak{g} \stackrel{\text{def}}{=} T_eG$ of G at the origin. In our case this is the space of $n \times n$ matrices $\mathfrak{g} = M_n$. Group G acts on its Lie algebra by conjugation.

$\mathfrak{g}$ has interesting subspaces

(1) regular semisimple part $\mathfrak{g}_{rs}$ (n distinct eigenvalues)
(2) regular part $\mathfrak{g}_r$ (centralizer of dimension n)
(3) Nilpotent part $\mathcal{N}$ (all eigenvalues are zero).

Lemma. (a) $\mathcal{N}$ is singular.

(b) The orbits of G in $\mathcal{N}$ are classified by partitions $\lambda \mapsto \mathcal{O}_\lambda = {}^G e_\lambda$. So, we get a stratification $\mathcal{N} = \sqcup_{\lambda\in\Pi_n} \mathcal{O}_\lambda$.

(c) $\mathcal{O}_{(1^n)} = \{0\}$.

(d) $\mathcal{O}_{(n)}$ consists of nilpotent matrices which have one Jordan block. This is exactly the set of all smooth points in $\mathcal{N}$, so it is called the *regular part* $\mathcal{N}_r$ of $\mathcal{N}$.

Example. For $\lambda = (n-1,1)$ we call $\mathcal{O}_\lambda$ the *subregular nilpotent orbit* and for $\lambda = (2,1^{n-1})$ we call $\mathcal{O}_\lambda$ the *minimal nilpotent orbit.*

3.2.3. *Flag variety* $\mathcal{F} \cong \mathcal{B}$. A flag in V is a sequence of subspaces $F = (F_0\subseteq F_1\subseteq\cdots\subseteq F_n\subseteq V)$ such that $\dim(F_i) = i$. (Then $F_0 = 0$ and $F_n = V$ so we can forget about these two.) Let $\mathcal{F}$ be the moduli of all flags in V. The standard basis $e_1, ..., e_n$ of V gives the standard flag F^0 with $F^0_i = \langle e_1, ..., e_i\rangle$.

Let $U = U_n$ be *unitary* matrices in $G = GL_n(\mathbb{C})$.

Lemma. (a) G acts transitively on $\mathcal{F}$ and so does U.

(b) The stabilizer B^0 of F^0 in G consists of all $g \in GL_n$ whose sub-diagonal elements are 0.

(c) The stabilizer of F^0 in U consists of all diagonal matrices $u = diag(\alpha_1, ..., \alpha_n)$ with $|\alpha_i| = 1$ for all i. □

Corollary. (a) $\mathcal{F} \cong G/B^0$ (so $\mathcal{F}$ is smooth).

(b) $\mathcal{F}$ is compact. □

Remark. Actually, $\mathcal{F}$ is the largest compact homogeneous space of G.

3.2.4. *Borel subgroups.* Let the *Borel space* $\mathcal{B}$ consist of all subgroups B of G which are conjugate to the upper-triangular subgroup B^0. These subgroups are called *Borel subgroups.*

An element u of G is said to be *unipotent* if all of its eigenvalues are 1. The *unipotent cone* $\mathcal{U}$ in G consists of all unipotent elements. A subgroup U of G is said to be unipotent if all of its elements are.

The *unipotent radical* N of a Borel subgroup B is defined as the largest unipotent subgroup of B. It exists and it equals $B\cap\mathcal{U}$. For instance, for the standard Borel B^0 its unipotent radical N^0 consists of all $g\in G$ which are zero beneath the diagonal and 1 on the diagonal.

Lemma. (a) The normalizer $N_G(B)$ of any Borel subgroup B is B itself.

(b) There are canonical identifications of $\mathcal{F}\xleftarrow{\cong}G/B^0\xrightarrow{\cong}\mathcal{B}$. The resulting identifications $\mathcal{F}\cong\mathcal{B}$ sends a flag $F\in\mathcal{F}$ to its stabilizer $B=G_F$ in G and in the opposite direction a Borel B is sent to the unique flag F that B stabilizes. $\square$

Remark. We will often identify $\mathcal{F}$ and $\mathcal{B}$ and G/B^0. One can think of this as a group theoretical presentation of the flag variety $\mathcal{F}$.

3.2.5. *Borel subalgebras.* We will also consider the Lie algebra $\mathfrak{b}^0=T_e(B^0)$ of the group B^0. We have $\mathfrak{b}^0\subseteq\mathfrak{g}$, i.e., $Lie(B^0)\subseteq Lie(G)=M_n$, it consists of all $x\in M_n$ whose sub-diagonal elements are 0.

All G-conjugates $\mathfrak{b}={}^g(\mathfrak{b}^0)$, $g\in G$, of $\mathfrak{b}^0$ are called *Borel subalgebras* of $\mathfrak{g}$. (Here “subalgebra” is short for a “Lie subalgebra”.)

Lemma. Passing from Borel subgroups B to its Lie algebra $\mathfrak{b}=Lie(B)$ gives a bijection of all Borel subgroups and all Borel subalgebras. $\square$

Remark. We can now identify $\mathcal{F}$ and $\mathcal{B}$ and the space of Borel subalgebras which is ${}^G(\mathfrak{b}^0)\cong G/B^0$.

3.2.6. *Springer resolution.* Let $\widetilde{\mathfrak{g}}\subseteq\mathfrak{g}\times\mathcal{F}$ consists of all pairs of a matrix $x\in\mathfrak{g}$ and a flag $F=(F_0\subseteq\cdots\subseteq F_n)$ such that x preserves the flag, i.e., $xF_i\subseteq F_i$.

The Grothendieck map $\pi:\widetilde{\mathfrak{g}}\to\mathfrak{g}$ is the composition $\widetilde{\mathfrak{g}}\subseteq\mathfrak{g}\times\mathcal{F}\to\mathfrak{g}$. The *Springer fibers* are fibers $\pi^{-1}(x)$ of π at elements $x\in\mathfrak{g}$. We see that $\pi^{-1}x$ lies inside $\mathcal{F}$ and consists of all flags F that x preserves. We denote a Springer fiber $\pi^{-1}{}_x$ by $\mathcal{F}_x$ or $\mathcal{B}_x$.

Let $\widetilde{\mathcal{N}}$ be the restriction of $\widetilde{\mathfrak{g}}$ to $\mathcal{N}\subseteq\mathfrak{g}$, i.e.,

$$\widetilde{\mathcal{N}}=\{(x,F)\in\mathcal{N}\times\mathcal{F};\ xF_i\subseteq F_i\}=\{(x,F)\in\mathfrak{g}\times\mathcal{F};\ xF_i\subseteq F_{i-1}\}.$$

Lemma. (a) If $s\in\mathrm{End}(V)$ be regular semisimple, i.e., it has n distinct eigenvalues $\lambda_1,...,\lambda_n$ then $\mathcal{F}_x$ is naturally an S_n-*torsor*, i.e., it has a natural action of S_n and this action is simply-transitive.

(b) $\mathcal{F}_0=\mathcal{F}$.

(c) If $e\in\mathcal{N}$ is regular nilpotent then $\mathcal{F}_e$ is just a point $\{F\}$ for $F_i=\mathrm{Ker}(e^i)$.

(d) If $e \in \mathcal{N}$ is subregular then the Springer fiber $\mathcal{F}_e$ is a union of spaces $C_1, ..., C_{n-1}$, all of which are copies of $\mathbb{P}^1$. For $i \neq j$ the intersection $C_i \cap C_j$ is empty unless i, j are neighbors, then the intersection is a point.

3.3. **Guessing the relation of $Irr(S_n)$ and Springer fibers.** In the subregular case $e \in \mathcal{O}_{sr} = \mathcal{O}_\lambda$ for $\lambda = (n-1, 1)$ the irreducible representation $\pi_{n-1,1}$ is the reflection representation R (the quotient of $\mathbb{C}^n$ by the diagonal $\mathbb{C}$). So, it has dimension $n-1$.

On the other hand, the Springer fiber $\mathcal{B}_e$ has $n-1$ "pieces" which are all $\mathbb{P}^1$. To make this phrase precise we define the set $Irr(X)$ of *irreducible components* of a space X as closures of connected components of the open part $X_{sm} \subseteq X$ consisting of all smooth points of X. For our $\mathcal{B}_e$ the singular points are just the intersection points of different copies C_i of $\mathbb{P}^1$. So, the connected components of $(\mathcal{B}_e)_{sm}$ are $\mathbb{A}^1, G_m, ..., G_m, \mathbb{A}^1$. Then $Irr(\mathcal{B}_e)$ is just the set $\{C_1, ..., C_{n-1}\}$ of all copies of $\mathbb{P}^1$ in $\mathcal{B}_e$. Now we can state a guess

Conjecture. For any λ and $e \in \mathcal{O}_\lambda$, $\dim(\pi_\lambda) = |Irr(\mathcal{B}_e)|$ or (more ambitiously) π_λ has a basis $Irr(\mathcal{B}_e)$, i.e.,

$$\pi_\lambda = \mathbb{C}[Irr(\mathcal{B}_e)].$$

3.3.1. *Continuity idea I.* The first thing we need for this conjecture is that S_n acts on the nilpotent Springer fibers $\mathbb{C}[Irr(\mathcal{B}_e)]$, $e \in \mathcal{N}$. However, we already know that S_n does act on the Springer fiber $\mathcal{B}_s$ when s is regular semisimple, hence also on $\mathbb{C}[Irr(\mathcal{B}_e)]$.

Remark. Since $\mathcal{B}_s$ is an S_n-torsor we have an S_n-isomorphism $\mathcal{B}_s \cong S_n$. Therefore $\mathbb{C}[Irr(\mathcal{B}_s)] \cong \mathbb{C}[S_n] \cong \mathcal{O}(S_n)$. For any finite group G the representation of G on $\mathcal{O}(G)$ is called the *regular representation* of G. It contains all information about G built it is large – not irreducible. □

Now we know that S_n acts on generic Springer fibers ($\mathfrak{g}_{rs} \subseteq \mathfrak{g}$ is open and dense). One can hope that this will by "some kind of continuity" make it act on all fibers, in particular the nilpotent ones.

3.3.2. *Topological interpretation.* Since the "continuity" hope require topology we try to do everything in topological terms.

Lemma. $\mathbb{C}[Irr(\mathcal{B}_e)]$ is the top homology $H_{top}(\mathcal{B}_e)$.

Proof. This uses a general property of Springer fibers that they are *equidimensional*, meaning that all irreducible components have the same dimension. For a general algebraic variety X one has

$$H_{top}(X, \mathbb{C}) = \mathbb{C}[Irr^{top}(X)]$$

where $Irr^{top}(X) \subseteq Irr(X)$ consists of all irreducible components of maximal dimension. □

Remark. The Springer fibers $\mathcal{B}_x$ are *paved*, i.e., can be written as $\sqcup_1^N X_i$ where each X_i is some affine space $\mathbb{A}^{d(i)}$. If one uses this linear algebra fact, the proof of the lemma simplifies as one needs to know very little about homology. $\square$

3.3.3. *Enter sheaves.* Now $\mathbb{C}[Irr(\mathcal{B}_x)]$ has a topological interpretation as $H_{top}(\mathcal{B}_x, \mathbb{C}) \subseteq H_*(\mathcal{B}_x)$. The next step in our continuity argument is that we need to organize all graded groups $H_*(\mathcal{B}_x)$ for $x \in \mathfrak{g}$ into a single "continuous" object that lies above$\mathfrak{g}$. This object will be a sheaf or more precisely a complex of sheaves, for simplicity one calls it the *Grothendieck sheaf* $\mathcal{G}$ on the space $\mathfrak{g}$.

We know that S_n acts on $\mathcal{B}_x$ for $x \in \mathfrak{g}_{rs}$. This will translate into: S_n acts on the restriction $\mathcal{G}|\mathfrak{g}_{rs}$. Our goal is to get S_n act on the whole sheaf $\mathcal{G}$ and this will then imply that S_n acts on $H_*(\mathcal{B}_x)$ for all $x \in \mathfrak{g}$.

What we will need for this is that the restriction $\mathcal{G}|\mathfrak{g}_{rs}$ in *some* sense controls the whole sheaf $\mathcal{G}$. It will actually turn out that all of $\mathcal{G}$ can be reconstructed from the restriction $\mathcal{G}|\mathfrak{g}_{rs}$ in an explicit way. This miracle will complete our continuity argument. However, in order to access this miracle by following one of major successes of mathematics in the last quarter of the last century, we will have to become *perverse.*

3.C Springer construction via constructible sheaves

3.4. **The Grothendieck sheaf $\mathcal{G}$ on $\mathfrak{g}$.** We define the *Grothendieck sheaf* $\mathcal{G}$ on the Lie algebra $\mathfrak{g}$ as the direct image for $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ of the constant sheaf $\Bbbk_{\widetilde{\mathfrak{g}}}$

$$\mathcal{G} \stackrel{\text{def}}{=} \pi_* \Bbbk_{\widetilde{\mathfrak{g}}} \in D_c(\mathfrak{g}, \Bbbk).$$

Here, $D_c(\mathfrak{g}, \Bbbk)$ is the *derived category of constructible sheaves on X with coefficients in* $\Bbbk$ (see fro this category and its functoriality properties). In particular, $\pi_* : D_c(\mathfrak{g}, \Bbbk) \to D_c(\mathfrak{g}, \Bbbk)$ denotes the *derived direct image*, so though we start with just a sheaf $\Bbbk_{\widetilde{\mathfrak{g}}}$, the result $\mathcal{G}$ is really a constructible complex of of sheaves. (Though we just call a "sheaf".)

Lemma. For any $x \in \mathfrak{g}$, the stalk of the Grothendieck sheaf $\mathcal{G}$ at x is the cohomology of the Springer fiber $\mathcal{B}_x$ at x

$$\mathcal{G}_x = H^*(\mathcal{B}_x, \Bbbk).$$

Remark. A priori, $\mathcal{G}_x$ is a complex in $D_c(pt, \Bbbk) = D(\Bbbk)$ while $H^*(\mathcal{B}_x, \Bbbk)$ is only a graded group. We will kill the difference by taking the cohomology groups of $\mathcal{G}_x$. Then the precise form of "equality" in the lemma is an isomorphism of graded groups

$$\mathrm{H}^*(\mathcal{G}_x) \cong H^*(\mathcal{B}_x, \Bbbk).$$

Proof. The fiber $\mathcal{B}_x$ of the map $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ at x appears in a Cartesian square (see example A.1.3).a)

$$\begin{array}{ccc} \mathcal{B}_x = \pi^{-1}x & \xrightarrow{i} & \widetilde{\mathfrak{g}} \\ {\scriptstyle p}\Big\downarrow & & {\scriptstyle \pi}\Big\downarrow . \\ x & \xrightarrow{k} & X \end{array}$$

Now, since π is proper we have $\pi_! = \pi_*$ and $p_! = p_*$. Therefore we can use Base Change to calculate the stalk $\mathcal{G}_x^*\mathcal{G}$

$$k^*\mathcal{G} = \ \Bbbk^*\pi_*\Bbbk_{\widetilde{\mathfrak{g}}} \ = \ \Bbbk^*\pi_!\Bbbk_{\widetilde{\mathfrak{g}}} \ \overset{\text{Base Change}}{\cong} \ p_!i^*\Bbbk_{\widetilde{\mathfrak{g}}}.$$

Also, notice that $i^*\Bbbk_{\widetilde{\mathfrak{g}}} \ \cong \ \Bbbk_{\mathcal{B}_x}$. (For any map $f : Y \to X$ we have $f^*\Bbbk_X = \Bbbk_Y$ because $f^*\Bbbk_X = f^*a_X^*\Bbbk = \ (a_X \circ f)^*\Bbbk = \ a_Y^*\Bbbk = \Bbbk_Y$.) So,

$$\mathrm{H}^*(\mathcal{G}_x) \ \cong \ H^*(p_*\Bbbk_{\mathcal{B}_x}) \ \cong \ H^*(\mathcal{B}_x, \Bbbk). \quad \square$$

Remarks. (0) So, the Grothendieck sheaf $\mathcal{G}$ is a single "continuous" object that puts together all cohomologies of Springer fibers. Since homologies are dual to cohomologies $\mathcal{G}$ stores the information of homology groups of all Springer fibers.

(1) The precise statement for homologies is proved as above, it contains a shift in degrees of complexes:

$$H_*(\mathcal{B}_x, \Bbbk) \ \cong \ \mathrm{H}^*(\mathcal{G}_x^!)[2\dim_{\mathbb{C}}(\mathfrak{g})]$$

where the *shriek stalk* $\mathcal{G}_x^!$ of $\mathcal{G}$ at $x \overset{k}{\in} \mathcal{G}$ is defined as the !-restriction $k^!\mathcal{G}$ to a point. $\square$

3.5. **Restriction of $\mathcal{G}$ to $\mathfrak{g}_{rs} \subseteq \mathfrak{g}$ controls the whole $\mathcal{G}$ in the setting of perverse sheaves.** Now, we would line to use the Grothendieck sheaf $\mathcal{G}$ Let $j : \mathfrak{g}_{rs} \subseteq \mathfrak{g}$ be the inclusion of the open dense locus of regular semisimple operators. Let us also consider the open subspaces $\widetilde{\mathfrak{g}}_{rs} \overset{\text{def}}{=} \ \pi^{-1}\mathfrak{g}_{rs}$ of $\widetilde{\mathfrak{g}}$, i.e., the restriction of $\widetilde{\mathfrak{g}}$ to $\mathfrak{g}_{rs}$. Denote the inclusion $\widetilde{j} : \widetilde{\mathfrak{g}}_{rs} \subseteq \widetilde{\mathfrak{g}}$.

Recall that S_n-acts on Springer fibers $\mathcal{B}_s$ for $s \in \mathfrak{g}_{rs}$, so it acts on the open subspace $\widetilde{\mathfrak{g}}_{rs}$ of $\widetilde{\mathfrak{g}}$. Moreover, the restriction $\pi_{rs} : \widetilde{\mathfrak{g}}_{rs} \to \mathfrak{g}_{rs}$ of $\pi : \widetilde{\mathfrak{g}} \to\to \mathfrak{g}$ is an S_n-equivariant map for the trivial action of S_n on $\mathfrak{g}_{rs}$.

This ensures that S_n acts on the restriction $j^*\mathcal{G}$ of $\mathcal{G}$ to $\mathfrak{g}_{rs}$. Indeed, S_n acts on $\Bbbk_{\widetilde{\mathfrak{g}}_{rs}}$, map π_{rs} is S_n equivariant and by Base Change

$$\mathcal{G}|_{\mathfrak{g}_{rs}} \ = \ j^*\mathcal{G} = j_*\pi_!\Bbbk_{\widetilde{\mathfrak{g}}} \ \cong \ (\pi_{rs})_!\widetilde{j}^*\Bbbk_{\widetilde{\mathfrak{g}}} \ = \ (\pi_{rs})_!\Bbbk_{\widetilde{\mathfrak{g}}_{rs}}.$$

What we really want is that S_n should act on the sheaf $\mathcal{G}$ itself Then at any $x \in \mathfrak{g}$ group S_n will act on the stalk $\mathcal{G}_x$, $x \in \mathfrak{g}$, i.e., hence also on the cohomology groups $H^*(\mathcal{B}_x, \Bbbk) = \mathrm{H}^*(\mathcal{G}_x)$ of the Springer fiber at x. Equivalently, S_n will act on the dual $H_*(\mathcal{B}_x, \Bbbk)$ of cohomology. So, in particular it will act on $H_{top}(\mathcal{B}_e, \Bbbk)$ for $e \in \mathcal{N}$.

Action on $\mathcal{G}$ is not obvious since S_n does not act on the space $\widetilde{\mathfrak{g}}$ that produces $\mathcal{G}$. This action will be a consequence of a very strong relation between $\mathcal{G}$ and its restriction $\mathcal{G}|_{\mathfrak{g}_{rs}}$ in the next theorem which uses certain procedure $j_{!*}$ (a functor) that extends sheaves from $\mathfrak{g}_{rs}$ to $\mathfrak{g}$. Functor $j_{!*}$ has been constructed and studied for the framework of *perverse sheaves* by Beilinson-Bernstein-Deligne. The perverse sheaves on a space X and with coefficients in $\Bbbk$-modules form an abelian category $\mathcal{P}(X,\Bbbk)$.[3] The perverse sheaves are a class of complexes of sheaves which have particularly good properties so they appear to be more fundamental than the sheaves themselves. The term "perverse" seems to mean that these objects are different from the accustomed norms of our knowledge (but are even more beautiful that the world we traditionally see).[4]

$j_{!*}$ is called the *intersection cohomology extension*(or IC-extension). The intersection cohomology is a version of homology and cohomology combined which has good properties for singular spaces: it satisfies Poincare duality. The discovery of intersection cohomology by Goresky-McPherson is also the origin of the idea of perverse sheaves.

(1) For an open $U\overset{j}{\subseteq}X$ one has $j_{!*}:\mathcal{P}(U,\Bbbk)\to\mathcal{P}(X,\Bbbk)$ so this is a new functoriality which applies to perverse sheaves. A famous example is the Grothendieck sheaf $\mathcal{G}$ (or more precisely, its shift $\mathcal{G}[\dim(\mathfrak{g})]$).

(1) $j_{!*}$ is "in between $j_!$ and $j_!$. More precisely, for any perverse sheaf $\mathcal{F}\in\mathcal{P}(U,\Bbbk)$ there are canonical morphisms $j_!\mathcal{F}\to j_{!*}\mathcal{F}\to j_*\mathcal{F}$.

Theorem. $\mathcal{G}$ can be reconstructed from its restriction $\mathcal{G}|_{\mathfrak{g}_{rs}}$ by the $!*$ extension:

$$j_{!*}(\mathcal{G}|_{\mathfrak{g}_{rs}})\ \cong\ \mathcal{G}.$$

Proof. Without defining $j_{!*}$ we will just say that this property of $\mathcal{G}=\pi_*\Bbbk_{\widetilde{\mathfrak{g}}}$ is a consequence of the property of the map $\pi:\widetilde{\mathfrak{g}}\to\mathfrak{g}$ that it is *small*, meaning that "fibers grow slowly"

Precisely, for a map $\pi:Y\to X$ denoted by $X_{\geq k}\subseteq X$ the subspace of all $x\in X$ such that $\dim(Y_x)\geq k$, the dimension of the fiber Y_x at x is at least k. Then we say that π is *semismall* if for each $k\in\mathbb{N}$, the codimension of $X_{\geq k}\subseteq X$ is at least $2k$

$$\dim(X)-\dim(X_{\geq k})\ \geq\ 2k.$$

For instance the codimension of points x with fibers of positive dimension is at least 2. In particular the generic fibers are of dimension zero.

We say that π is *small* if it is semismall and the codimension of points with fibers of positive dimension is at least 3. □

[3] The terminology has been defined by Gabber and the the defining properties were first formulated much earlier by Kashiwara as properties of solutions of holonomic systems of linear PDEs.

[4] Though widely useful, perverse sheaves are still mysterious. Beilinson's choice of a question for this century is *What are perverse sheaves?*

Example. For $\mathfrak{g} = sl_2$ the Springer fibers are finite over all points except for 0. Here, $\mathcal{B}_e = \mathbb{P}^1$ has dimension one. SO we see that this Grothendieck map is indeed small.

Remarks. (0) In the end the *continuity mechanism* that extends the S_n-action on generic Springer fibers $\mathcal{B}_s$ for $s \in \mathfrak{g}_{rs}$ to an action on *homology* of <u>all</u> Springer fibers is given by the construction $j_{!*}$ in perverse sheaves.

(1) The proof uses no results on perverse sheaves, only the definitions.

Corollary. (a) S_n acts on the Grothendieck sheaf.

(b) For any nilpotent e, $H_{top}(\mathcal{B}_e, \mathbb{Z}) = \mathbb{Z}[Irr(\mathcal{B}_e)]$ is a representation of S_n.

Proof. (a) follows from the theorem – as S_n acts on the restriction $\mathcal{G}|_{\mathfrak{g}_{rs}} = j^*\mathcal{G}$, the functoriality of $j_{!*}$ makes it act on $j_{!*}(\mathcal{G}|_{\mathfrak{g}_{rs}}) \cong \mathcal{G}$.

(b) follows as indicated in the discussion before the theorem. □

3.5.1. *Perverse sheaves.* Functor $j_{!*}$ has been constructed for the framework of *perverse sheaves* by Beilinson-Bernstein-Deligne. The perverse sheaves are a class of complexes of sheaves which have particularly good properties so they appear to be more fundamental than the sheaves themselves.(5) The term "perverse" seems to mean that these objects are different from the accustomed norms of our knowledge (but are even more beautiful that the world we traditionally see).(6)

The perverse sheaves on a space X and with coefficients in $\mathbb{k}$-modules form an abelian category $\mathcal{P}(X, \mathbb{k})$ constructed by Beilinson-Bernstein-Deligne. However, the definition has originally been made by Mekhbout in the framework of the Riemann-Hilbert correspondence. Moreover, the defining properties were first formulated (much earlier) by Kashiwara as properties of solutions of holonomic systems of linear PDEs.

The development of the theory of perverse sheaves by Beilinson–Bernstein-Deligne originates in the discovery of *intersection cohomology* by Goresky-McPherson. The intersection cohomology is a combination of homology and cohomology which has good properties for singular spaces: it satisfies Poincare duality which was previously only known for smooth manifolds. The functor $j_{!*}$ is also called the *intersection cohomology extension*(or IC-extension).

The popularity of perverse sheaves is largely due to spectacular applications in representation theory starting with the proof of the Kazhdan-Lusztig conjecture.

5 A famous example of a perverse sheaf is the Grothendieck sheaf $\mathcal{G}$ (or more precisely, its shift $\mathcal{G}[\dim(\mathfrak{g})]$).

6 Though widely useful, perverse sheaves are still mysterious. Beilinson's choice of a question for this century is *What are perverse sheaves?*

Remarks. (0) For an open $U\overset{j}{\subseteq}X$ one has a functor $j_{!*} : \mathcal{P}(U,\Bbbk) \to \mathcal{P}(X,\Bbbk)$. So, $j_{!*}$ is a new functoriality which applies to perverse sheaves.

(1) $j_{!*}$ is "in between $j_!$ and $j_!$. More precisely, for any perverse sheaf $\mathcal{F} \in \mathcal{P}(U,\Bbbk)$ there are canonical morphisms $j_!\mathcal{F} \to j_{!*}\mathcal{F} \to j_*\mathcal{F}$.

3.6. **Irreducible representations of S_n from the Grothendieck sheaf.** Any map $\pi : X \to Y$ defines the Cartesian square $\Sigma = X\times_Y X\subseteq X^2$.

3.6.1. *Ginzburg algebra A_π of a map $\pi : X \to Y$.* Any map $\pi : X \to Y$ defines the Cartesian square $\Sigma = X\times_Y X\subseteq X^2$. (We can think of it as the square $(X \to Y)\times_Y (X \to Y)$ of X viewed as a space $X \to Y$ over Y.)

A correspondence between spaces A and B is a space $\mathcal{C}$ with maps $A \xleftarrow{p} \mathcal{C} \xrightarrow{q} B$. We will see that the Cartesian squares are a particular kind of self-correspondences that produce algebras.

For a finite set Z we denote by $\Bbbk[Z]$ the free $\Bbbk$-module with basis Z. We can think of it as $\Bbbk$-valued functions $\mathcal{O}(Z) = Map(Z,\Bbbk)$ or distributions $\mathcal{D}(Z) \overset{\text{def}}{=} \mathcal{O}(Z)^*$ on X.

Lemma. (a) For any equivalence relation $\Sigma\subseteq X^2$ on a finite set X, $\Bbbk[\Sigma]$ is an algebra for $(x,y) * (u,v) = \delta_{y,u}(x,v)$.

(b) $X\times_Y X\subseteq X^2$ is an equivalence relation on X. So, it defines an algebra $A_\pi = \Bbbk[X\times_Y X]$ associated to the map π.

Proof. (a) A pair $(x,y) \in X^2$ lie in $X\times_Y X$ iff $\pi(x) = \pi(y)$ and this is clearly an equivalence relation on X.

(b) When $\Sigma = X^2$, i.e., $x \sim y$ for any two elements of X, then $*$ is the matrix algebra $M_X(\Bbbk)$. So, it suffices to see that if the relation σ is transitive then $\Bbbk[\mathbb{S}]\subseteq\Bbbk[X^2$ is closed under the multiplication $*$, if Σ is reflexive then $\Bbbk[\Sigma]$ contains the unity in the matrix algebra. (When Σ is symmetric then the subalgebra $\Bbbk[\Sigma]\subseteq M_X(\Bbbk)$ is closed under the transpose of matrices.) $\square$

Proposition. For any map $\pi : X \to Y$ of finite sets:

1. For any $S\subseteq Y$ the part $X_Y \overset{\text{def}}{=} \pi^{-1}(Y)$ of X that lies over Y gives an A_π-module $\Bbbk[X_Y]$.
2. For any point $y \in Im(\pi)\subseteq Y$, the A_π-module $L_y \overset{\text{def}}{=} \Bbbk[X_y]$ given by the fiber $X_y = \pi^{-1}y\subseteq X$ is irreducible.
3. The map $Im(f) \ni y \mapsto L_y \in Irr(A_\pi)$ is a bijection from $f(X)$ to the set of isomorphism classes of irreducible modules for the algebra A_π.
4. Algebra A_π is a sum of matrix algebras $\oplus_{y\in f(X)} \mathrm{End}_\Bbbk(L_y)$. $\square$

Corollary. The constant sheaf $\Bbbk_X$ on X gives a sheaf $\pi_*\Bbbk_X$ of $\Bbbk$-modules on Y.

(a) Its stalk at $y \in Y$ is $\Bbbk[X_y]$.

(b) The algebra A_π associated to the map π is the endmorphism algebra of this sheaf

$$\mathrm{End}(\pi_*\Bbbk_X) \ \cong \ A_\pi.$$

3.6.2. *Beyond finite sets.* This machinery becomes particularly powerful when one considers maps $\pi : X \to Y$ of algebraic varieties. Then $\Bbbk_X \in D_c(X, \Bbbk)$ gives $\pi_*\Bbbk_X \in D_c(Y, \Bbbk)$ and we define a $\Bbbk$-algebra associated to π as

$$A_\pi \ \overset{\text{def}}{=} \ \mathrm{End}_{D_c(Y,\Bbbk)}(\pi_*\Bbbk_X).$$

One can enlarge this algebra using the fact that we are working in a triangulated category $D_c(Y, \Bbbk)$, Then $A_\pi = \mathbb{A}^0_\pi$ is a part of of the algebra

$$A^\bullet = \oplus_{n\in\mathbb{Z}} A^n_\pi \quad \text{with} \quad A^n_\pi \overset{\text{def}}{=} \mathrm{Ext}^n(\pi_*\Bbbk_X, \pi_*\Bbbk_X) \ \text{ for } \ \mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) \ \overset{\text{def}}{=} \ \mathrm{Hom}_{D_c(Y,\Bbbk)}(\mathcal{F}, \mathcal{G}[n]).$$

Its properties are then the We X is smooth and π is proper.

Lemma. The Grothendieck sheaf $\mathcal{G}$

(a) $\Bbbk[W] \xrightarrow{\cong} \mathrm{End}(\mathcal{G})$.

(b) The group algebra of S_n is the top Borel-Moore homology of the Steinberg space

$$\Bbbk[S_n] \ \cong \ H^c_{top}(St_{\mathfrak{g}}).$$

(c) When $\Bbbk \supseteq \mathbb{C}$ then

$$\mathcal{G} \ \cong \ \oplus_{\pi \in Irr(W)} \pi \otimes \mathcal{G}_{\pi^*}.$$

(d)

3.7. **Conclusion.** Interesting objects usually have geometric realizations. (For instance S_n is the symmetry group of the Grothendieck sheaf and its group algebra $\mathbb{Z}[S_n]$ is the Borel-Moore homology of the Steinberg space.)

Geometric techniques are the most powerful techniques of study of finer aspects of representation theory such as characters of irreducible representations.

3.8. **Appendix. The total homology of fibers gives induced representations.** The precise relation is that there is a geometric construction of the action of S_n on the (co)homology of Springer fibers and there is an isomorphism of representations $H_*(\mathcal{F}_{y_\lambda}) \cong Gr_F\ (\mathcal{I}nd_{S_\lambda}^{S_n}\ \tau_\lambda)$ for a certain filtration F on the induced representation.

Part 2. Semisimple Lie groups and Lie algebras

Section 4 sketches the notion of *Lie groups* and the relation to their infinitesimal incarnations, the *Lie algebras*.

The next section 5 encodes the structure of the group $SL_n(\mathbb{C})$ and its Lie algebra $sl_n(\mathbb{C})$ into the set of *roots* which one views as objects of "linear combinatorics". Based on this example in section 6 we consider the class of semisimple Lie groups and Lie algebras, their encoding into *systems of roots* and how this leads to the classification of semisimple Lie algebras.

4. Lie groups and algebras

4.1. Lie groups.

4.1.1. *Manifolds.* We considers classes of manifolds over a field $\mathbb{k}$ which for us will be $\mathbb{R}$ or $\mathbb{C}$ (but could also be $\mathbb{Q}_p$ etc). A class of manifolds over a field $\mathbb{k}$ is described by the corresponding class $\mathcal{O}$ of functions on such manifolds. We will let $\mathcal{O}$ be one of the following C^n for $n = 0, 1, 2, ..., \infty$, analytic functions or holomorphic functions.

An $\mathcal{O}$-manifold is a topological space which is locally isomorphic to $\mathbb{k}^n$ and the transition functions are in the class $\mathcal{O}$.

For such manifold M we define *vector fields* $V(M)$ on M as the derivations $Der_{\mathbb{k}}[\mathcal{O}(M)]$ of functions, i.e., linear operators $\partial : \mathcal{O}(M) \to \mathcal{O}(M)$ such that $\partial(fg) = \partial f \cdot g + f\partial g$.

A point $a \in M$ is algebraically encoded as a homomorphism of rings $ev_a : \mathcal{O}(M) \to \mathbb{k}$ given by the evaluation $ev_a(f) = f(a)$. This defines the class $Der^a_{\mathbb{k}}[\mathcal{O}(M), \mathbb{k}]$ derivations of $\mathcal{O}(M)$ with values in $\mathbb{k}$ and with respect to homomorphism ev_a, these of linear operators $\xi : \mathcal{O}(M) \to \mathbb{k}$ such that $\xi(fg) = \xi f \cdot g(a) + f(a)\xi g$. The tangent space at a is $T_aM \stackrel{\text{def}}{=} Der^a_{\mathbb{k}}[\mathcal{O}(M), \mathbb{k}]$.

Clearly, any vector field ∂ on M defines tangent vectors $\partial_a \in T_aM$ at points $a \in M$ by $\partial_a f \stackrel{\text{def}}{=} (\partial f)(a)$.

The cotangent space at a is $T^*_aM \stackrel{\text{def}}{=} \mathcal{I}_a/\mathcal{I}^2_a$ for the ideal $\mathcal{I}_a = \{f \in \mathcal{O}(M);\ f(a) = 0\}$ of the point a. The differential at a is the linear operator $d_a : \mathcal{O}(M) \to T^*_aM$ by $d_af \stackrel{\text{def}}{=} [f - f(a)] + \mathcal{I}^2_a$.

Example. If $M = \mathbb{k}^n$ then $V(M) = \sum_1^n \mathcal{O}(M)\frac{\partial}{\partial x_i}$ while $T_aM = \oplus_1^n (\frac{\partial-}{\partial x_i})_a$ and $T^*_aM = \oplus_1^n d_ax_i$.

Lemma. (a) Vector spaces T_aM form a vector bundle TM over M. We call it the *tangent vector bundle*.

(b) Vector fields $V(M)$ are exactly the sections $\Gamma(M, TM)$ of the tangent vector bundle TM.

(c) T_a^*M is dual of the vector space T_aM.

Proof. It suffices to check all these claims for $M = \Bbbk^n$. Then (a) and (b) are clear and (c) comes from the pairing $T_aM \times T_a^*M \to \Bbbk$ defined for $\xi \in T_aM$ and $f \in \mathcal{I}_a$ by by $\langle \xi, f + \mathcal{I}^2 \rangle \stackrel{\text{def}}{=} \xi(f)$. This is well defined (since $\xi = 0$ on $\mathcal{I}_a^2$) and $\langle (\frac{\partial -}{\partial x_i})_a, d_a x_j \rangle = \delta_{ij}$.. $\square$

4.1.2. *Examples.* (0) The notion of manifolds is local, i.e., the $\mathcal{O}(O)$-manifold structures on a topological space M form a sheaf.

(1) If $M \subseteq \Bbbk^n$ is given by $f_1 = \cdots = f_c = 0$ then M is a manifold provided that that the differentials $d_a f_1, ..., d_a f_c$ are independent vectors in $T_a^* \Bbbk^n$ for any $a \in M$.

(2) A particular case is when f_i's a re polynomials, then M is said to be a (smooth) algebraic variety.

Remark. $\mathcal{O}$-manifolds form a category, the homomorphisms are mappings $f : M \to Y$ which are locally, i.e., in when viewed in terms of charts on M and N, functions in class $\mathcal{O}$.

4.1.3. *Lie groups.* An $\mathcal{O}$-Lie group is a group $(G, \cdot)$ with an $\mathcal{O}$0manifold structure such that the multiplication $\cdot : M \times M \to M$ is a map of $\mathcal{O}$-manifolds.

4.1.4. *Examples.* Our examples will all be algebraic varieties.

(0) A vector space V over $\Bbbk$ is a Lie group.

(1) Group $GL_n(\Bbbk)$ is a Lie group. The manifold structure comes from $GL_n(\Bbbk)$ being open in $M_n(\Bbbk) \cong \Bbbk^{n^2}$. One can also view $GL_n(\Bbbk)$ as a closed subset of $\Bbbk^{n^2+1} \cong M_n(\mathbb{K}) \times \Bbbk$, consisting of all (g, z) such that $\det(g) \cdot z = 1$.

This makes each $GL(V)$ into a Lie group. The simplest example is the multiplicative group $G_m \stackrel{\text{def}}{=} GL_1$.

(2) The special linear group $SL(V)$ is given by $\det(g) = 1$ in $\operatorname{End}(V)$.

(3) *Symplectic groups.* A *symplectic form* on a vector space V is a bilinear form $\omega : V \times V \to \Bbbk$ which is non-degenerate and skew, i.e., $\omega(u, v) = -\omega(v, u)$. $GL(V)$ acts on symplectic forms by $(g\omega)(u, v) \stackrel{\text{def}}{=} \omega(g^{-1}u, g^{-1}v)$. The symplectic group $Sp(\omega)$ is the stabilizer $(GL(V))_\omega$ of ω in $GL(V)$. So, it consists of all linear operators $g \in GL(V)$ such that $\omega(gu, gv) = \omega(u, v)$ for $u, v \in V$.

The standard symplectic form on $\Bbbk^{2n}$ is $\omega(u, v) = \sum_1^n u_i v_{i+n} - u_{i+n} v_i$, Then $Sp(\omega)$ is denoted Sp_{2n}.

(3) *Orthogonal groups.* An *inner product* on a vector space V is a bilinear form $h : V\times V \to \mathbb{k}$ which is non-degenerate and symmetric, i.e., $\omega(u, v) = \omega(v, u)$. $GL(V)$ acts on symplectic forms by $(g\omega)(u, v) \stackrel{\text{def}}{=} \omega(g^{-1}u, g^{-1}v)$. The orthogonal group $O(h)$ is the stabilizer $(GL(V)_h$ of h in $GL(V)$.

The standard inner product on $\mathbb{k}^n$ is $h(u, v) = \sum_1^n u_iv_i$, then $O(h)$ is denoted O_n.

Remarks. (a) While $Sp(\omega)\subseteq SL(V)$, this is not true for orthogonal groups so we get *special orthogonal groups* $SO(h) \stackrel{\text{def}}{=} O(\hbar)\mathcal{SL}(\mathcal{V})$ and in particular $SO_n \stackrel{\text{def}}{=} O_n \cap SL_n$.

(b) All symplectic structures over $\mathbb{R}$ and $\mathbb{C}$ are isomorphic to the standard one. Over $\mathbb{C}$ all inner products are isomorphic to the standard one. Over $\mathbb{R}$ all inner products are isomorphic to one of the form $\sum_1^p x_iy_i - \sum_{p+1}^n x_jy_j$.

4.2. **Lie algebras.** 8 A *Lie algebra* over a $\mathbb{k}$ is a vector space $\mathfrak{g}$ together with a bilinear operation $[-, -]$ which is antisymmetric, i.e., $[y, x] = -[x, y]$ and satisfies the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Remark. This is just the infinitesimal form of the associativity property.

4.2.1. *Examples.* (0) Any associative algebras $(A, \cdot)$ gives a Lie algebra $(A, [-, -])$ for the commutator operation. The basic example is that for a vector space V, $\mathrm{End}(V)$ with the commutator bracket is the Lie algebra called $gl(V)$.

(1) A subspace $\mathfrak{h}$ of a Lie subalgebra $\mathfrak{g}$ is (said to be) a *Lie subalgebra* if it is closed under the bracket in $\mathfrak{g}$. Such $\mathfrak{h}$ is naturally a Lie algebra.

(2) For any associative $\mathbb{k}$-algebra A the set of derivations of A

$$Der_{\mathbb{k}}(A) \stackrel{\text{def}}{=} \{\alpha \in \mathrm{End}_{\mathbb{k}}(A);\ \alpha(ab) = \alpha(a)b + a\alpha(b)\}$$

is a Lie subalgebra of the associative algebra $\mathrm{End}_{\mathbb{k}}(A)$.

The basic example is for any manifold M, the vector space $V(M) \stackrel{\text{def}}{=} \mathrm{Der}_k[\mathcal{O}(M)]$ is a Lie algebra called *vector fields on* M.

(3) If a Group G acts on a manifold M then it acts on vector fields $V(M)$ and the G-invariant vector fields $V(M)^G\subseteq V(M)$ form a Lie subalgebra.

4.2.2. *Relation of Lie groups and Lie algebras.*

Proposition. (a) For any Lie group G its tangent space T_eG (at the neutral element) is canonically a Lie algebra.

(b) Any map of Lie groups $\pi : G \to G'$ differentiates to a map of Lie algebras $d_e\pi : T_eG \to T_eG'$ (denoted $d_e\pi : \mathfrak{g} \to \mathfrak{g}'$).

Proof. (a) Group G^2 acts on the set G by $(u,v)x \stackrel{\text{def}}{=} uxv^{-1}$, we say that $G{\times}1$ acts by left translations $L_u(x) = ux$ and $1{\times}G$ by right translations $R_v(x) = xv^{-1}$. These actions commute so we get an action of $G{\times}G$ on G by $(g,h)u = guh^{-1}$.(7) Therefore G acts on the Lie algebra $V(G)$ of vector fields on G in three ways L, R, C.

We define the *left invariant vector fields on* G as the invariants $V(G)^{G\times 1}$ of the left multiplication action. One has $V(G)^{G\times 1} \cong T_eG$ as one can evaluate any vector field $\partial \in V(G)$ at $e \in G$ to get $\partial_e \in T_eG$, and in the opposite direction any tangent vector $\xi \in T_eG$ extends uniquely to a left invariant vector field $\widetilde{\xi}$ such that at any $g \in G$ one has $\widetilde{x}_g = (g,1)\xi$, i.e., $\widetilde{\xi}(f)\ (g) \stackrel{\text{def}}{=} \xi((g,1)^{-1}f)$.

Now, T_eG becomes a Lie algebra via the isomorphism $V(G)^{G\times 1} \cong T_e(G)$. For $x, y \in T_eG$ the bracket is the evaluation at $e \in G$ of the bracket of left invariant vector fields $\widetilde{x}, \widetilde{y}$ (calculated in the Lie algebra of vector fields):

$$[x,y] \stackrel{\text{def}}{=} \big([\widetilde{x},\widetilde{y}]_{V(G)}\big)_e.$$

(b) is found by from the fact that for $x \in T_G$ the invariant vector fields $\widetilde{x}$ on $\mathbb{G}$ and $\widetilde{(d_e\pi)x}$ on G' are related by the differential $d\pi$ of the map π. $\square$

We define the *Lie algebra of a Lie group* G (usually denoted $\mathfrak{g}$) to be T_eG with the bracket coming from left invariant vector fields.

Example. Since $GL(V)$ is open in $\mathrm{End}(V)$ we have an isomorphism $T_eGL(V) \cong \mathrm{End}(V)$. Now one checks that the Lie algebra structure on $\mathrm{End}(V)$ that comes from $\mathrm{End}(V) \cong T_eGL(V)$ is just the commutator bracket on $\mathrm{End}(V)$.

Remark. One can also use the right multiplication action of G on G to get another Lie algebra structure on T_eG (which is now identified with $V(G)^{1\times G}$).

4.2.3. *The exponential map $exp_G : \mathfrak{g} \to G$.* The key observation is the following.

Proposition. For each $x \in T_eG$ there exists a unique homomorphism of Lie groups $\theta_x : \mathbb{R} \to G$ such that the differential $d_e\theta_x$ is x. $\square$

(7) The third action of G on G is by conjugation $C_g(u) = gug^{-1}$, this corresponds the diagonal embedding $G \hookrightarrow G{\times}G$.

Theorem. (a) There is a unique map of manifolds $exp_G : \mathfrak{g} \to G$ such that

(1) For any $x \in \mathfrak{g}$ the map $\mathbb{R} \to G$ by $s \mapsto \stackrel{\text{def}}{=} exp_G(sx)$ is a map of groups. (In particular, $exp(0) = e$.)
(2) The differential $d_0(exp_G) : T_0(\mathfrak{g}) \to T_e(G)$ is the identity on $\mathfrak{g}$. (In particular π is an isomorphism on some neighborhoods of 0 in $\mathfrak{g}$ and of e in G.)

(b) The exponential map is functorial, i.e., for any map $\pi : G \to G'$ of Lie groups one has a commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d_e\pi"] \arrow[d, "exp_G"'] & \mathfrak{g}' \arrow[d, "exp_{G'}"] \\ G \arrow[r, "\pi"] & G'. \end{tikzcd}$$

□

Proof. (a) The proposition shows that the only possibility for such $exp_G : T_eG \to G$ is $exp_G(x) \stackrel{\text{def}}{=} \theta_x(1)$. Now, one needs to check that the dependence on x is a map of manifolds.

(b) The uniqueness of θ_y implies that $\pi(\theta_x) = \theta_{d_e\pi x}$ and then (b) follows. □

Example. For $G = GL(V)$ the exponential $exp_{GL(V)} : gl(V) \to GL(V)$ is the usual exponential $exp : \text{End}(V) \to \text{End}(V)^* = GL(V)$ in the algebra $\text{End}(V)$. (because exp satisfies the characterizations of $exp_{GL(V)}$ in the theorem).

Corollary. (a) If G is connected, for any map of Lie groups $\pi : G \to G'$, the map of Lie algebras $d_e\pi : \mathfrak{g} \to \mathfrak{g}'$ determines the homomorphism π.

(b) If G' is a Lie subgroup of G then the inclusion $i : G' \to G$ differentiates to an embedding $d_ei : \mathfrak{g}' \to \mathfrak{g}$ hence $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, then $exp_{G'} : \mathfrak{g}' \to G'$ is the restriction of $exp_G : \mathfrak{g} \to G$ to $\mathfrak{g}' \subseteq \mathfrak{g}$.

Proof. (a) $d_e\pi : \mathfrak{g} \to \mathfrak{g}'$ determines π on some neighborhood V of e in G, hence also on the subgroup of G generated by V. However, this subgroup is easily seen to be the connected component of G. □

Example. Now we understand the exponential map for all Lie subgroups of groups $GL(V)$.

4.2.4. *Commutators in G and $\mathfrak{g}$.* Let us denote by $O(t^n)$ any function of t with values in a vector space such that for some $\varepsilon > 0$, function $t^{-n}O(t^n)$ is is bounded and in class $\mathcal{O}$ for $0 < t > \varepsilon$.

Proposition. For $x, y \in \mathfrak{g}$

(1)
$$exp_G(tx)exp_G(ty) \ = \ exp_G[t(x+y) \ + \ \frac{t^2}{2}[x,y] \ + \ O(t^3)].$$
(2)
$$exp_G(-tx)\ exp_G(-ty)\ exp_G(-tx)\ exp_G(-ty) \ = \ exp_G[t^2[x,y] \ + \ O(t^3)].$$
(3)
$$exp_G(tx)exp_G(ty)exp_G(-tx) \ = \ exp_G[ty \ + \ t^2[x,y] \ + \ O(t^3)].$$

Corollary. (a) $[x, y]$ is the tangent vector at $s = 0$ of the commutator function from $\mathbb{R}$ to G
$$exp_G(-\sqrt{s}x)exp_G(-\sqrt{s}y)exp_G(\sqrt{s}x)exp_G(\sqrt{s}y).$$

(b) Two Lie groups are *locally isomorphic* iff their Lie algebras are isomorphic. □

Remark. If $\Bbbk\supseteq\mathbb{R}$ the expression in the corollary is defined for $s \geq 0$. A general formulation is the [art (2) of the theorem.

Example. (i) If G is abelian then the bracket $[-,-]$ on $\mathfrak{g}$ is zero. (Any vector space V becomes a Lie algebra with the zero bracket. Such Lie algebras are called *abelian.*)

(ii) $\mathfrak{g}$ knows nothing about G outside the connected component of G. For instance if G is discrete than $\mathfrak{g} = 0$.

4.2.5. *Adjoint actions.* The conjugation action of a Lie group G on itself is called the *adjoint* action and denoted $(Adg)(u) = \ {}^g u = gug^{-1}$. Since it fixes $e \in G$, the differentiation in the u-direction at e gives a representation of G on the vector space $\mathfrak{g}$. It is called the *adjoint representation* and still denoted $(Adg)x = \ {}^g x$ for $x \in \mathfrak{g}$. As conjugation on G preserves the group structure on G, each Adg preserves the Lie algebra structure on $\mathfrak{g}$, i.e., ${}^g[x, y] = [{}^g x, {}^g y]$.

Finally, we know that the differentiation of the adjoint representation ${}^g x$ of G on $\mathfrak{g}$ yields a representation of the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}$. We will call it the adjoint representation of $\mathfrak{g}$ on itself and denote it $(ad\ x)y$ for $x, y \in \mathfrak{g}$.

Lemma. The adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ is given by the bracket in $\mathfrak{g}$:
$$(ad\ y)x = [y, x].$$

Proof. The claim is that $\frac{d}{ds}_{s=0}\ {}^{\exp_G(sy)}x$ equals $[y, x]$. This follows from the lemma 4.2.4. □

4.2.6. *Relation of* $\mathfrak{g}$ *and* G *when* G *is simply connected, i.e.,* $\pi_0(G) = 0 = \pi_1(G)$. Here, $\pi_0(G) = 0$ means that G is connected and $\pi_1(G) = 0$ means that there are non nontrivial loops in G, i.e., that any loop can be squeezed into a point.

Lemma. If G is simply connected then for any Lie group G', any map of Lie algebras $\tau : \mathfrak{g} \to \mathfrak{g}'$ defines a unique map of Lie groups $\pi : G \to G'$ with $d_e\pi = \tau$.

Proof. Let $\mathcal{U}, U$ be open neighborhoods of $0 \in \mathfrak{g}$ and $e \in G$ such that exp_G gives an isomorphism $\mathcal{U} \xrightarrow{\cong} U$.

We define $\pi : U \to G'$ so that for $g \in U$ and $x \in \mathcal{U}$ with $exp_G(x) = g$ we have $\pi(g) = exp_{G'}(\tau x)$. The subgroup generated by U is the connected component of G, i.e., all of G in our case. It remains to prove the absence of contradictions, i.e., that if $exp_G(x_1) \cdots exp_G(x_n)$ equals $exp_G(y_1) \cdots exp_G(y_m)$ then $exp_{G'}(\tau x_1) \cdots exp_{G'}(\tau x_n)$ equals $exp_{G'}(\tau y_1) \cdots exp_{G'}(\tau y_m)$. This is where we use $\pi_1(G) = 0$. $\square$

4.2.7. *Representations of Lie groups.* A representation of an $\mathcal{O}$-Lie group G on a finite dimensional vector space V over $\Bbbk$ is a homomorphism of $\mathcal{O}$-Lie groups $\pi : G \to GL(V)$.

A representation of a Lie algebra $\mathfrak{g}$ over $\Bbbk$ on a finite dimensional vector space V over $\Bbbk$ is a homomorphism of Lie algebras $\pi : \mathfrak{g} \to gl(V)$.

Lemma. (a) A representation π of a Lie group G on V differentiates to a representation $d_e\pi$ of the Lie algebra $\mathfrak{g}$ on V. This is a functor $Rep(G) \to Rep(\mathfrak{g})$

(b) If G is simply connected this is an isomorphism of categories. (This means that any representation of $\mathfrak{g}$ lifts uniquely to a representation of G and for $U, V \in Rep(G)$ the canonical inclusion $\mathrm{Hom}_G(U, V) \subseteq \mathrm{Hom}_{\mathfrak{g}}(U, V)$ is an equality.)

Proof. This is a special case of the above statements. $\square$

Remark. Any complex manifold (meaning for $\mathcal{O}$ = holomorphic functions) is in particular a real manifold (for $\mathcal{O} = C^{\mathcal{Y}}$-functions). So, any complex Lie group G is in particular a real Lie group. In particular, for a real Lie group G there is a notion of a representation of G on a complex vector space V – these are homomorphisms of real Lie groups $G \to GL_{\mathbb{C}}(V)$.

4.3. **Complexification.**

4.3.1. *Complexification: There are more things over* $\mathbb{R}$ *than over* $\mathbb{C}$. The comparison is clear on the level of algebraic varieties.

Let M be a real algebraic variety defined by polynomial equations $M \overset{\text{def}}{=} \{x \in \mathbb{R}^n;\ f_1(x) = \cdots = f_c(x) = 0\}$ (so $f_i \in \mathbb{R}[x_1, ..., x_n]$), Then its *complexification* is the set of solutions over $\mathbb{C}$

$$M_{\mathbb{C}} \overset{\text{def}}{=} \{z \in \mathbb{C}^n;\ f_1(z) = \cdots = f_c(z) = 0\}.$$

Then we say that M is a *real form* of a complex variety $M_{\mathbb{C}}$. The same for groups (i.e., $M_{\mathbb{C}}$ is then a group).

Example. (1) $GL_1(\mathbb{R}) = \mathbb{R}^*$ is clearly a real form of $GL_1(\mathbb{C})$. However S^1 is another real form since it is given inside $\mathbb{C} \cong \mathbb{R}^2$ by $x^2 +_y^2 = 1$, so its complexification $(S^1)_{\mathbb{C}}$ is given inside $\mathbb{C}^2$ by the same equations. However, one can change the coordinates on $\mathbb{C}^2$ to $u = x + iy$ and $v = x - iy$ so that the equation becomes $uv = 1$ which is $GL_1(\mathbb{C})$.

Here, $GL_1(\mathbb{R})$ is said to be a *split* real form of $GL_1(\mathbb{C})$ (meaning that it behaves very much like its complexification) while S^1 is quite different because it is compact.

(2) More generally, $GL_n(\mathbb{C})$ has a split real form $GL_n(\mathbb{R})$ and a compact real form U_n. (This is the stabilizer in $GL_n(\mathbb{C})$ of the hermitian form $\sum\ |z_i|^2$ which is the same as the matrices $u \in GL_n(\mathbb{C})$ such that $uu^* = 1$ for $u^* = (\overline{u})^{tr}$.)

(3) On $\mathbb{C}^n$ there is a single non-degenerate quadratic form $\sum\ z_i^2$ (up to isomorphism, i.e., a change of coordinates). However, on $\mathbb{R}^n$ there are $n+1$ such forms $\sum_{i=1}^p\ x_i^2 = \sum_{j=p+1}^n\ x_j^2$, distinguished by their *signature* invariants $p - q$.

Remarks. (0) In each of these cases things that are different over $\mathbb{R}$ may become isomorphic over $\mathbb{C}$.

(1) For a real group G we would first consider its complex representations and then we could try to refine this understanding by considering the real representations of G (as real forms of known complex representations).

(2) The same for the semisimple Lie algebras. Their classification over $\mathbb{C}$ is going to be the first step towards classification over $\mathbb{R}$.

4.3.2. *Complexification: Invariant subspaces.*

Lemma. Let (π, U) be a real form of a connected complex algebraic group G. For a complex vector subspace V' of a representation V of G the following is equivalent

(1) V' is G-invariant,
(2) V' is $\mathfrak{g}$-invariant,
(3) V' is $\mathfrak{u}$-invariant,
(4) V' is U-invariant.

Proof. (1)$\Rightarrow$ (2) is clear since for any $x \in \mathfrak{g}$ the linear operator $d_e\pi x$ is a derivative of the family of linear operators $\pi(exp_G(sx))$. The same argument works for (4)$\Rightarrow$ (3).

(2)$\Rightarrow$ (1) follows since G is connected. Recall that $exp_G : \mathcal{N} \xrightarrow{\cong} N$ for some neighborhoods $\mathcal{N}$ and N of 0 and e in $\mathfrak{g}$ and G. For $x \in \mathcal{N}$ we have $\pi(exp_G x) = exp_{GL(V)}(d_e\pi)x)$ which is the usual exponential $exp[d_e\pi)x]$ in linear operators. So, $\mathfrak{g}$ invariance implies $\mathcal{N}$ -invariance and then also G-invariance.

Also, (2)$\Leftrightarrow$(3) is obvious since $\mathfrak{g}$ is the complexification $\mathfrak{u}_{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$ of $\mathfrak{u}$ and $d_e\pi$ is $\mathbb{C}$-linear. Finally (1)$\Rightarrow$(4) since $G = U_{\mathbb{C}}$ contains U. $\square$

Corollary. A representation V of G is irreducible iff it is irreducible for $\mathfrak{g}$.

4.4. **Semisimple and reductive Lie groups and algebras.** Now we define semisimple and reductive Lie algebras.

4.4.1. *Semisimple Lie algebras over* $\Bbbk$. A subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is an *ideal* if $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$.

A (finite dimensional) Lie algebra $\mathfrak{g}$ is said to be *simple* if its only ideals are the trivial ones: 0 and $\mathfrak{g}$ itself.

A Lie algebra $\mathfrak{g}$ is said to be *simple* if its only ideals are the trivial ones: 0 and $\mathfrak{g}$ itself.

It is said to be *semisimple* if it is a sum of simple Lie algebras.

Remarks. (0) One can think of "semisimple" as *maximally non-abelian.*

(1) The classification of semisimple Lie algebras over complex numbers is entertaining. It is outlined later in 6.

4.4.2. *Semisimple Lie groups.* We will say that a Lie group G is semisimple if it is connected and its Lie algebra is semisimple.

To any Lie group G one can associate its Lie algebra $\mathfrak{g}$ as the tangent space T_eG at unity. The reverse direction from $\mathfrak{g}$ to G is in general more complicated. However, for a semisimple Lie algebra $\mathfrak{g}$ it is easy to find a group corresponding to $\mathfrak{g}$.

Theorem. The group $\mathrm{Aut}(\mathfrak{g}) \subseteq GL(\mathfrak{g})$ of automorphisms of the Lie algebra has Lie algebra $\mathfrak{g}$. $\square$

Remarks. (0) The point of the theorem is that the semisimple Lie algebras are "maximally non-commutative", so the Lie algebra $\mathfrak{g}$ itself is recorded in its adjoint action on the vector space $\mathfrak{g}$. For this reason we can expect that a group associated to $\mathfrak{g}$ will be recorded in its action on the vector space $\mathfrak{g}$.

One can actually understand of all connected groups with a given semisimple Lie algebra $\mathfrak{g}$.

Proposition. (a) The smallest connected Lie group with the Lie algebra $\mathfrak{g}$ is the connected component $\mathrm{Aut}_{LieAlg}(\mathfrak{g})_0$ of $\mathrm{Aut}_{LieAlg}(\mathfrak{g})$. It is called the *adjoint* group associated to G and sometimes it is denoted G_{ad}.

(b) The largest connected group G with the Lie algebra $\mathfrak{g}$ is the *universal cover* of $G_{ad} = \mathrm{Aut}_{LieAlg}(\mathfrak{g})_0$. It is called the *simply connected* group associated to G and sometimes denoted G_{sc}.

(c) The center $Z(G_{sc})$ of G_{sc} is finite (and it coincides with the fundamental group $\pi_1(G_{ad})$ of the adjoint version). All connected groups G with the Lie algebra $\mathfrak{g}$ correspond to all subgroups $\mathcal{Z}$ of $Z(G_{sc})$, a subgroup $\mathcal{Z}$ gives the group $G_{sc}/\mathcal{Z}$.

Remark. Each of connected groups G with Lie algebra $\mathfrak{g}$ has $G_{ad} = \mathrm{Aut}_{LieAlg}(\mathfrak{g})_0$ as a quotient, so G acts on $\mathfrak{g}$ via this quotient map $G \to G_{ad}$.

Example. For $\mathfrak{g} = sl_n$ the simply connected group G_{sc} associated to $\mathfrak{g}$ is SL_n. The adjoint group G_{ad} is $SL_n/Z(SL_n)$, it is isomorphic to $GL_n/Z(GL_n)$ which is called the *projective general linear group* and denoted PGL_n.

4.4.3. *Reductive semisimple Lie algebras over* $\Bbbk$. Here, "reductive" is a small generalization of "semisimple".

A Lie algebra $\mathfrak{g}$ is *reductive* if it is a direct sum of a semisimple Lie algebra $\mathfrak{s}$ and an abelian Lie algebra $\mathfrak{a}$.

Remark. Another characterization is: $\mathfrak{g}$ is reductive if its adjoint representation is *completely reducible* (this is where the name comes from).

We will say that a Lie group G is *reductive* if it is connected, the connected component of its center is a torus and its Lie algebra is semisimple.

Example. GL_n and gl_n are reductive. Group G_m is reductive but the additive group G_a is not (though their Lie algebras are isomorphic). Semisimple implies reductive.

Remarks. (0) Reductive complex groups have the nice property that their finite dimensional representations are semisimple (see 4.5.3 below).

(1) Semisimple group SL_n is closely related to the reductive group GL_n.

(2) Reductive groups are necessary for study of semisimple groups because important subgroups of a semisimple group are often reductive but not semisimple (the main example are the Levi subgroups, for instance the Cartan subgroups).

4.5. **Semisimplicity properties of complex representations.** Here we first notice that a complex representation of a compact Lie group is semisimple. Then it will imply the same property for semisimple complex Lie groups. (Of course for the additive group the 2-dimensional representation $G_a \ni x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is not semisimple.)

4.5.1. *Haar measures on locally compact groups.* A *left Haar measure* on a locally compact topological group A is a measure μ which is

- left invariant, i.e., for $g \in G$ we have $L_g\mu = \mu$, i.e., one has $\int_A f(ga)\, d\mu(a) = \int_A f(a)\, d\mu(a)$ for compactly supported continuous functions f on G; and
- μ is positive in the sense that for any open non-empty U one has $\mu(U) > 0$.

Theorem. On any locally compact topological group A there exists a left Haar measure and it is unique up to a positive multiple.

Proof. We will only consider the case when A is a Lie group (this is all we need). Let us consider it as a real Lie group. We know that any differential form of top degree $\omega \in \Omega^{top}(A)$ defines a measure μ_ω on A. So, it suffices to find an invariant $\omega \in \Omega^{top}(A) = \Gamma[A, \wedge^{top} T^*A]$, however these are in bijection with vectors in $\wedge^{top} T_e^* A]$. □

4.5.2. *Invariant inner products.*

Lemma. Let (π, V) be a representation of a compact Lie group U over $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$. If $\Bbbk = \mathbb{R}$ then there exists a U-invariant inner product h on V. If $\Bbbk = \mathbb{C}$ then there exists a U-invariant hermitian inner product h on V.

Proof. We start with any (hermitian) inner product h_0 on V and for $x, y \in V$ we define $h(x, y) \overset{\text{def}}{=} \int_U h_0(ux, uy)\, d\mu(u)$ for a left Haar measure μ on U. Then $h(vx, vy) = h(x, y)$ for $v \in U$. □

Corollary. (a) For a compact Lie group U any representation V is semisimple.

(b) If a connected complex algebraic group G has an inner form U which is compact then any representation V of G is semisimple.

Proof. (a) If h is a U-invariant (hermitian) inner product on V then for any subrepresentation $V' \subseteq V$ its orthogonal complement $(V')^\perp$ is U-invariant. So, V' has a complementary subrepresentation $(V')^\perp$.

(b) Again, let h be a U-invariant hermitian inner product on V. Then for any G-subrepresentation $V' \subseteq V$ we know that its orthogonal complement $V'' = (V')^\perp$ is a U-invariant complex vector subspace. By lemma 4.3.2 it is also G-invariant. □

4.5.3. *Compact real forms.*

Lemma. (a) Group $SL_n(\mathbb{C})$ has a compact $\mathbb{R}$-form $SU(n)$.

(b) Finite dimensional representations of G are semisimple.

Proof. (a) implies (b) by corollary 4.5.2 $\square$

Remark. Actually any semisimple complex group G has a maximal compact subgroup U and U is a real form of G. (Moreover, this is true for a larger class of *reductive complex groups.*)

4.6. **Tori.** A *torus over* $\mathbb{C}$ is a Lie group isomorphic to $(G_m)^n$, i.e., to $(\mathbb{C}^*)^n$. We will also consider the corresponding notion of *toral Lie algebras* For any torus T its Lie algebra $\mathfrak{t}$ will be toral.

Here "toral" means something like: "works like a torus". The meaning of that will be that all representations of a torus are semisimple and so are the "relevant" representation a toral Lie algebra.

So, the main observation is that any representation of a torus T is semisimple. In fact it is a sum of eigenspaces $V = \oplus_{\chi\in X^*(T)} V^T_\chi$ corresponding to characters χ of T.

We will now translate this property of T into that of its Lie algebra $\mathfrak{t}$ because it is easier to work with Lie algebras. Aa consequence we see that any representation V of T is also semisimple as a representation of its Lie algebra $\mathfrak{t}$, i.e., it is a sum of eigenspaces $V = \oplus_{\lambda\in X^*(T)} V^{\mathfrak{t}}_\lambda$ corresponding to linear functionals λ on $\mathfrak{t}$. Moreover, these are the same decompositions as $V^T_\chi = V^{\mathfrak{t}}_\lambda$ when λ is the differential $d_e\chi$ of the character χ.

This involves the embedding $X^*(T) \hookrightarrow \mathfrak{t}^*$ by taking the differential at $e \in T$. Moreover, $\mathfrak{t}^*$ will be just the $\mathbb{C}$-vector space generated by the lattice $X^*(T)$, i.e., $\mathfrak{t}^* = X^*(T)\otimes_{\mathbb{R}}\mathbb{C}$. So, with a little caution (i.e., by concentrating on the subgroup $X^*(T)$ of $\mathfrak{t}^*$, T and $\mathfrak{t}$ will work the "same" for us).

4.6.1. *The (co)character lattices* $X_*(T)$ *and* $X^*(T)$. We define the abelian groups of *characters* of T as $X^*(T) \stackrel{\text{def}}{=} \text{Hom}_{Lie}(T, G_m)$ and *cocharacters* of T as $X_*(T) = \text{Hom}_{Lie}(T, G_m)$.

Lemma. (a) An isomorphism $\iota : {G_m}^I \xrightarrow{\cong} T$ gives $X^*(T) \cong \mathbb{Z}[I] \cong X_*(T)$.

(b) $X^*(T)$ and $X_*(T)$ are naturally dual lattices by the composition pairing $X^*(T)\times X_*(T) \to \text{Hom}(G_m, G_m) \cong \mathbb{Z}$.

(c) $X^*(T) = Irr(T)$ and any representation V of a torus T is semisimple hence it is canonically of the form[8]

$$V \;\cong\; \oplus_{\chi\in X^*(T)} \, [V:\chi]\otimes\chi.$$

[8] Here one thinks of $\chi \in X^*(T)$ as a 1-dimensional representation $(\chi, \mathbb{C})$ of T. Also, $[V : \chi]$ denotes $\text{Hom}_T(\chi, V)$.

Proof. In (a) and (b) it suffices to consider the case $T = G_m$ and then $X^*(G_m)$ and $X_*(G_m)$ are both $\mathrm{Hom}_{Lie}(G_m, G_m)$. So it suffices to show that this is $\mathbb{Z}$.

Certainly $n \in \mathbb{Z}$ gives $\chi_n(z) = z^n$. The differential is $d_0\chi_n = n$.

On the other hand, we know that for any $\chi\mathrm{Hom}_{Lie}(G_m, G_m)$ the differential $d_e\chi$ is a linear operator on $Lie(G_m) = \mathbb{C}$, i.e., multiplication by some $s \in \mathbb{C}$. Since exp_{G_m} is the usual exponential in $\mathbb{C}$, for $\alpha \in Lie(G_m) = \mathbb{C}$ we have $\pi(exp(\alpha)) = exp(de_\chi\alpha) = exp(s\alpha)$. When $\alpha = 2\pi i$ this gives $1 = exp(s\alpha)$ hence $s\alpha \in \mathrm{Ker}(exp) = 2\pi i\mathbb{Z}$ and $s \in \mathbb{Z}$.

(c) Semisimplicity follows from the fact that $T \cong (\mathbb{C}^*)^n$ has a compact real form $T_c \cong (S^1)^n$. This gives the decomposition with $\chi \in Irr(T)$. However, since T is abelian, $Irr(T)$is the 1-dimensional representations, i.e., $X^*(T)$. □

4.6.2. *Toral subalgebras.* This part can be skipped as it explains how the parallel definitions in the world of Lie algebras are little bit more involved. The difference is that one is forced to use the adjoint representation.

A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is said to be *toral* if for each $s \in \mathfrak{t}$ the operator $ad_{\mathfrak{g}}s$ on $\mathfrak{g}$ is semisimple. A Lie algebra $\mathfrak{h}$ is said to be *toral* if it is toral as a subalgebra of itself, i.e., if for all its elements are $ad_{\mathfrak{h}}$-semisimple. (Clearly if $\mathfrak{h}$ is a toral subalgebra of some $\mathfrak{g}$ then it is a toral Lie algebra.)

Lemma. Toral Lie algebras are the same as abelian Lie algebras.

Proof. If $\mathfrak{h}$ is abelian then for any $x \in \mathfrak{h}$ operator $ad_{\mathfrak{t}}x$ is zero, hence it is semisimple.

Now let $\mathfrak{h}$ be toral. For $x \in \mathfrak{h}$, $ad_{\mathfrak{g}}$-semisimplicity says that $\mathfrak{g}$ is the sum of α-eigenspaces $\mathfrak{g}^x_\alpha$. So we need to see that for $\alpha \neq 0$ the α-eigenspace $\mathfrak{g}^x_\alpha$ of any $x \in \mathfrak{g}$ is 0.

If $y \neq 0$ is an eigenvector $[x, y] = \alpha y$ then $ad(y)x = -\alpha y$, hence $ad(y^2)x = 0$. The semisimplicity of $ad(y)$ then guarantees that $ad(y)x = 0$ hence $\alpha = 0$. □

4.6.3. *Comparison.* For any torus T we will see that its Lie algebra $\mathfrak{t}$ is controlled by the lattice $X_*(T)$ of cocharacters of T.

Lemma. (a) $X_*(T)$ lies inside $\mathfrak{t}$ and $\mathfrak{t} \cong X_*(T)\otimes_{\mathbb{Z}} \mathbb{C}$. Also, $X^*(T)$ lies inside $\mathfrak{t}^*$ and $\mathfrak{t}^* \cong X^*(T)\otimes_{\mathbb{Z}} \mathbb{C}$.

(b) In any representation V of T, its Lie algebra $\mathfrak{t}$ acts by semisimple operators.

Proof. (a) Any cocharacter $\eta : G_m \to T$ differentiates to $d_1\eta : Lie(G_m) = \mathbb{C} \to \mathfrak{t}$ so it gives $d_1\eta(1_{\mathbb{C}}) \in \mathfrak{t}$. Clearly, $d_1\eta(1_{\mathbb{C}})$ determines $d_1\eta$ hence also η.

Also, any character $\chi : T \to G_m$ differentiates to $d_e\chi : \mathfrak{t} \to Lie(G_m) = \mathbb{C}$ which is a linear functional on T. Checking the two isomorphisms reduces to the case $T = G_m$.

(b) The decomposition $V = \oplus_{\chi\in X^*T}\ [V : \chi]\otimes\chi$ into a sum of 1-dimensional representations of T is also valid for $\mathfrak{t}$. □

Example. If T is a torus subgroup of a complex Lie group G then its Lie algebra $\mathfrak{t}$ is a toral subalgebra of $\mathfrak{g}$. (We use claim (b) where V is the restriction of Ad_G to T.)

Remark. For a representation V of a torus T any $\chi \in X^*(T)$ defines the T-eigenspace V_χ^T consisting of all $v \in V$ such that $tv = \chi(t){\cdot}v$ for $t \in T$. (This is the subspace we have denoted above by $[V : \chi]{\otimes}\chi$.) The set $\mathcal{W}_T(V)$ of *weights* of T in V consist of all $\chi \in X^*(T)$ that appear in V, i.e., $V_\chi^T \neq 0$.

For a representation V of a Lie algebra $\mathfrak{h}$ any $\lambda \in \mathfrak{h}^*$ defines the fh-eigenspace $V_\lambda^{\mathfrak{h}}$ consisting of all $v \in V$ such that $sv = \langle \lambda, v \rangle{\cdot}v$ for $s \in \mathfrak{h}$. Again, the weights $\mathcal{W}_{\mathfrak{h}}(V) \subseteq \mathfrak{h}^*$ consist of all λ with $V_\lambda^{\mathfrak{h}} \neq 0$.

Now, notice that for a representation V of T we have $V_\chi^T = V_{d_e\chi}^{\mathfrak{t}}$ hence in particular, T-weights and $\mathfrak{t}$-weights are identified by sending χ to $d_e\chi$.

Remark. We will usually denote for any map of lie groups $\pi : G \to G'$ its differential d_π simply by π.

4.7. **Cartan subgroups.** A *Cartan subgroup* of a complex Lie group G is any maximal torus $T \subseteq G$.

A *torus over* $\mathbb{R}$ is any algebraic group $\mathcal{T}$ over $\mathbb{R}$ whose complexification is a torus over $\mathbb{C}$. Now we can also define *Cartan subgroup*s of real Lie groups G as maximal tori $T \subseteq G$.

Similarly, a *Cartan subalgebra* of a Lie algebra $\mathfrak{g}$ is defined as any maximal toral subalgebra of $\mathfrak{g}$.

Proposition. If $T \subseteq G$ is a Cartan subgroup then $\mathfrak{t} \subseteq \mathfrak{g}$ is a Cartan subalgebra. $\square$

A proof is elementary but we will be content to confirm this in examples of interest.

4.7.1. *Roots of a Cartan subgroup (subalgebra).* The *roots* $\Delta^T(\mathfrak{g})$ of a Cartan subgroup T of G are the weights of T in $\mathfrak{g}$ that are nontrivial, i.e., $\chi \neq 1$. So, $\mathcal{W}^T(\mathfrak{g}) = \Delta^T(\mathfrak{g}) \sqcup 0$.

The *roots* $\Delta^{\mathfrak{t}}(\mathfrak{g})$ of the Cartan subalgebra $\mathfrak{t}$ are the nonzero weights of $\mathfrak{t}$ in $\mathfrak{g}$. We know that $\Delta^T(\mathfrak{g}) \subseteq X^*(T)$ is identified with $\Delta^{\mathfrak{t}}(\mathfrak{g}) \subseteq \mathfrak{t}^*$ (by taking the differential at e).

We will usually think of roots on the level of Lie algebras: $\Delta = \Delta^{\mathfrak{t}}(\mathfrak{g}) \subseteq X^*(T) \subseteq \mathfrak{t}^*$. Moreover, we will consider $X^*(T)$ inside a real vector space

$$\mathbb{V} = X^*(T)_{\mathbb{R}} \stackrel{\text{def}}{=} X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}.$$

This is an $\mathbb{R}$-form of its complexification

$$\mathfrak{t}^* = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} = (X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{V} \otimes_{\mathbb{R}} \mathbb{C}.$$

It will turn out that this is naturally an Eucledean vector space and that roots have nice properties with respect to this Eucledean geometry.

Remark. We will get familiar with properties of roots in SL_n, or equivalently in the Lie algebra sl_n. The upshot is that "any computation in $\mathfrak{g}$ can be reduced to calculating with roots". It will turn out that the "same" mechanism generalizes to all semisimple Lie group over $\mathbb{C}$.[9]

4.7.2. *Borel subgroups and subalgebras.* A lie algebra $\mathfrak{a}$ is said to be *solvable* if it has a finite filtration by ideals $0 = \mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_N = \mathfrak{a}$, such that the graded pieces $\mathfrak{a}_p/\mathfrak{a}_{p-1}$ are abelian Lie algebras.

The *Borel subalgebras* of a Lie algebra $\mathfrak{g}$ are defined as *maximal solvable subalgebras.* We also define Borel subgroups as a subgroups whose Lie algebras are Borel subalgebras.

5. The structure of $SL_n(\mathbb{C})$ and $sl_n(\mathbb{C})$

5.0.1. *The structure of a semisimple Lie algebras* $\mathfrak{g}$. It can be encoded combinatorially on several levels.

First a choice of a Cartan subalgebra $\mathfrak{h}$ defines *a system of roots* $\Delta = \Delta_{\mathfrak{h}}(\mathfrak{h})$ (these are the nonzero eigenvalues of $\frac{g}{2}$ in $\mathfrak{g}$). This is a finite subset of an Euclidean vector space which remembers $\mathfrak{g}$. So, this can be thought of as incarnation of $\mathfrak{g}$ in linear combinatorics.

Next, for any choice of a *base* $\Pi \subseteq \Delta$ of a root system Δ its Cartan matrix (the normalized inner products in Π) remembers Δ. Finally, the corresponding *Dynkin graph* $\mathcal{D}$ is just a graphical representation of the Cartan matrix. So, $\mathcal{D}$ can be viewed as an incarnation of $\mathfrak{g}$ in ordinary combinatorics.

5.0.2. *Usefulness of combinatorial encodings. A. Classification of semisimple Lie algebras.* It reduces to classification of Dynkin graphs, i.e., to combinatorics.

B. Calculation. Knowing how to calculate in the Lie algebra $\mathfrak{g}$ is mostly the same as knowing how to calculate in the root system Δ.

C. Classification of irreducible representations and calculations with these. We think of these in terms of the linear combinatorics of root systems.

5.0.3. *The parallel existence of semisimple Lie algebras on several levels.* The following are equivalent manifestations of one idea

(1) A semisimple complex Lie algebra $\mathfrak{g}$.
(2) A simply connected complex semisimple Lie group G.
(3) A simply connected compact real Lie group U.
(4) A root system (Δ, V).
(5) A Dynkin graph Γ.

A small mystery: while we have good understanding of

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[9] Actually, once we go beyond the simply laced types the mechanism will become slightly more subtle.

- 1 and 2: this is just the relation of a Lie algebra and its simply connected group;
- 2 and 3: U is a maximal compact subgroup of G and G is a complexification of U;
- 1 and 4, as well as 4and 5, were described above.

we do no really understand other relations directly. For instance, I like to think of vertices of a Dynkin graph as particles and the bonds in the graph as interactions between particles. Then the question is how to find $\mathfrak{g}$ directly from this particle picture?

The following are equivalent manifestations of one idea

(1) A semisimple complex Lie algebra $\mathfrak{g}$.
(2) A simply connected complex semisimple Lie group G.
(3) A simply connected compact real Lie group U.
(4) A root system (Δ, V).
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A small mystery. We have good understanding of some of these relations:

- 1 and 2: this is just the relation of a Lie algebra and its simply connected group;
- 2 and 3: U is a maximal compact subgroup of G and G is a complexification of U;
- 1 and 4, as well as 4 and 5: were described above.

However, we do not really understand other relations directly. For instance, I like to think of vertices of a Dynkin graph as particles and the bonds in the graph as interactions between particles. Then the question is how to find $\mathfrak{g}$ directly from this particle picture?

5.0.4. *The case* $\mathfrak{g} = sl_n$. The corresponding group is $G = SL_n(\mathbb{C})$. We will see that for sl_n the roots are just a name for off diagonal positions in an $n{\times}n$ matrix. However, it turns out that they satisfy a useful formalism called *system of roots*.

5.1. **Summary: sl_n is controlled by roots.** sl_n is the standard example of the class of *semisimple* Lie algebras (which we will define later). A key feature of Lie algebras $\mathfrak{g}$ in this class is that "everything" is captured by combinatorial data called the *system of roots of* $\mathfrak{g}$.

The combinatorial data come from consider certain maximal abelian Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ called *Cartan subalgebra*. By considering $\mathfrak{g}$ as a representation over $\mathfrak{h}$ we find the finite set $\Delta{\subseteq}\mathfrak{h}^*$ of *roots* of the Lie algebra $\mathfrak{g}$. It is defined as the nonzero *weights* (i.e., eigen-functionals) of the action of $\mathfrak{h}$ on $\mathfrak{g}$. The structure of the root system on the set Δ essentially refers to *angles* between the roots and to lengths of roots. Here roots are considered as vectors in the vector space $\mathfrak{h}^*$ endowed with a certain inner product.

Each root α gives a copy $\mathfrak{s}_\alpha$ of sl_2 that lies inside $\mathfrak{g}$. The subalgebras $\mathfrak{s}_\alpha$ generate $\mathfrak{g}$, so the Lie algebra structure of $\mathfrak{g}$ is captured by the relation of Lie subalgebras $\mathfrak{s}_\alpha$. These are in turn determined by *angles* and *lengths* for the corresponding roots.

5.2. **Group $SL_n(\mathbb{C})$ and its Lie algebra sl_n.** We consider the complex Lie group $G = SL_n(\mathbb{C})$ and its Lie algebra $\mathfrak{g} = sl_n(\mathbb{C})$. (We often omit $\mathbb{C}$ from notations.)

Lemma. For $n > 1$, $SL_n(\mathbb{C})$ is simply connected.

Proof. **A.** *Reduction to $SU(n)$.* On matrices define the operation $x^* = (\overline{x})^{tr}$ (complex conjugation and transpose). This is an anti-involution, i.e., $(xy)^* = y^*x^*$. On the group $G = SL_n(\mathbb{C})$ we have an involution $\theta(x) = x^{-*}$ (meaning $(x^*)^{-1}$). This is an anti-holomorphic map and one can view it as a nonstandard action of the Galois group $\Gamma(\mathbb{C}/\mathbb{R})$ on $SL_n(\mathbb{C})$. Its fixed points are the unitary matrices, they form a compact real form $U = SU(n)$ of $G = SL_n(\mathbb{C})$.

The corresponding involution on $\mathfrak{g} = sl_n(\mathbb{C})$ is $\theta(x) = -x^*$. Its fixed points are the anti-hermitian matrices, they form the Lie algebra $\mathfrak{u} = su(n)$ of U. We have $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ for the space $\mathfrak{p}$ of hermitian matrices in $sl_n(\mathbb{C})$. The first fact we will need is that the following map is a homeomorphism:

$$\textbf{(Cartan decomposition)} \qquad U \times i\mathfrak{p} \xrightarrow{\iota} \iota(u,x) \stackrel{\text{def}}{=} u\ exp_G(x).$$

This does reduce the problem to the smaller subgroup $SU(n)$.

B. *The case of $SU(2)$.* This will follow when we identify this group with the 3-sphere S^3. Each $U(n)$ acts simply transitively on the space of all *orthonormal bases* $u_1, ..., u_n$ in $\mathbb{C}^n$. From here one finds that $SU(2)$ acts simply transitively on the set of all unit vectors in $\mathbb{C}^2$ and this is S^3.

C. *The general case of $SU(n)$.* By using the same fact as in **B** one finds that $SU(n)$ is iterated bundle of spheres S^p with $p > 2$. This suffices. ☐

Now we know that the finite dimensional representations are the same for G and $\mathfrak{g}$. The Lie algebra is a much simpler object so we will learn its structure theory and use it to find the irreducible representations.

Lie algebra $\mathfrak{g} = sl_n$ lies inside a larger Lie algebra $\mathfrak{g}_0 = gl_n$ and this will occasionally be convenient.

5.2.1. *Lie algebra sl_2.* In this fundamental special case everything will be very explicit. Lie algebra sl_2 has a standard basis e, h, f below. This is a pattern that in some sense persists for all semisimple Lie algebras $\mathfrak{g}$, however h becomes a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and the pair e, h becomes a Borel subalgebra $\mathfrak{b}$.

On the other hands copies of sl_2 will turn out to be the basic building block for all semisimple Lie algebras $\mathfrak{g}$, i.e., $\mathfrak{g}$ is in some sense glued from subalgebras isomorphic to sl_2.

Lemma. (a) The following elements of sl_2 form a basis (called the *standard basis*)

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) This basis satisfies

$$[h,e] = \ 2e, \quad [h,f] = \ -2f, \quad [e,f] = h. \quad \square$$

(c) sl_2 is the free Lie algebra on three generators e,h,f and the relations from (b).

Corollary. e and f have "symmetric" roles in sl_2. (There is a unique automorphism of the Lie algebra sl_2 that takes e,h,f to $f,-h,e$.) $\square$

5.2.2. *Cartan and Borel subalgebras of sl_n.* The following subspaces of $\mathfrak{g}_0 = gl_n$ have special names.

- Diagonal matrices $\mathfrak{h}_0 \overset{\text{def}}{=} \begin{pmatrix} * & & & & \\ & * & & & \\ & & \ddots & & \\ & & & * & \\ & & & & * \end{pmatrix}$ ("Cartan subalgebra");
- Upper triangular matrices $\mathfrak{b}_0 \overset{\text{def}}{=} \begin{pmatrix} * & * & \cdots & * \\ & * & & \vdots \\ & & \ddots & \vdots \\ & & * & * \\ & & & * \end{pmatrix}$ ("Borel subalgebra");
- Strictly upper triangular matrices $\mathfrak{n}_0 \overset{\text{def}}{=} \begin{pmatrix} 0 & * & \cdots & * \\ & 0 & & \vdots \\ & & \ddots & \vdots \\ & & 0 & * \\ & & & 0 \end{pmatrix}$

  ("the nilpotent radical of the Borel subalgebra $\mathfrak{b}_0$").

Notice that $\mathfrak{b}_0 = \mathfrak{h}_0 \oplus \mathfrak{n}_0$.

The same terminology is used for intersections with $\mathfrak{g} = sl_n$: $\mathfrak{h} = \mathfrak{h}_0 \cap \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{b} = \mathfrak{b}_0 \cap \mathfrak{g}$ is a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{n} = \mathfrak{n}_0 \cap \mathfrak{g} = \mathfrak{n}_0$ is the nilpotent radical of $\mathfrak{b}$, and we have $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$.

Lemma. (a) $\mathfrak{g}_0, \mathfrak{b}_0, \mathfrak{h}_0, \mathfrak{n}_0$ are associative algebras.

(b) $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{n}$ are Lie algebras.

(c) $\mathfrak{b}$ is really a Borel subalgebra of $\mathfrak{g}$, i.e., a maximal solvable subalgebra (the definition is in 4.7.2).

Proof. (a) Consider the lines $L_i = \Bbbk e_i$ and subspaces $F_i = L_1 \oplus \cdots \oplus L_i$ of $V = \Bbbk^n$. Then

- $\mathfrak{h}_0$ consists of all $A \in M_n(\Bbbk)$ such that $AL_i \subseteq L_i$;
- $\mathfrak{b}_0$ consists of all $A \in M_n(\Bbbk)$ such that $AF_i \subseteq F_i$;
- $\mathfrak{h}_0$ consists of all $A \in M_n(\Bbbk)$ such that $AF_i \subseteq F_{i-1}$.

(b) Now $\mathfrak{g}, \mathfrak{b}_0, \mathfrak{h}_0, \mathfrak{n}_0$ are all known to be Lie subalgebras of $\mathfrak{g}_0$. So the intersections $\mathfrak{b}, \mathfrak{h}, \mathfrak{n}$ are also Lie subalgebras. $\square$

Remarks. (0) Symmetrically, one has subalgebras $\mathfrak{b}_0^-, \mathfrak{b}^-$ of lower triangular matrices and $\mathfrak{n}_0^- = \mathfrak{n}^-$ of strictly lower triangular matrices

(1) In groups we also have analogous objects $G_0 \subseteq B_0 = N \ltimes H_0$ and $G \subseteq B = N \ltimes H$.

Example. In sl_2 we have $\mathfrak{g} = \{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\}$ with $\mathfrak{h} = \Bbbk h$, $\mathfrak{n} = \Bbbk e$, $\mathfrak{h}^- = \Bbbk f$. So, the Lie subalgebras $\mathfrak{h}, \mathfrak{n}, \mathfrak{n}^-$ of $\mathfrak{g}$ will play the role analogous to that of the basis h, e, f of sl_2.

Lemma. H is a Cartan subgroup of G and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ (and the same for G_0).

5.2.3. *Roots of sl_n.* Recall that for a representation V of $\mathfrak{g}$ and $\lambda \in \mathfrak{h}^*$, the λ-weight space in V is

$$V_\lambda \stackrel{\text{def}}{=} \{v \in V : \ hv = \ \langle \lambda, h \rangle \cdot v \ \text{ for all } h \in \mathfrak{h}\}.$$

We say that λ is a *weight* of V if $V_\lambda \neq 0$. Let $\mathcal{W}(V)$ be the set of weights in V.

Remark. A basis of $\mathfrak{h}_0^*$ is given by linear functionals ε_i^o such that $\langle \varepsilon_i^o, diag(a_1, ..., a_n) \rangle = a_i$. We denote by ε_i the restriction of ε_i^o to $\mathfrak{h} \subseteq \mathfrak{h}_0$, so $\sum \varepsilon_i = 0$ (since $\sum \varepsilon_i^o$ is the trace on $\mathfrak{h}_0$) and $\varepsilon_1, ..., \varepsilon_{n-1}$ is a basis of $\mathfrak{h}^*$.

5.3. **Roots of sl_n.** Here we use the above Cartan subgroup H of $G = SL_n$ and the corresponding Cartan Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. These produce the set of roots $\Delta = \Delta_{\mathfrak{h}}(\mathfrak{g})$ (the non-zero weights of $\mathfrak{h}$ in $\mathfrak{g}$).

We will see how Δ controls the Lie algebra structure of $\mathfrak{g}$, its action on representations. Also, any root $\alpha \in \Delta$ defines an element $\check{\alpha}$ of the Lie algebra $\mathfrak{h}$ and subalgebra $\mathfrak{s}_\alpha \cong sl_2$.

Lemma. (a) The roots of sl_n are all linear functionals $\alpha_{ij} \stackrel{\text{def}}{=} \varepsilon_i - \varepsilon_j \in \mathfrak{h}^*$ for $1 \leq i \neq j \leq n$. The corresponding weight spaces (now called *root spaces*) are (for the standard basis of matrices E_{ij})

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \Bbbk E_{ij}.$$

(b) $\mathcal{W}(sl_n) = \Delta(sl_n) \sqcup \{0\}$ and $\mathfrak{g}_0 = \mathfrak{h}$.

(c) We have $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

(d) The roots in $\mathfrak{n}$ (i.e. the roots α such that $\mathfrak{g}_\alpha \subseteq \mathfrak{n}$) are all $\varepsilon_i - \varepsilon_j$ with $i < j$. We denote these by Δ^+ and call them the *positive roots*. Then $\mathfrak{n} = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$.

5.3.1. *Roots and representations.* Here we notice how roots control the action of $\mathfrak{g}$ on representations.

Lemma. (a) For a representation V of $\mathfrak{g}$ and $\lambda \in \mathcal{W}(V)$, $\alpha \in \mathcal{W}(\mathfrak{g}) = \Delta \sqcup 0$ we have

$$\mathfrak{g}_\alpha \ V_\lambda \subseteq V_{\lambda+\alpha}.$$

(a') In particular for $\alpha, \beta \in \Delta \sqcup 0$

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}.$$

(b) For $\alpha \in \Delta$, any $x \in \mathfrak{g}_\alpha$ acts on any finite dimensional representation of $\mathfrak{g}$ as a nilpotent operator. In particular x itself is a nilpotent matrix. □

5.3.2. *Roots and commutators.* Here we will refine the lemma 5.3.1.b on how roots control the Lie algebra structure of $\mathfrak{g}$.

Lemma. (a) If $\alpha, \beta, \alpha+\beta$ are roots then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(b) If $\alpha, \beta \in \Delta$ but $\alpha+\beta \notin \Delta$ and $\beta \neq -\alpha$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.

(c) For $\alpha \in \Delta$, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a one dimensional subspace of $\mathfrak{h}$.

Proof. It is easy to see that $E_{ij}E_{pq} = \delta_{j,p}E_{iq}$, hence $[E_{ij}, E_{pq}] = \delta_{j,p}E_{iq} - \delta_{i,q}E_{pj}$,

(a) We know that $\alpha = \varepsilon_i - \varepsilon_j$ and $\beta = \varepsilon_p - \varepsilon_q$ with $i \neq j$ and $p \neq q$. Then $\alpha+\beta$ is a root (i.e., of the form $\varepsilon_r - \varepsilon_s$ with $r \neq s$), iff $j = p$ and $i \neq q$ or (symmetrically) $q = i$ and $j \neq p$. In the first case $\alpha+\beta = \alpha_{ij} + \alpha_{iq} = \varepsilon_i - \varepsilon_q = \alpha_{iq}$ and $[E_{ij}, E_{jq}] = E_{iq}$. In the second case $\alpha+\beta = \alpha_{ij} + \alpha_{pi} = \varepsilon_p - \varepsilon_j = \alpha_{pj}$ and $[E_{ij}, E_{pi}] = -E_{pj}$. So, in both cases $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(b) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ but the conditions are that $\alpha+\beta \notin \Delta \sqcup 0 = \mathcal{W}(\mathfrak{g})$ hence $\mathfrak{g}_{\alpha+\beta} = 0$

(c) $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$. □

5.3.3. *sl_2 subalgebra $\mathfrak{s}_\alpha \subseteq \mathfrak{g}$ and $\check{\alpha} \in \mathfrak{h}$ associated to a root α.* For a root $\alpha \in \Delta$ let

$$\mathfrak{s}_\alpha \stackrel{\text{def}}{=} \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}.$$

Lemma. (a) $\mathfrak{s}_\alpha$ is a Lie subalgebra.

(b) There is a Lie algebra map $\psi : sl_2 \to \mathfrak{g}$ such that $0 \neq \psi(e) \in \mathfrak{g}_\alpha$, $\psi(f) \in \mathfrak{g}_{-\alpha}$. Any such ψ gives an isomorphism $\psi : sl_2 \to \mathfrak{s}_\alpha$.

(c) The image $\psi(h)$ is independent of the choice of ψ. We denote it $\check{\alpha}$.

(d) Then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathfrak{s}_\alpha \cap \mathfrak{h}$ has basis $\check{\alpha}$.

Proof. (a) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_{\alpha+-\alpha} = \mathfrak{h}$ and $\mathfrak{h}$ preserves each $\mathfrak{g}_\phi$. Anyway, (a) follows from (b).

(b) A root $\alpha = \alpha_{ij}$, i.e., a choice of indices $i \neq j$, gives an embedding of of Lie algebras $\phi : sl_2 \hookrightarrow sl_n$ by $\phi(e) = E_{ij}$, $\phi(f) = E_{ji}$, $\phi(h) = E_{ii} - E_{jj}$.

For another choice ψ, $\psi e \in \mathfrak{g}_\alpha = \Bbbk\phi e$ we have $\psi(e) = a\phi e,\ \psi(f) = b\psi f$ for some scalars a, b. Then $\psi h = ab\phi(h)$ as

$$\psi h = \psi[e,f] = [\psi e, \psi f] = [a\phi e, b\phi f] = ab\phi[e,f]h = ab\phi(h).$$

So it remains to notice that $ab = 1$ since

$$2\psi e = \psi[h,e] = [\psi h, \psi e] = [ab\,\phi h, a\phi e] = a^2 b[\phi h, \phi e] = a^2 b\,\phi[h,e] = a^2 b\,2\phi e = ab\,2\psi e.$$

Finally, $\mathfrak{s}_\alpha \cap \mathfrak{h} = (\mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha})\mathfrak{h} = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \Bbbk[\phi e, \phi f] = \Bbbk\phi h = \Bbbk\check{\alpha}$ for $\alpha = \alpha_{ij}$. $\square$

Remark. $\check{\alpha_{ij}} = E_{ii} - E_{jj}$ was noticed in the proof of the lemma.

5.3.4. *Real form $\mathfrak{h}^*_{\mathbb{R}}$ of $\mathfrak{h}^*$, root lattice Q and the positive cone Q_+.* Recall that $\mathfrak{h}_0^*$ has a basis ε_i dual to the basis E_{ii} of $\mathfrak{h}_0$ $(1 \le i \le n)$. Their restrictions to $\mathfrak{h}$ are linear functionals $\varepsilon_i = \varepsilon_i^o|_{\mathfrak{h}}$ on $\mathfrak{h}$ with $\sum_1^n \varepsilon_i = 0$. While ε_i for $1 \le i < n$ is a basis of $\mathfrak{h}^*$, we will actually use another basis of $\mathfrak{h}^*$ given by the *simple* roots

$$\Pi \overset{\text{def}}{=} \{\alpha_i \overset{\text{def}}{=} \alpha_{i,i+1} = \varepsilon_i - \varepsilon_{i+1};\ i = 1, ...n-1\}.$$

Lemma. Π is a basis of $\mathfrak{h}^*$.

Proof. We use the relation $-\varepsilon_n = \sum_{i<n} \varepsilon_i = \sum_{i<n} \alpha_{in} + \varepsilon_n$. Solving for ε_n we get that ε_n lies in $\text{span}_{\mathbb{Q}}\Delta$.

However, $\text{span}_{\mathbb{Q}}\Delta = \text{span}_{\mathbb{Q}}\Pi$ since $\Delta^+ \subseteq \text{span}_{\mathbb{N}}\Pi$ (for $i < j$ one has $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$), hence $\Delta \subseteq \text{span}_{\mathbb{Z}}\Pi$. Now $\text{span}_{\mathbb{Q}} + \Pi = \text{span}_{\mathbb{Q}}\Delta$ contains ε_n, hence also all $\varepsilon_i = \varepsilon_n + \alpha_{ni}$. $\square$

Now, inside $\mathfrak{h}^*$ we define

- the real vector subspace $\mathfrak{h}^*_{\mathbb{R}} \overset{\text{def}}{=} \oplus_1^{n-1} \mathbb{R}\alpha_i$ generated by simple roots,
- the subgroup $Q \overset{\text{def}}{=} \oplus_1^{n-1} \mathbb{Z}\alpha_i$ generated by simple roots,
- the semigroup $Q_+ \overset{\text{def}}{=} \oplus_1^{n-1} \mathbb{N}\alpha_i$ generated by simple roots.

We have $\mathfrak{h}^* \supseteq \mathfrak{h}^*_{\mathbb{R}} \supseteq Q \supseteq Q_+$. By the preceding proof we know that $Q = \text{span}_{\mathbb{Z}}\Delta$ so we call it the *root lattice* and $Q_+ = \text{span}_{\mathbb{N}}\Delta^+$ so we call it the *positive cone*.

Lemma. For $\lambda, \mu \in \mathfrak{h}^*$ let $\lambda \le \mu$ mean that $\mu - \lambda \in Q_+$. This is a partial order on $\mathfrak{h}^*$. $\square$

5.3.5. *The inner product on $\mathfrak{h}^*_{\mathbb{R}}$.* We will first define it by a formula and then we will deduce it from an obvious inner product on $\mathfrak{h}^*_{0,\mathbb{R}}$.

Lemma. (a) On $\mathfrak{h}_{\mathbb{R}}^*$ there is a unique inner product such that (α_i, α_j) is

- 2 if $i = j$,
- -1 if i, j are neighbors, i.e., $|j - i| = 1$,
- 0 otherwise.

(b) The inner products of roots are $(\alpha_{ij}, \alpha_{pq}) = \ \delta_{i,p} - \delta_{i,q} - \delta_{j,p} + \delta_{j,q}$. In more details

- $(\alpha_{ij}, \alpha_{pq}) = 0$ when $\{i.j\}$ and $\{p, q\}$ are disjoint.
- $(\alpha_{ij}, \alpha_{iq}) = 1$ when $q \notin \{i, j\}$;
- $(\alpha_{ij}, \alpha_{ij}) = 2$.

Proof. (a) We can embed the vector space $\mathfrak{h}_{\mathbb{R}}^*$ into $\mathfrak{h}_{0,\mathbb{R}}^* \stackrel{\text{def}}{=} \oplus_1^n \mathbb{R}\varepsilon_i^o$ so that α_i goes to $\varepsilon_i^o - \varepsilon_{i+1}^o$. Then point is that on $\mathfrak{h}_{0,\mathbb{R}}^*$ we have an obvious inner product $(-,-)$ with orthonormal basis ε_i^o. It restricts to an inner product on $\mathfrak{h}_{\mathbb{R}}^*$ such that

$$(\alpha_{ij}, \alpha_{pq}) = \ \delta_{i,p} - \delta_{i,q} - \delta_{j,p} + \delta_{j,q}.$$

Now all formulas in (b) are clear. □

5.3.6. *Lie algebra structure in therms of angles between roots.*

Corollary. (a) All roots $\alpha \in \Delta(sl_n)$ have the same length $(= \sqrt{2})$.

(b) All possibilities for the angle θ between two roots α, β in $\Delta(sl_n)$ are

(1) $\theta = 2\pi/3$ iff $\alpha + \beta$ is a root;
(2) $\theta = \pi/3$ iff $\alpha - \beta$ is a root;
(3) $\theta = \pi/2$ iff neither of $\alpha \pm \beta$ is a root and $\beta \neq \pm\alpha$.
(4) $\theta = 0$ iff $\beta = \alpha$;
(5) $\theta = \pi$ iff $\beta = -\alpha$.

(c) For $\beta \neq \pm\alpha$ the following are equivalent (i) $\theta = 2\pi/3$; (ii) $(\alpha, \beta) = -1$; (iii) $\alpha + \beta$ is a root; (iv) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$; (v) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. The cosine of the angle between α, β is $\frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}} = \ \frac{1}{2}(\alpha, \beta)$.

(1) If $\alpha + \beta$ is a root then the pair α, β (or β, α) equals α_{ij}, α_{jk} for some distinct i, j, k. Then $(\alpha, \beta) = -1$ and the cosine is $-\frac{1}{2}$.

(2) If $\alpha - \beta$ is a root then α, β (or $-\alpha, -\beta$) are of the form α_{ij}, α_{kj} for distinct i, j, k. Then $(\alpha, \beta) = 1$ and the cosine is $\frac{1}{2}$.

(3) If neither of $\alpha \pm \beta$ is a root and $\beta \neq \pm\alpha$ then our roots are of the form α_{ij}, α_{pq} for disjoint i, j and p, q. Then $(\alpha, \beta) = 0$.

(4-5) Clearly if $\beta = \alpha$ then $\theta = 0$ and $\beta = -\alpha$ gives $\theta = \pi$.

By now we have proved implications in (1-5) (from RHS to LHS). This implies equivalences. $\square$

6. **Classification of semisimple Lie algebras**

6.1. **Intro.**

6.1.1. *The structure of a semisimple Lie algebras* $\mathfrak{g}$. It can be encoded combinatorially on several levels.

First a choice of a Cartan subalgebra $\mathfrak{h}$ defines *a system of roots* $\Delta = \Delta_{\mathfrak{h}}(\mathfrak{h})$ (these are the nonzero eigenvalues of $\frac{g}{2}$ in $\mathfrak{g}$). This is a finite subset of an Euclidean vector space which remembers $\mathfrak{g}$. So, this can be thought of as incarnation of $\mathfrak{g}$ in linear combinatorics.

Next, for any choice of a *base* $\Pi \subseteq \Delta$ of a root system Δ its Cartan matrix (the normalized inner products in Π) remembers Δ. Finally, the corresponding *Dynkin graph* $\mathcal{D}$ is just a graphical representation of the Cartan matrix. So, $\mathcal{D}$ can be viewed as an incarnation of $\mathfrak{g}$ in ordinary combinatorics.

6.1.2. *Usefulness of combinatorial encodings. A. Classification of semisimple Lie algebras.* It reduces to classification of Dynkin graphs, i.e., to combinatorics.

B. Calculation. Knowing how to calculate in the Lie algebra $\mathfrak{g}$ is mostly the same as knowing how to calculate in the root system Δ.

C. Classification of irreducible representations and calculations with these. We think of these in terms of the linear combinatorics of root systems.

6.1.3. *The parallel existence of semisimple Lie algebras on several levels.* The following are equivalent manifestations of one idea

(1) A semisimple complex Lie algebra $\mathfrak{g}$.
(2) A simply connected complex semisimple Lie group G.
(3) A simply connected compact real Lie group U.
(4) A root system (Δ, V).
(5) A Dynkin graph Γ.

A small mystery: while we have good understanding of

- 1 and 2: this is just the relation of a Lie algebra and its simply connected group;
- 2 and 3: U is a maximal compact subgroup of G and G is a complexification of U;
- 1 and 4, as well as 4and 5, were described above.

we do no really understand other relations directly. For instance, I like to think of vertices of a Dynkin graph as particles and the bonds in the graph as interactions between particles. Then the question is how to find $\mathfrak{g}$ directly from this particle picture?

The following are equivalent manifestations of one idea

(1) A semisimple complex Lie algebra $\mathfrak{g}$.
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A small mystery. We have good understanding of some of these relations:

- 1 and 2: this is just the relation of a Lie algebra and its simply connected group;
- 2 and 3: U is a maximal compact subgroup of G and G is a complexification of U;
- 1 and 4, as well as 4 and 5: were described above.

However, we do not really understand other relations directly. For instance, I like to think of vertices of a Dynkin graph as particles and the bonds in the graph as interactions between particles. Then the question is how to find $\mathfrak{g}$ directly from this particle picture?

6.1.4. *The case* $\mathfrak{g} = sl_n$. The corresponding group is $G = SL_n(\mathbb{C})$. We will see that for sl_n the roots are just a name for off diagonal positions in an $n \times n$ matrix. However, it turns out that they satisfy a useful formalism called *system of roots*.

6.2. **Root systems.** Here we consider the abstract notion of *root systems*. This is a combinatorial equivalent of the notion of semisimple Le algebras over $\mathbb{C}$. However, we will only consider this relation in the case of sl_n (lemma 6.2.2). The general statement is:

Theorem. For a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$, the set of roots $\Delta \subseteq \mathfrak{h}^*_{\mathbb{R}}$ is a root system.

6.2.1. *Reflections.* A reflection in vector space V requires a pair $v \in V$, $v^* \in V^*$, however if V is an Euclidean real vector space than a single vector $\in V$ will suffice. We present both points of view since both will appear in applications to Lie algebras and root systems.

For a vector space V a pair of a vector $v \in V$ and a "covector" $v^* \in V^*$ such that $\langle v^*, v \rangle = 2$ defines a liner map $s_{v,v^*} : V \to V$ by $s_{v,v^*} x \stackrel{\text{def}}{=} x - \langle v^*, x \rangle v$.

Lemma. (a) $s_{v^*,v}$ is identity on the hyperplane $(v^*)^\perp \subseteq V$ and $s_{v,v^*} = -1$ on $\Bbbk v$.

(b) $s^2_{v,v^*} = id_V$. *Proof.* In (a), for $x = v$ we have $s_{v^*,v} v = v - \langle v^*, v \rangle v = v - 2v = -v$. (b) follows. $\square$

We say that s_{v,v^*} is a *reflection* in the hyperplane $(v^*)^\perp$.

Remarks. (0) We say that s_{v,v^*} is a *reflection* in the hyperplane $(v^*)^\perp$.

(1) [Orthogonal reflections in Euclidean vector spaces.] If V is a real vector space with an inner product then a non-zero vector $\alpha \in V$ defines a vector and an operator s_α on V by

$$\check{\alpha} \stackrel{\text{def}}{=} \frac{2}{(\alpha,\alpha)}\alpha \ \in V s_\alpha x \ = \ x - (\check{\alpha}, x)\alpha \ = \ x - 2\frac{(\alpha,x)}{(\alpha,\alpha)}\alpha.$$

For the linear functional $\check{\alpha}^* \stackrel{\text{def}}{=} (\check{\alpha}, -) \in V^*$ we have $\langle \check{\alpha}^*, \alpha\rangle = (\check{\alpha}, \alpha) = 2.$[10]

(2) Notice that the s_α is *orthogonal*, i.e., it preserves the inner product on V. (Because $V = \mathbb{R}\alpha \oplus H_\alpha$ is an orthogonal decomposition and s_α is ± 1 on summands.) Actually, s_α is the unique orthogonal reflection in H_α.

6.2.2. *Root systems.* A *root system* in a real vector space V with an inner product is a finite subset $\Sigma \subseteq V - 0$ such that

- For each $\alpha \in \Sigma$, reflection s_α preserves Σ.
- For $\alpha, \beta \in \Sigma$, $\langle \alpha, \check{\beta}\rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.
- Σ spans V.

We say that a root system is *reduced* if $\alpha \in \Sigma$ implies that $2\alpha \notin \Sigma$. The non-reduced root systems appear in more complicated representation theories. When we say *root system* we will mean a reduced root system.[11]

The *rank* r of the root system Δ is defined as $\dim(V)$.

The sum of two root systems (V_i, Δ_i) is $(V_1 \oplus V_2, \Delta_1 \sqcup \Delta_2)$. We say that a root system (V, Δ) is *irreducible* if it is not a sum.

Lemma. (a) Roots Δ of sl_n form an irreducible reduced root system in the Euclidean space $\mathfrak{h}^*_{\mathbb{R}}$.

(b) $\check{\alpha} = \alpha$ for each root.

Proof. Most of the properties are clear form the list α_{ij} of roots. For instance there are finitely many roots and none is zero. Already the roots α_i span $\mathfrak{h}^*_{\mathbb{R}}$.

For each $\alpha \in \Delta$, we have $\check{\alpha} = \frac{2}{(\alpha,\alpha)}\alpha = \alpha$. Therefore, for $\alpha, \beta \in \Delta$ we have $(\alpha, \check{\beta}) = (\alpha, \beta)$ which is one of $0, \pm 1$ so it is an integer.

Finally, to see that reflections s_α preserve Δ we consider $s_\alpha\beta = \beta - (\check{\alpha}, \beta)\alpha = \beta - (\alpha,\beta)\alpha$. If $\beta \perp \alpha$ this is $\beta \in \Delta$. If $\alpha \pm \beta$ is a root then $(\alpha, \beta) = \mp$, hence $s_\alpha\beta = \beta - \mp\alpha = \beta \pm \alpha \in \Delta$. $\square$

[10] The inner product identifies V and V^* by $v \mapsto (v, -)$. This identifies $\check{\alpha} \in V$ with $\check{\alpha}^* \in V^*$.

[11] Non-reduced root systems appear in more complicated representation theories. For instance for the non-split real groups.

6.2.3. *Positive roots and bases.* In the study of representations of a semisimple Lie algebra $\mathfrak{g}$ it is essential to choose a "direction" in $\mathfrak{g}$. The precise meaning of that is a choice of a Borel subalgebra $\mathfrak{b}\subseteq\mathfrak{g}$.[(12)] On the level of roots this is described as a *system of positive roots* or a *base of a root system.* We recall these definitions, the meaning in sl_n and the equivalence of these two notions.

For a root system Σ a subset $\Sigma^+\subseteq\Sigma$ is called a *system of positive rots* if $\Sigma = \Sigma^+ \sqcup -\Sigma^+$ and Σ^+ is closed under addition within Σ, i.e.,

- If $\alpha,\beta\in\Sigma^+$ and $\alpha+\beta\in\Sigma$ then $\alpha+\beta\in\Sigma^+$.

Then $\Delta^\pm = \Delta\cap \pm span_{\mathbb{N}}(\Pi)$ are called the *positive* and *negative* roots. We often write "$\alpha > 0$" for "$\alpha\in\Delta^+$".

The existence and construction of systems of positive roots is given by

6.2.4. *Sublemma.* (a) We say that $\gamma\in V$ is *regular* if $(\gamma,\alpha)\neq 0$ for $\alpha\in\Delta$ (i.e., if it lies in one of the chambers in V). For any regular γ

$$\Delta^+_\gamma \overset{\text{def}}{=} \{\alpha\in\Delta; (\gamma,\alpha)>0\}$$

is a system of positive roots.

(b) In particular, any root system has a base.

A *base* of a root system Δ is a subset $\Pi\subseteq\Delta$ such that Π is an $\mathbb{R}$-basis of V and

$$\Delta \subseteq span_{\mathbb{N}}(\Pi) \sqcup -span_{\mathbb{N}}(\Pi).$$

Once a basis Π is chosen its elements are called *simple roots* .

Lemma. (a) For the sl_n root system $(\mathbb{V},\Delta)$, the roots $\Delta_{\mathfrak{h}}(\mathfrak{n})$ in the Lie subalgebra $\mathfrak{n}$ (the strictly upper triangular matrices) are

$$\Delta^+ \overset{\text{def}}{=} \Delta_{\mathfrak{h}}(\mathfrak{n}) = \{\alpha_{ij};\ i<j\}.$$

This is a system of positive roots.

(b) For $\alpha_i = \alpha_{i,i+1}$, the subset $\Pi = \{\alpha_1,...,\alpha_{n-1}$ is the set of simple roots for Δ^+.

Proof. The roots in either $\mathfrak{n}$ or $\mathfrak{b}$ are all α_{ij} with $i<j$. Since $-\alpha_{ij}=\alpha_{ji}$, $-\Delta(\mathfrak{n})$ is given by the condition $j<i$, This makes $\Sigma = \Sigma^+\sqcup -\Sigma^+$ clear. If $\alpha,\beta,\alpha+\beta\in\Delta$ then (after possibly exchanging the order of α and β), we have $\alpha=\alpha_{ij},\ \beta=\alpha_{jk}$. If $\alpha,\beta\in\Sigma^+$ then $i<j$ and $j<k$, hence $\alpha+\beta=\alpha_{ik}$ with $i<k$. □

Corollary. $\mathfrak{g}_\alpha$ with $\alpha\in\Pi$ generate the Lie subalgebra $\mathfrak{n}$.

The above lemma is an example of the following.

[12] In general, Borel subalgebras are defined as maximal solvable subalgebras: 4.7.2.

Proposition. (a) A system of positive roots Δ^+ in Δ, defines a partial order $x \leq y$ in V by $y - x \in \text{span}_{\mathbb{N}}(\Delta^+)$. The the subset $\Pi \subseteq \Delta^+$ consisting of all minimal elements of Δ^+ is a basis of the root system Δ.

(b) If Π is a basis of the root system Δ then the $\Delta^+ \stackrel{\text{def}}{=} \text{span}_{\mathbb{N}}(\Pi) \cap \Delta$ is a system of positive roots.

(c) The two procedures in (a) and (b) give inverse bijections between all systems of positive roots and all bases of Δ. □

6.2.5. *Systems of positive roots and Borel subalgebras.* The following explains the meaning of systems of positive roots.

Lemma. Let $(\Delta, \mathfrak{h}^*_{\mathbb{R}})$ be the root system of a semisimple Lie algebra $\mathfrak{g}$ with a Cartan $\mathfrak{h}$. Then the Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ that contain $\mathfrak{h}$ are in bijection with the systems of positive roots $\Delta^+ \subseteq \Delta$.

Proof. The basic observation is that the Lie subalgebras $\mathfrak{c}$ of $\mathfrak{g}$ that contain $\mathfrak{h}$ correspond to subsets C of Δthat are closed for addition in the sense that $\alpha, \beta \in C$ and $\alpha + \beta \in \Delta$ implies that $\alpha + \beta \in C$. The relation is by $\mathfrak{c} = \frac{g}{2} \oplus \oplus_{\alpha \in\ C}\ \mathfrak{g}_\alpha$.

Now the claim of the lemma follows easily. □

Corollary. In this case the Weyl group also acts simply transitively on all Borel subalgebras of $\mathfrak{g}$ that contain $\mathfrak{h}$. □

6.2.6. *Chambers in V.* For a root system Δ in V, a *chamber* in V is a connected component of $V - \cup_{\alpha \in \Delta}\ H_\alpha$ where $H_\alpha = \alpha^\perp$ is the hyperplane orthogonal to vector α.

Lemma. For a root system (V, Δ) the systems of positive roots and chambers are "dual" notions in the sense that.

(a) A choice of system of positive roots Δ^+ defines a chamber by

$$\mathcal{C} \stackrel{\text{def}}{=} \{v \in V;\ (\alpha, v) > 0 \ \text{ for } \ \alpha \in \Delta^+\}.$$

(b) Any chamber $\mathcal{C}$ defines a system of positive roots by

$$\Delta^+ \stackrel{\text{def}}{=} \{\alpha \in \Delta;\ (\alpha, v) > 0 \ \text{ for } \ v \in \mathcal{C}\}..$$

(c) These are inverse bijections. □

6.2.7. *The* Weyl group W *of the root system* Δ. This is the subgroup of $GL(V)$ generated by the reflections s_α for $\alpha \in \Delta$.

Lemma. (a) W preserves Δ.

(b) W is finite. □

Corollary. For $G = SL_n$ the Weyl group is S_n.

Proof. For $\alpha = \alpha_{ij}$ the reflection s_α acts on the generators ε_i of $\mathfrak{h}^*$ as the transposition s_{ij}. So, W is the group generated by transpositions, i.e., S_n. □

Theorem. The Weyl group W acts simply transitively on each of these three sets of objects.

- (i) systems of positive roots Δ^+ in Δ;
- (ii) bases Π of Δ;
- (iii) chambers in V.

These actions are compatible with the bijections between (i) and (ii) given in proposition 6.2.3 and between (i) and (iii) given in lemma 6.2.6. □

Remarks. (0)

Remark. One consequence is that *all* bases of Δ behave the same so it suffices to consider one.

(1) It is quite helpful to draw all chambers in sl_2 and sl_3 and see how S_2 and S_3 act simply transitively on chambers.

(2) In sl_n the standard chamber is given by all $\sum\ c_i\varepsilon^i$ such that $c_1 < \cdots < c_n$. By reordering the set $1, ..., n$ we get all chambers.

Proposition. Consider a root system $(\Delta, \mathbb{V})$ that comes from a semisimple complex group G and its Cartan subgroup H. Then the Weyl group $W(\Delta, \mathbb{V})$ coincides with the Weyl group $W(G, H)$ defined as $N_G(H)/H$ (the quotient of the normalizer of the Cartan by the Cartan itself).

Here, the group $N_G(H)/H$ acts on H by conjugation since H is commutative. Then it also acts on $X^*(H)$ and it preserves $\Delta\subseteq X^*(H)$. The claim is that inside $GL(\mathfrak{h}^*)$ it coincides with the Weyl group of the root system.

Example. In the case of $G = GL_n$ we observed that $W(G, H)$ is S_n since $N_G(H)$ is the semidirect product $H \ltimes S_n$. defined as $N_G(H)/H$ (3.1.4). For $G = SL_n$ the normalizer of H does not contain S_n however, for any root $\alpha \in \Delta$ there is an element $\dot{s}_\alpha$ in SL_n such that it normalizes H and acts on $\mathfrak{h}^*$ as the reflection s_α. In this way we still get that $N_G(H)/H$ realizes all reflections $s_\alpha,\ \alpha \in \Delta$; hence the whole Weyl group $W(\Delta, \mathbb{V}) = S_n$.

Proof. If we are in SL_2 then such element is given by $\dot{s}_\alpha = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Notice that it has order 4 and its square is the element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ that generates the center of SL_2. The claim follows for any root α in SL_n since the root defines an embedding of SL_2 in S_n. □

6.2.8. *Bases and Dynkin diagrams.* To a base Π of a root system (V, Δ) we associate its *Cartan matrix* $C : \pi^2 \to \mathbb{Z}$ defined by

$$C_{\alpha\beta} \stackrel{\text{def}}{=} (\alpha, \check{\beta}).$$

We also encode it as the *Dynkin diagram* of Π. It is a graph whose vertices are given by the set Π of simple roots. If $|\alpha| \geq |\beta|$ we connect α to β with $|(\alpha, \check{\beta})|$ bonds. If $|\lambda| > |be|$ we also put an arrow from α to β over these bonds.

Lemma. Draw the Dynkin diagram for sl_n is $1 - -2 - -\cdots\_n - 1$. (It is called A_{n-1}.)

6.2.9. *Duality operation for root systems and Langlands duality.*

Lemma. For a root system Δ in V the subset

$$\check{\Delta} \stackrel{\text{def}}{=} \{\check{\alpha};\ \alpha \in \Delta\}$$

is also a root system in V. It is called the *dual* root system. □

Remark. We have seen that for sl_n we have $\check{\Delta} = \Delta$. This is true for all *simply laced* Lie algebras.

Remark. (a) If (Δ, V) corresponds to a semisimple Lie algebra $\mathfrak{g}$ then $(\check{\Delta}, V)$ corresponds to a semisimple Lie algebra called the *Langlands dual* $\check{\mathfrak{g}}$ of $\mathfrak{g}$.

(b) One of the key efforts in mathematics is to understand the *Langlands conjectures* these conjectures in Number Theory have representation theoretic formulation. In the simplest case when the groups involved are abelian the Langlands conjectures restate the Class Field Theory a key part of Number Theory.

They predict – in a very precise and often involved way – how representation theory of a group G with Lie algebra $\mathfrak{g}$ is easily understood from the *Langlands dual* group $\check{G}$ with the Lie algebra $\check{\mathfrak{g}}$.

(c) In this sense the simplicity of identification $\check{\Delta} = \Delta$ for *simply laced* Lie algebras is misleading. If $G = SL_n$ then $\check{G}$ is different, it is actually isomorphic to $PGL_n \stackrel{\text{def}}{=} GL_n/G_m$. Moreover, there is no way to construct $\check{G}$ from G without Geometric Representation Theory.

(d) Related conjectures in physics are called *dualities* (S-duality, mirror symmetry, symplectic duality[13]... Each of these says that two seemingly unrelated objects contain the same information. We do not understand a single one of these dualities.

A huge amount of details are known.

[13] This one was proposed by Braden and his coauthors.

6.3. **Appendix, Classification of root systems.** We are not particularly interested in the complete list of root systems but in special examples. From this point of view the following facts are not of our central interest.

6.3.1. *Bases and Dynkin diagrams.*

Lemma. For $\alpha \neq \beta$ in a base Π we have

(a) $(\alpha, \beta) \leq 0$, i.e., the angle is $\geq \pi/2$;

(b) $\alpha - \beta$ is not a root. $\square$

To base Π we associate its *Cartan matrix* $C : \pi^2 \to \mathbb{Z}$ defined by

$$C_{\alpha\beta} \stackrel{\text{def}}{=} (\alpha, \check{\beta}).$$

We also encode it as the *Dynkin diagram* of Π. It is a graph whose vertices are given by the set Π of simple roots. If $|\lambda| \geq |be|$ we connect α to β with $|(\alpha, \check{\beta})|$ bonds. If $|\lambda| > |be|$ we also put an arrow from α to β over these bonds.

Notice that if α and β are not connected in the Dynkin diagram iff $\alpha \perp \beta$. For the reason the Dynkin diagram of a sum of root systems Δ_i is a disjoint union of Dynkin diagrams of Δ_i's,

Theorem. (a) A root system Δ is completely determined by its Dynkin diagram.

(b) A root system Δ is irreducible iff its Dynkin diagram is connected.

(c) The irreducible root systems fall into 4 infinite series called A_n, B_n, C_n, D_n for $n = 1, 2....$ and 5 more ("exceptional") root systems called E_6, E_7, E_8, F_4, G_2.

6.3.2. *Classical series of root systems.* The following are all infinite series of irreducible root systems.

Let Here $E = \oplus_1^n \mathbb{R}\varepsilon_i$ for orthonormal ε_i. **Type A.** Here, $V = \sum c_i\varepsilon_i$ with $\sum c_i = 0$. The roots are all $\pm\varepsilon_i \pm \varepsilon_j$ where $1 \leq i < j \leq n$. The rank is $n-1$ and root system is called A_{n-1}. We have see that these are roots of the Lie algebra sl_n with respect to the diagonal Cartan $\mathfrak{h}$.

Notice that $\check{\Delta} = \Delta$ as $\check{\alpha} = \alpha$ for each root α.

Type B. Here $V = E$ and Δ consists of all $\pm\varepsilon_i$ and $\pm\varepsilon_i \pm \varepsilon_j$ for $i < j$.

Type C. Here $V = E$ and Δ consists of all $\pm 2\varepsilon_i$ and $\pm\varepsilon_i \pm \varepsilon_j$ for $i < j$.

Type D. ...

6.4. **Classification of semisimple Lie algebras.**

Theorem. (a) For a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$, the set of roots $\Delta\subseteq\mathfrak{h}^*_{\mathbb{R}}$ is a root system.

(b) The root system of $\mathfrak{g}$ determines $\mathfrak{g}$ up to an isomorphism.

(c) Any root system comes from a semisimple Lie algebra.

Remarks. (i) The root system of a sum $\oplus\ \mathfrak{g}_i$ of semisimple Lie algebras $\mathfrak{g}_i$ is the sum of root systems of the summands $\mathfrak{g}_i$. In particular, the Dynkin diagram of $\oplus\ \mathfrak{g}_i$ is a disjoint union of Dynkin diagrams of $\mathfrak{g}_i$'s.

(ii) A semisimple Lie algebra is simple iff its root system is irreducible, i.e., iff its Dynkin diagram is connected.

Corollary. The semisimple Lie algebras over $\Bbbk = \mathbb{C}$ are classified the same as root systems or Dynkin diagrams.

Remarks. (0) Each root $\alpha \in \Delta$ encodes an sl_2-subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}$. The geometry of the root system gives all information on how the sl_2-subalgebras $\mathfrak{s}_\alpha$ are related and how to reconstruct $\mathfrak{g}$ from these subalgebras.

6.4.1.

Theorem. Simple Lie algebras over $\mathbb{C}$ are classified by "Dynkin graphs" of one of the types A,B,C,D,E,F,G. $\square$

Remarks. (0) Dynkin graphs are actual graphs in the ADE cases, these are called *simply laced cases*. The remaining cases are obtained from ADE cases by operation of *folding*, i.e., taking a "quotient" by a cyclic symmetry group Γ ($\mathbb{Z}_2$ or $\mathbb{Z}_3$), these are called *multiply laced cases*. Here, "multiple" means that one edge will be drawn with multiplicity (2 or 3) and an arrow which should be viewed as inequality $>$ as it indicates which of the two vertices it connects is considered as being "longer".

(2) The infinite series are A_n, B_n, C_n, D_n (for $n = 1, 2, ...$ and with some small overlaps), these are called *classical*. These correspond to classical groups $SL_{n+1}, O_{2n}, Sp_{2n}, O_{2n+1}$ whose meaning is obvious. Finite series E_6, E_7, E_8, F_4, G_2 are called *exceptional*. They are more mysterious because they tend to appear in some subtle but attractive geometries. For instance group $E_8{\times}E_8$ is one of 5 equivalent ways to describe String Theory.

(3) The vertices in the Dynkin graph associated to a semisimple Lie algebra $\mathfrak{g}$ are found in $\mathfrak{g}$ as *simple roots*. The notion of roots is an encoding of a semisimple Lie algebra $\mathfrak{g}$ in terms of its *Cartan subalgebra* $\mathfrak{t}$, see ...

Remark. We will not prove this theorem (the proof has a "combinatorial" part), but we will get some understanding of how it works.

6.5. **Appendix. More on root systems.** Again, this material on general root systems is here only for encyclopedic reasons.

6.5.1. *Rank ≤ 2 root systems.* The rank two root systems are of interest because they capture all possible relative positions of two roots α, β.

Lemma. (a) If rank is 1 the root system is isomorphic to A_1.

(a) If rank is 2 the root system is isomorphic to $A_1 \oplus A_1,\ B_2 = C_2,\ G_2$. □

Corollary. The angles between two roots can be $\pi,\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{4},\ \frac{\pi}{6}, 0$ and also $\frac{2\pi}{3},\ \frac{3\pi}{4},\ \frac{5\pi}{6}$.

6.5.2. *The α-string of roots through β.*

Proposition. For $\alpha, \beta \in \Delta$ not proportional, we have

$$\langle \beta, \check{\alpha} \rangle \ = \ r - s$$

where

- s is the maximum of all $p \in \mathbb{N}$ such that $\beta + p\alpha \in \Delta$ and
- r is the maximum of all $q \in \mathbb{N}$ such that $\beta - q\alpha \in \Delta$.

(b) For any $p \in [-r, s]$ $\beta + p\alpha$ is a root.

Remark. Root strings are of length ≤ 4. Length 4 is found in G_2 only.

Part 3. Representations of semisimple Lie groups and Lie algebras

The key results of the course are the classification of finite dimensional representations of semisimple Lie groups and Lie algebras in section 7.2.

The category of finite dimensional $\mathfrak{g}$ modules lies in a larger category of representations of a semisimple Lie algebra $\mathfrak{g}$ called *category $\mathcal{O}$*. These are all $\mathfrak{g}$ modules generated by finitely many primitive vectors. (ere $\mathcal{O}$ stands for "ordinary" meaning the "most obvious" representations). The basic results on category $\mathcal{O}$ are recalled in section 8. However, the classification of finite dimensional representations in section 7.2 has already introduced the *Verma modules* which are standard objects of category $\mathcal{O}$. Moreover, in that section we have also proved a classification of all irreducible objects in category $\mathcal{O}$.

Category $\mathcal{O}$ is more subtle than the finite dimensional representations – it is not semisimple and the understanding of characters of irreducibles is not possible without some use of Hodge theory.

7. **Finite dimensional representations of semisimple Lie algebras [NOT YET in the FINAL FORM]**

We start in 7.1 with the complete description of the category of finite dimensional representation theory of sl_2.

sl_n

7.1. **Finite dimensional representation theory of sl_2.**

We have considered the Lie algebra sl_2 in 5.2.1.

7.1.1. *Weights and primitive vectors.* These are basic organizational concepts in representation theory of sl_2. A *weight* λ in a representation V of $\mathfrak{g} = sl_2$ means an eigenvalue λ of h in V. Then $V_\lambda \stackrel{\text{def}}{=} \{v \in V;\ hv = \lambda v\}$ is the *weight* λ *subspace* of V and $\dim(V_\lambda)$ is called the *multiplicity of the weight* λ in V.

Lemma. (a) $eV_\lambda \subseteq V_{\lambda+2}, \quad fV_\lambda \subseteq V_{\lambda-2}, \quad hV_\lambda \subseteq V_\lambda$.

(b) $V = \sum_{\lambda\in\mathbb{C}} V_\lambda = V$.

(c) All weights are integers.

Proof. (a) is easy to check. (b) and (c) follow from the fact that finite dimensional representations are the same for the group $SL_2(\mathbb{C})$ and the Lie algebra sl_2. □

7.1.2. *Primitive vectors.* The *primitive vectors of weight* λ in V are the non-zero vectors in

$$V_\lambda^o \stackrel{\text{def}}{=} \{v \in V_\lambda;\ ev = 0\}$$

(also called the *highest weight vectors of weight* λ). For a primitive vector v we define the vectors

$$v_n \stackrel{\text{def}}{=} \frac{f^n}{n!}v \ \in V, \quad .$$

So, $v_0 = v$, $v_1 = fv$ etc.

Lemma. (a)

$fv_k = (k+1)v_{k+1}, \quad hv_k = (\lambda - 2k)v_k \quad k \in \mathbb{N}$ and $ev_k = (\lambda+1-k)v_{k-1}, \quad (k > 0)$.

(b) The nonzero vectors v_n are independent.

(c) The sum $\sum\ \mathbb{C}v_n \subseteq V$ is a $\mathfrak{g}$-submodule.

(d) If $v_n \neq 0$ but $v_{n+1} = 0$ then $\lambda = n$.

Proof. (a) is a direct computation by induction. The formulas in (a) imply (b-d). □

Corollary. (a) If $V \neq 0$ then V has a primitive vector.

(b) The weights of all primitive vectors are natural numbers.

(c) If v is a primitive vector of weight $n \in \mathbb{N}$ then $v_0, ..., v_n$ are independent and $v_i = 0$ for $i > n$.

Proof. (a) Since V is finite dimensional there are finitely many weights and they are integers, so some weight λ is the largest. Now any $0 \neq v \in V_\lambda$ is primitive since $eV_\lambda \subseteq V_{\lambda+2} = 0$.

(b) follows from the part (d) of the lemma.

(c) Since the nonzero vectors v_p are independent and V is finite dimensional, there is some $m \in \mathbb{N}$ such that $v_m \neq 0$ and $v_{m+1} = 0$. Then $v_i = 0$ for $i > m$ and $v \neq 0$ for $i \leq m$. By (d) we have $m = n$. $\square$

7.1.3. *Classification of irreducible finite dimensional modules for* $\mathfrak{g} = sl_2$. We will construct irreducible representations $L(n)$, $n \in \mathbb{N}$, by formulas. (These formulas will turn out to be from lemma 7.1.2.a.) Then we will prove that these are the only irreducible representations.

Lemma. (a) For any $n \in \mathbb{N}$ there is a representation $L(n)$ of sl_2 of dimension $n+1$ given by matrices

$$E = \begin{pmatrix} 0 & n & & & & \\ & 0 & n-1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 2 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \quad H = \begin{pmatrix} n & & & & & \\ & n-2 & & & & \\ & & n-4 & & & \\ & & & \ddots & & \\ & & & & 2-n & \\ & & & & & -n \end{pmatrix}, \quad F = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & n-1 & 0 & \\ & & & & n & 0 \end{pmatrix}.$$

(b) $L(n)$ is irreducible and has a primitive vector of weight n.

(c) The standard basis $e_0, ..., e_n$ of $\mathbb{k}^{n+1}$ is of the form $v_0, ..., v_n$ for the primitive vector e_0 of weight n.

Proof. (a) One just checks that the operators E, H, F satisfy the commutator relations of $e, h, f \in \mathfrak{g}$.

(b) On any $\mathfrak{g}$-invariant subspace $U \subseteq L(n)$ operator H is semisimple, so $U \neq 0$ implies that U contains one of standard basis vectors $e_0, ..., e_n$ of $L(n+1) = \mathbb{k}^{n+1}$. But then the formulas for the action of e, f imply that U contains all e_i. Also, the formulas show that e_0 is a primitive vector of weight n.

The final claim (c) just says that formulas in (a) are the same as in lemma 7.1.2.a. $\square$

Theorem. $L(n)$, $n \in \mathbb{N}$, is a classification of irreducible finite dimensional representations of sl_2.

Proof. All $L(n)$ are irreducible by the lemma. They are non-isomorphic since the dimension of $L(n)$ is $n+1$. So, it remains to check that each finite dimensional irreducible representation V of sl_2 is isomorphic to precisely one of $L(n)$.

However, such V contains a primitive vector v by lemma 7.1.2. Actually, the weight of v is some $n \in \mathbb{N}$ by corollary . Then again, by lemma V contains a submodule spanned by vectors $v_0, ...v_n$ so by irreducibility this is all of V. Finally, V is isomorphic to $L(n)$ by the observation in 7.1.3. □

Corollary. (a) Any irreducible finite dimensional sl_2-module V has a unique primitive vector (up to an invertible scalar). Its weight n is the highest weight of V. All weight multiplicities in V are ≤ 1.

(b) The dual V^* of any irreducible finite dimensional representation V of sl_2 is isomorphic to V.[14] □

Remark. So, the irreducible modules are classified by their highest weights by $\mathbb{N} \ni n \mapsto L(n)$.

7.2. **Finite dimensional representations of semisimple Lie algebras: Announcement.**

7.2.1. *Generalizing the sl_2 situation: Cartans, Borels and dominant weights.* The key observation in the classification of sl_2 representations is that finite dimensionality of a representation implies that it has finitely many weights; so if you walk in any direction (e or f) you will eventually fall off a cliff – and the last thing you will see is a primitive vector (for e or for f respectively).

This generalizes for finite dimensional representations of any semisimple complex Lie group G or its Lie algebra $\mathfrak{g}$.

(0) We start by replacing the element h of sl_2 with a *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ (corresponding to $H \subseteq G$). In the new context, a *weight* of a representation V of $\mathfrak{g}$ is an element $\lambda \in \mathfrak{h}^*$ such that the corresponding $\mathfrak{h}$-eigenspace $V_\lambda = \{v \in V;\ hv = \langle\lambda, h\rangle \cdot v\ (h \in \mathfrak{h})\}$ is non-zero. If V is a representation of G then its weights lie in the lattice of H-integral weights $X^*(H) \subseteq \mathfrak{h}^*$.

(1) Then one replaces $e \in sl_2$ and its relation to h, with a *Borel subalgebra* $\mathfrak{b}$ of $\mathfrak{g}$ that contains $\mathfrak{h}$. This provides a choice of a *direction* in $\mathfrak{g}$.

On the level of roots, the set of roots Δ of $\mathfrak{h}$ in $\mathfrak{g}$ generates the *root sublattice* $Q \subseteq X^*(H)$. The "direction" $\mathfrak{b}$ is seen here as the *positive cone* $Q^+ \subseteq Q$ which is the semigroup generated by the roots $\Delta^+ = \Delta_{\mathfrak{h}}(\mathfrak{b})$ of $\mathfrak{h}$ in $\mathfrak{b}$.

Moreover, this direction has another incarnation on the level of H-weights which is the *dominant cone* $X^+(T)^+$ consisting of all $\lambda \in X^*(H)$ that are "positive with Q^+" in the sense that for any $\alpha \in \Delta^+$ one has $\langle\lambda, \check{\alpha}\rangle \in \mathbb{N}$. (I use the pairing of $\lambda \in \mathfrak{h}^*$ and $\check{\alpha} \in \mathfrak{h}$.)

[14] This is a special property of sl_2, it is not true for sl_n.

Remark. In the case of sl_2 and $G = SL_2(\mathbb{C})$ we have $\mathfrak{h} = \mathbb{C}h$ and $\mathfrak{b} = \mathbb{C}h \oplus \mathbb{C}e$. Then $X^*(H) = \mathbb{Z}\rho$ where $\rho\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) = a$. $\alpha = 2\rho$. Notice that $h = \check{\alpha}$ and $\langle \rho, \check{\alpha}\rangle = 1$. In particular, the set $\mathbb{N}$ of primitive weights of h (i.e., the set of weights of primitive vectors of h in finite dimensional representations) are now interpreted as $\mathbb{N}\rho \subseteq X^*(H) = \mathbb{Z}\rho$. We think of $\mathbb{N}\rho$ as the set of primitive weights of $\mathfrak{h}$ in finite dimensional representations, they parameterize the irreducible representations of sl_2 which we now denote $L(n\rho) \stackrel{\text{def}}{=} L(n)$.

Also, $\Delta = \{\pm\alpha\} \supseteq \Delta^+ = \{\alpha\}$ for the root $\alpha = \alpha_{12} = \varepsilon^1 - \varepsilon^2$ such that $\alpha = 2\rho$. So, $Q \subseteq X^*(H)$ is $2\mathbb{Z}\rho \subseteq \mathbb{Z}\rho$. Now we see that the dominant weights $\mathbb{N}\rho$ parameterize irreducible finite dimensional representations. □

We will now say this in more details, in particular for sl_n.

7.2.2. *The coroot lattice $\check{Q} \subseteq \mathfrak{h}$.* Recall that to any root $\alpha \in \Delta$ we have associated an element $\check{\alpha}$ of $\mathfrak{h}$. We call such elements of $\mathfrak{h}$ the *coroots*. Inside of the set of coroots $\check{\Delta} \stackrel{\text{def}}{=} \{\check{\alpha};\ \alpha \in \Delta\}$ we have the subset $\check{\Delta}^+ \stackrel{\text{def}}{=} \{\check{\alpha};\ \alpha \in \Delta^+\}$ of *positive coroots* and the subset $\check{\Pi} \stackrel{\text{def}}{=} \{\check{\alpha};\ \alpha \in \Pi\}$ of *simple coroots*.

We define the *coroot lattice* $\check{Q} \subseteq \mathfrak{h}$ to be the subgroup generated by all coroots $\check{\alpha},\ \alpha \in \Delta$. Its *positive cone* is the subsemigroup $\check{Q}$ generated by all positive coroots $\check{\alpha},\ \alpha \in \Delta^+$.

Lemma. $\check{Q} = \oplus_{\alpha\in\Pi} \mathbb{Z}\check{\alpha}$ and $\check{Q}_+ = \oplus_{\alpha\in\Pi} \mathbb{N}\check{\alpha}$.

Proof. □

7.2.3. *The weight lattice $P \subseteq \mathfrak{h}^*$.* We define the subgroup $P \subseteq \mathfrak{h}^*$ of *integral weights* to consist of all $\lambda \in \mathfrak{h}^*$ that are integral with coroots, i.e.,

$$P \stackrel{\text{def}}{=} \{\lambda \in \mathfrak{h}^*;\ \langle \lambda, \check{\alpha}\rangle \in \mathbb{Z} \text{ for } \alpha \in \Delta.$$

We will often omit the word "integral", so we will call P the *weight lattice*.

We will also use the sub semigroup P^+ of *dominant weights*, these are required to be non-negative on each positive coroot.

$$P^+ \stackrel{\text{def}}{=} \{\lambda \in \mathfrak{h}^*;\ \langle \lambda, \check{\alpha}\rangle \in \mathbb{N} \text{ for } \alpha \in \Delta^+\}.$$

An example will be the *fundamental weights* $\omega_1, ..., \omega_{n-1}$ defined as the basis of $\mathfrak{h}^*$ dual to the *simple coroots basis* $\check{\alpha}_1, ..., \check{\alpha}_p$ of $\mathfrak{h}$.

Lemma. (a) $\omega_i = \varepsilon_1 + \cdots + \varepsilon_p$.

(b) $P = \oplus_1^{n-1} \mathbb{Z}\omega_i$ and $P^+ = \oplus_1^{n-1} \mathbb{N}\omega_i$.

Proof. □

7.2.4. *Primitive vectors.* For a representation V of $\mathfrak{g}$ a vector $v \neq 0$ is *primitive* if it lies in V_λ for some $\lambda \in \mathfrak{h}^*$ and $\mathfrak{n}v = 0$.

Lemma. (a) A vector v is killed by $\mathfrak{n}$ iff it is killed by all simple root spaces $\mathfrak{g}_\alpha,\ \alpha\in\Pi$.

(b) For a primitive vector $v\in V_\lambda$ the $\mathfrak{g}$-submodule generated by v is the subspace $\sum\ (\mathfrak{n}_-)^n v$ of V.

Proof. (a) is clear since we know that the simple root spaces generate the Lie algebra $\mathfrak{n}$.

(b) We start by listing all weights of $\mathfrak{g}$ as a sequence $\beta_1,...,\beta_M,0,\gamma_1,..\gamma_M$ so that β_i's are negative roots and γ_i's are positive roots.

By the next proposition the $\mathfrak{g}$-submodule generated by v is the sum of subspaces

$$\mathfrak{g}_{\beta_1}^{p_1}\cdot\cdot\cdot\mathfrak{g}_{\beta_M}^{p_M}\ \mathfrak{g}_0^r\ \mathfrak{g}_{\gamma_1}^{q_1}\cdot\cdot\cdot\mathfrak{g}_{\gamma_M}^{p_M}v$$

over all choices of of powers $p_i,r,q_i\in\mathbb{N}$. Now, if one of q_i is >0 then the whole expression is zero since positive roots kill a primitive vector. So, we only need to consider $\mathfrak{g}_{\beta_1}^{p_1}\cdot\cdot\cdot\mathfrak{g}_{\beta_M}^{p_M}\ \mathfrak{g}_0^r v$. Since $\mathfrak{g}_0$ preserves the line through v we can assume that $r=0$. □

Remark. The weight of a primitive vector $v\in V_\lambda$ is said to be a *highest weight*. Here, "highest" refers to the partial order on $\mathfrak{h}^*$ defined by positive roots. Then the precise meaning is that λ is the highest weight in the submodule generated by v. (This follows from the part (b) of the lemma.

7.2.5. *Finite dimensional representations of sl_n.* The following theorem describes the basic structure of finite dimensional representations. Because sl_n is the sum $\sum_{\alpha\in\Delta}\ \mathfrak{s}_\alpha$ of subalgebras isomorphic to sl_2, the theorem will follow from results for sl_2.

Theorem. V be a finite dimensional representation of $\mathfrak{g}=sl_n$.

(a) Any $h\in\mathfrak{h}$ acts semisimply on V, equivalently $\oplus_{\lambda\in\mathfrak{h}^*}\ V_\lambda$ is all of V. More precisely, all weights in V are integral, hence

$$V\ =\ \oplus_{\lambda\in P}\ V_\lambda.$$

(b) Any $x\in\mathfrak{n}$ acts nilpotently on V. Moreover, for $p>>0$ we have $\mathfrak{n}^p V=0$.

(c) If $V\neq 0$ then V has a primitive vector.

(d) The weight of any primitive vector is dominant.

Proof. (a) We can restrict the action of $\mathfrak{g}$ on V to any root subalgebras $\mathfrak{s}_\alpha$, $\alpha\in\Delta$. Then, via $sl_2\xrightarrow{\cong}\mathfrak{s}_\alpha$ our V becomes a representation of sl_2. However, we know that $h\in sl_2$ acts semisimply in any finite dimensional representation of sl_2. Now, because the standard isomorphisms $sl_2\xrightarrow{\cong}\mathfrak{s}_\alpha$ takes h to $\check\alpha$, we know that $\check\alpha$ acts semisimply on V.

Finally, $\mathfrak{h}$ has a basis $\check\alpha_i=E_{ii}-E_{i+1,i+1}$ of simple coroots. The actions of these on V form a family of commuting semisimple operators, so they have a simultaneous diagonalization. This proves that $V=\oplus_{\lambda\in\mathfrak{h}^*}\ V_\lambda$.

To see that any weight λ in V is integral notice that for any root $\alpha\in\Delta$, the number $\langle\lambda,\check\alpha\rangle$ is an eigenvalue of the action of $\check\alpha$ on V. These are integers because all eigenvalues of the action of $h \in sl_2$ in any finite dimensional representation of sl_2 are known to be integers.

(b) By $\mathfrak{n}^p V$ we mean the subspace of V spanned by all $x_1 \cdots x_p v$ for $x_i \in \mathfrak{n}$ and $v \in V$. So, the subspace $\mathfrak{n}^p V \subseteq V$ is a sum of subspaces $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} V_\lambda$ over all choices of $\phi_i \in \Delta^+$ and all weights λ in V.

The set $\mathcal{W}(V)$ of weights in V is finite because the $\dim(V) < \infty$.

Recall that $\mathfrak{g}_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$, hence $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} V_\lambda \subseteq V_{\lambda + \sum_1^p \phi_i}$. However, the set $\mathcal{W}(V)$ of weights in V is finite because the $\dim(V) < \infty$. So, for $p >> 0$, and any $\lambda \in \mathcal{W}(V)$ we have that $\lambda + \sum_1^p \phi_i$ is not in $\mathcal{W}(V)$, hence $V_{\lambda + \sum_1^p \phi_i} = 0$.

For such p we have $\mathfrak{n}^p V = 0$, hence in particular for $x \in \in \mathfrak{n}$ we have $x^p V = 0$.

(c) Again, we use the fact that the set $\mathcal{W}(V)$ of weights in V is finite. Therefore, it contains a maximal element λ for the partial order on $\mathfrak{h}^*$ defined by $\lambda \leq \mu$ if $\mu - \lambda \in Q_+ = \mathrm{span}_{\mathbb{N}}(\Delta^+) = \oplus_1^{n-1} \mathbb{N}\alpha_i$. For such λ we have $\lambda + \alpha \notin \mathcal{W}(V)$ for any $\alpha > 0$, hence $\mathfrak{g}_\alpha V_\lambda \subseteq V_{\lambda+\alpha} = 0$. So, $\mathfrak{n} V_\lambda = 0$ and therefore any vector in V_λ is primitive.

(d) Let λ be the weight of some primitive vector v in V. Then for any positive root α we have $\mathfrak{g}_\alpha v \subseteq \mathfrak{n} v = 0$. Since the standard isomorphisms $sl_2 \xrightarrow{\cong} \mathfrak{s}_\alpha$ take e to $\mathfrak{g}_\alpha$, we see that for the action of sl_2 on V via $sl_2 \xrightarrow{\cong} \mathfrak{s}_\alpha \subseteq \mathfrak{g}$ we have $ev = 0$ and $hv = \langle \lambda, \check{\alpha} \rangle \cdot v$. So, v is also a primitive vector for the action of sl_2 on V, so its sl_2-weight $\langle \lambda, \check{\alpha} \rangle$ must be in $\mathbb{N}$. □

Remark. If u and v are primitive vectors of weights λ and μ in representations U and V then $u \otimes v$ is a primitive vector of weight $\lambda + \mu$ in $U \otimes V$. □

7.3. **Classification of irreducible finite dimensional representations of $\mathfrak{g} = sl_n$.** We know that any irreducible finite dimensional representation V of $\mathfrak{g}$ has a primitive vector with a dominant weight.

Theorem. (a) For each dominant weight $\lambda \in P^+$ there is exactly one (up to isomorphism) irreducible finite dimensional representation with a primitive vector of weight λ. We denote it $L(\lambda)$.

(b) $L(\lambda)$, $\lambda \in P^+$ is the complete list of irreducible finite dimensional representations of $\mathfrak{g}$.

Proof. (b) follows from (a) since we know that any irreducible finite dimensional representation V of $\mathfrak{g}$ has a primitive vector with a dominant weight.

Claim (a) consists of two parts

- *Existence*: for $\lambda \in P^+$ there exists an irreducible finite dimensional representation with a primitive vector of weight λ.
- *Uniqueness*: If L, L' are two irreducible finite dimensional representation with a primitive vector of weight λ then $L' \cong L$.

We will next prove existence and postpone the proof of uniqueness until the general setting of semisimple Lie algebras.

7.3.1. *Questions.* For the sl-module $V = \mathbb{k}^n$ what are the highest weights of irreducible modules (i) V, (ii) $\wedge^p V$, (iii) the adjoint representations $\mathfrak{g}$, (iv) $S^p V$?

7.4. **The $\mathfrak{g}$-submodule generated by a vector.** The proof of the following proposition will later be a motivation for introducing the enveloping algebras of Lie algebras.

Lemma. Let v be a primitive vector of weight λ in a$\mathfrak{g}$-module V. Denote by S the $\mathfrak{g}$-submodule generated by v.

(a) $S = U\mathfrak{n}^{-}\cdot v$.

(b) The weights of S lie in $\lambda - Q^{+}$, i.e., for any $\mu \in \mathcal{W}(S)$ one has $\mu \leq \lambda$. Moreover, $V_\lambda = \mathbb{k}v$. (So, λ is the highest weight of S.)

(c) S has a unique irreducible quotient L. One has $\dim(L_\lambda) = 1$.

Proof. (a) *The first proof.* The $\mathfrak{g}$-submodule generated by any vector v is the the subspace of V spanned by all $x_1 \cdots x_p v$ for $x_i \in \mathfrak{g}$. This is the same as $U\mathfrak{g}\cdot v$.

We know that the multiplication $U\mathfrak{n}^{-}\otimes U\mathfrak{b} \to U\mathfrak{g}$ is surjective, so $S = U\mathfrak{g}\cdot v = U\mathfrak{n}^{-}\cdot(U\mathfrak{b}\cdot v) = U\mathfrak{n}^{-}\cdot\mathbb{k}v = U\mathfrak{n}^{-}\cdot v$.

(b) follows from (a). For this we choose a basis $x_1, ..., x_N$ of $\mathfrak{n}^{-}$ so that x_i lies in $\mathfrak{g}_{\phi_i}$, where $\phi_1, ..., \phi_N$ is any ordering of roots in $\Delta(\mathfrak{n}^{-}) = -\Delta(\mathfrak{n}) = -\Delta^{+}$. Then the monomials $x_1^{e_1} \cdots x_N^{e_N}$ span $U\mathfrak{n}_{-}$ and $x_1^{e_1} \cdots x_N^{e_N}$ lies in $(U\mathfrak{n}_{-})_{\sum\ e_i\phi_i}$. Moreover, $\sum\ e_i\phi_i$ is 0 iff all e_i are 0.

(c) Quotients Q of S correspond to submodules S' of S. A quotient Q is irreducible iff the submodule S' is a maximal proper submodule. Therefore, an equivalent formulation is that

- (i) S has exactly one maximal proper submodule $\mathcal{S}$ and that
- (ii) $\mathcal{S}$ does not have weight λ (so that for $L = S/\mathcal{S}$ we have $L_\lambda = S_\lambda/\mathcal{S}_\lambda$ is the line S_λ.

For this we notice that

$$(*) \text{ For any proper submodule } S' \subseteq S,\ S' \text{ does not contain weight } \lambda.$$

Clearly, if $S'_\lambda \subseteq S_\lambda = \mathbb{k}v$ would be nonzero then S' would contain v and then S' would contain all of S.

Now it is clear that there exists the largest proper submodule $\mathcal{S}$ of S – this is just the sum of all proper submodules S'. This $\mathcal{S}$ is proper since $\mathcal{S}_\lambda = \sum\ S'_\lambda = 0$. $\square$

We say that λ is the *highest weight* of S in the sense that it is the largest weight in S for the partial order defined by Q_+. (For this reason we also call primitive vectors the *highest weight vectors.*)

Corollary. (a) An irreducible finite dimensional $\mathfrak{g}$-module L has precisely one primitive vector. Its weight λ is the highest weights of L.

(b) If for a given $\lambda \in P^+$ there exists a finite dimensional representation with a primitive vector of weight λ, then there exists an irreducible finite dimensional representation with the highest weight λ.

If for a given $\lambda \in P^+$ there exists a finite dimensional representation U with a primitive vector of weight λ, then there exists an irreducible finite dimensional representation V with a primitive vector of weight λ.

Proof. (a) Since L is irreducible it has a primitive vector v. The submodule S generated by v is not zero so it is all of L. Therefore λ is the highest weight in $S = L$. So, any primitive vector lies in L_λ for the highest weight λ in L. However, L_λ is one dimensional by the lemma.

(b) If v is a primitive vector of weight λ in a finite dimensional representation V then we get an irreducible representation of highest weight λ by taking the unique irreducible quotient of the submodule generated by v. $\square$

7.4.1. *The second proof of the part (a) of the theorem.* This can be skipped – we write the same proof but without introducing the enveloping algebra. So, this version can be viewed as a motivation for introducing the enveloping algebras in the first place.

Proposition. Let us write all weights of $\mathfrak{g}$ as a sequence $\beta_1, ..., \beta_N$. Then for any representation V the $\mathfrak{g}$-submodule generated by a given vector v is the sum of subspaces $\mathfrak{g}_{\beta_1}^{p_1} \cdots \mathfrak{g}_{\beta_N}^{p_N} v$ over all choices of of powers $p_i \in \mathbb{N}$.

Proof. The $\mathfrak{g}$-submodule generated by v is the the subspace of V spanned by all $x_1 \cdots x_p v$ for $x_i \in \mathfrak{g}$. We can think of it as the sum of subspaces $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} v$ over all choices of $\phi_i \in \mathcal{W}(\mathfrak{g}) = \Delta \sqcup 0$. Let $\mathcal{V}_q$ be the sum of all such subspaces $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} v$ with $p \leq q$. It contains $\mathcal{U}_q$ which is the sum of subspaces $\mathfrak{g}_{\beta_1}^{p_1} \cdots \mathfrak{g}_{\beta_N}^{p_N} v$ for all choices of of powers p_i such that $\sum p_i \leq q$. We will prove by induction in q that $\mathcal{U}_q \subseteq \mathcal{V}_p$ is equality.

If the sequence $\phi_1, ..., \phi_p$ is compatible with the chosen order on $\mathcal{W}(\mathfrak{g})$ then $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} v$ is of the above form $\mathfrak{g}_{\beta_1}^{p_1} \cdots \mathfrak{g}_{\beta_N}^{p_N} v$. If not then there are some neighbors ϕ_{i-1}, ϕ_i which are in the wrong order. However, for x and y in $\mathfrak{g}_{\phi_{i-1}}$ and $\mathfrak{g}_{\phi_i}$,

$$\pi(x)\pi(y) = \pi(y)\pi(x) + [\pi(x), \pi(y)] = \pi(y)\pi(x) + \pi[x, y].$$

So, we can replace the product of length two $\mathfrak{g}_{\phi_{i-1}} \mathfrak{g}_{\phi_i}$ with the product in the opposite order $\mathfrak{g}_{\phi_i} \mathfrak{g}_{\phi_{i-1}}$ (again of length 2), at the price of adding a term which is a product of length 1. $\square$

7.5. **Existence of irreducible representations.** Here we prove the first part of the theorem 7.3.

Lemma. For each dominant weight $\lambda \in P^+$ there exists an irreducible finite dimensional representation L with a primitive vector of weight λ. (Then λ is the highest weight in L.)

Proof. From homeworks we know that when λ is one of the fundamental ω_i then such representation is given by $\wedge^i \mathbb{k}^n$. Denote by v_{ω_i} its primitive vector.

Now for any dominant weight $\lambda \in P^+$ we have $\lambda = \sum_1^{n-1} \lambda_i\omega_i$ with $\lambda_i \in \mathbb{N}$. Then in $\otimes_1^{n-1} (\wedge^i\Bbbk^n)^{\otimes\lambda_i}$ the vector $\otimes_1^{n-1} v_i^{\otimes\lambda_i}$ is primitive of weight λ (see the remark in 7.2.5). $\square$

7.6. **Uniqueness of irreducible representation with a given highest weight.**

7.6.1. *Irreducible finite dimensional representations of* $\mathfrak{b}$*.*

Lemma. (a) $\mathfrak{n}$ is an ideal in $\mathfrak{b}$.

(b) $[\mathfrak{b},\mathfrak{b}] = \mathfrak{n}$ and $\mathfrak{b}^{\mathrm{ab}} \cong \mathfrak{h}$.

Proof. $\mathfrak{h}$ is commutative, i.e., $[\mathfrak{h},\mathfrak{h}] = 0$. Also for a root α we have $[\mathfrak{h},\mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ since $\mathfrak{h}$ acts on $\mathfrak{g}_\alpha$ by $\alpha \in \mathfrak{h}^*$ which is not zero. This implies that $[\mathfrak{h},\mathfrak{b}] = \mathfrak{n}$. Together with $[\mathfrak{n},\mathfrak{n}]\subseteq\mathfrak{n}$ ($\mathfrak{n}$ is a subalgebra) this implies that $[\mathfrak{b},\mathfrak{b}] = \mathfrak{n}$.

Now $\mathfrak{b}^{\mathrm{ab}} = \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] = \mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$. $\square$

Remark. Using $\mathfrak{b}\twoheadrightarrow\mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$ we get $\mathfrak{h}^*\hookrightarrow\mathfrak{b}^*$. The meaning is that a linear functional λ on$\mathfrak{h}$ extends to $\mathfrak{b}$ by zero on $\mathfrak{n}$.

Proposition. (a) Any $\lambda \in \mathfrak{h}^*$ gives a 1-dimensional representation $\Bbbk_\lambda^{\mathfrak{b}}$ of $\mathfrak{b}$. The vector space is $\Bbbk$ and $\mathfrak{b}$ acts on it by λ viewed as a functional on $\mathfrak{b}$, i.e., $x{\cdot}1_\Bbbk = \langle\lambda, x\rangle\ 1_\Bbbk$.

(b) This is the complete classification of 1 dimensional representations of $\mathfrak{b}$.

Proof. (b) is a case of lemma **??** since $\mathfrak{b}^{\mathrm{ab}} = \mathfrak{h}$. $\square$

7.6.2. *Verma modules for* $\mathfrak{g} = sl_n$. The *Verma module* with the highest weight λ is defined as the induced module[15]

$$M(\lambda) \stackrel{\text{def}}{=} Ind_{\mathfrak{b}}^{\mathfrak{g}}\Bbbk_\lambda^{\mathfrak{b}} \stackrel{\text{def}}{=} U\mathfrak{g}\otimes_{U\mathfrak{b}} \Bbbk_\lambda^{\mathfrak{b}}.$$

The most obvious vector in $M(\lambda)$ is $v_\lambda = 1_{U\mathfrak{g}}\otimes 1_\Bbbk$.

Lemma. (a) v_λ is a primitive vector with weight λ.

(b) v_λ generates M_λ.

7.6.3. *Corollary.* (1) $\mathfrak{h}$ acts semisimply on $M(\lambda)$ and the weights $\mathcal{W}(M(\lambda))$ lie in $\lambda - Q_+$ (i.e., weights are $\le \lambda$). Moreover the λ weight space $M(\lambda)_\lambda$ is $\Bbbk v_\lambda$.

(2) $M(\lambda)$ has the largest proper submodule $M(\lambda)^+$. Equivalently, it has a unique irreducible quotient, we denote it $L(\lambda)$. $L(\lambda)$ is also generated by a primitive vector of weight λ (the image of v_λ which we again denote v_λ) and $L(\lambda)_\lambda = \Bbbk v_\lambda$.

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[15] The notation we initially used used in class was M_λ.

7.6.4. *The universal property of Verma modules.* The categorical formulation of the following lemma is that the object $M(\lambda)\in\mathfrak{m}(\mathfrak{g})$ represents the functor $-^o_\lambda:\ \mathfrak{m}(fg)\to Vec_{\Bbbk}$ of taking the primitive vectors.

Lemma. For any $\mathfrak{g}$-module V there is a canonical isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}[M(\lambda),V]\ \underset{\cong}{\overset{\iota}{\to}}\ V^o_\lambda.$$

Here, $\iota(\phi)\ =\ \phi(v_\lambda)\in V^o_\lambda$.

Proof. We use the Frobenius reciprocity, i.e., the fact that the induction $U\mathfrak{g}\otimes_{\mathfrak{b}}-$ is the left adjoint of the forgetful functor $\mathcal{F}^{\mathfrak{b}}_{\mathfrak{g}})$:

$$\mathrm{Hom}_{\mathfrak{g}}[M(\lambda),V]=\ \mathrm{Hom}_{U\mathfrak{g}}[U\mathfrak{g}\otimes_{U\mathfrak{b}}\Bbbk^{\mathfrak{b}}_\lambda),V]\ \cong\ \mathrm{Hom}_{U\mathfrak{b}}(\Bbbk^{\mathfrak{b}}_\lambda,V).$$

A linear map $\psi:\Bbbk^{\mathfrak{b}}_\lambda\to V$ is the same as a choice of a vector $v=\psi(1_{\Bbbk})$ in V. Now, ψ is an $\mathfrak{h}$-map iff $\mathfrak{h}$ acts on v by λ, and ψ is an $\mathfrak{n}$-map iff $\mathfrak{n}$ kills v. So, ψ is an $\mathfrak{b}$-map iff $v\in V^o_\lambda$,

Now one checks that the isomorphism $\iota:\ \mathrm{Hom}_{\mathfrak{g}}[M(\lambda),V]\xrightarrow{\cong}V^o_\lambda$ that we have constructed acts by the formula in the lemma. □

Corollary. For any $\lambda\in\mathfrak{h}^*$ there is a unique irreducible $\mathfrak{g}$-module L which has a primitive vector of weight λ. This L is the unique irreducible quotient $L(\lambda)$ of the Verma $M(\lambda)$.

Proof. For existence of L we note that the above $L(\lambda)$ satisfies the properties. We will also see that any irreducible $\mathfrak{g}$-module L which has a primitive vector v of weight λ is isomorphic to $L(\lambda)$.

First, to a primitive vector v in L there corresponds some homomorphism $\phi:\mathbb{M}(\lambda)\to L$. Since $v\neq 0$ we have $\phi\neq 0$. Then $0\neq Im(\phi)$ is a submodule of L, since L is irreducible we have $Im(\phi)=L$, i.e., L is an irreducible quotient of $M(\lambda)$. But there is only on irreducible quotient of $M(\lambda)$ and it is $L(\lambda)$. □

7.7. **Proof of the classification of irreducible finite dimensional representations of sl_n (theorem 7.3). A.** The first claim in this theorem is that for each dominant weight $\lambda\in P^+$ there is exactly one irreducible finite dimensional representation$L$ with a primitive vector of weight λ.

The existence of L was proved in 7.5. The uniqueness is a special case of the corollary 7.6.4. This corollary also says that such L is the representation $L(\lambda)$ constructed as the unique irreducible quotient of $M(\lambda)$.

B. The second claim in the theorem is that $L(\lambda)$ for $\lambda\in P^+$ is the classification of irreducible finite dimensional representations of $\mathfrak{g}$, i.e., that

- (i) any irreducible finite dimensional representation L is isomorphic to one of $L(\lambda)$'s and

- (ii) there are no repetitions in the list, i.e., the only way that $L(\lambda) \cong L(\mu)$ is when $\lambda = \mu$.

For (i) notice that since L is irreducible we have $L \neq 0$. Then we know that since L is finite dimensional and $\neq 0$ it has a primitive vector of some weight $\lambda \in P^+$. Then the corollary 7.6.4 guarantees that L is $L(\lambda)$.

For (ii), recall that λ is the highest weight in $L(\lambda)$, so if $L(\lambda) \cong L(\mu)$ then they have the same highest weight hence $\lambda = \mu$. $\square$

7.8. **Classification of finite dimensional representations of a semisimple Lie algebra.** A Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ gives the corresponding root system Δ. Any root $\alpha \in \Delta$ gives a vector $\check{\alpha} \in \mathfrak{h}$.

We define the *integral weights* $P\subseteq\mathfrak{h}^*$ to consist of all $\lambda \in \mathfrak{h}^*$ such that $\langle\lambda, \check{\Delta}\rangle\subseteq \mathbb{Z}$.

A choice of base Π of Δ gives the *dominant integral weights* cone $P^+\subseteq P$ consisting of all $\lambda \in \mathfrak{h}^*$ such that $\langle\lambda, \check{\Pi}\rangle\subseteq \mathbb{N}$.

7.8.1. *Borel subalgebras and Verma modules.* A choice of a system of positive roots $\Delta^+\subseteq\Delta$ give a Borel subalgebra

$$\mathfrak{b} = \mathfrak{h}\oplus\mathfrak{n} \quad \text{for} \quad \mathfrak{n} = \oplus_{\alpha\in\Delta^+}\ f g_\alpha.$$

Then any $\alpha \in \mathfrak{h}^*$ defines an $\mathfrak{g}$-module, the Verma module

$$M(\lambda) \ \stackrel{\text{def}}{=}\ U\mathfrak{g}\otimes_{U\mathfrak{b}} \Bbbk_\lambda.$$

Here $\mathfrak{k}_\lambda$ denotes the 1-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts by λ and $\mathfrak{n}$ by zero.

The base Π of Δ corresponding to Δ^+ defines Let us

Theorem. (a) Any Verma module M_λ has a unique irreducible quotient L_λ.

(b) $L(\lambda)$ is finite dimensional iff $\lambda \in P^+$, i.e., iff λ is the the dominant integral cone.

(c) All irreducible finite dimensional representations are exactly all $L(\lambda)$, $\lambda \in P^+$.

Proof. We have proved this theorem for sl_n. In the general case most of the proof is the same. The difference is in one direction in part (b), where we need to show that for $\lambda \in P^+$ the representation $L(\lambda)$ is finite dimensional.

For sl_n we proved this by an explicit construction of $L(\lambda)$ that starts with the fundamental weights $\lambda = \omega_i$. This method does not extend well to the general case because we do not understand the fundamental representations so well.

It is actually easier to construct irreducible finite dimensional representations of a semisimple Lie algebra $\mathfrak{g}$ using the associated group G (then to do it using the Lie algebra itself). This approach is sketched in 7.9. □

7.9. **The Borel-Weil-Bott construction of irreducible representations.**

7.9.1. *The flag variety $\mathcal{B}$ of a semisimple algebra.* For a semisimple Lie algebra $\mathfrak{g}$ let $G = G_{sc}$ be the simply connected group associated to G. Inside G one finds subgroups B, H, N with Lie algebras $\mathfrak{b}, \mathfrak{h}, \mathfrak{n}$.[(16)]

The quotient $\mathcal{B} \stackrel{\text{def}}{=}\ G/B$ is called the *flag variety* of $\mathfrak{g}$.[(17)]

[16] Say, B consists of all $g \in G$ that preserve the subspace $\mathfrak{b}$ of $\mathfrak{g}$ and H consists of all $g \in G$ that fix each element of $\mathfrak{h}$. Then N is the unique subgroup of B complementary to H.

[17] The letter $\mathcal{B}$ stands for "Borel". The reason is that the flag variety $\mathcal{B}$ can be identified with the set of all conjugates ${}^g\mathfrak{b}\subseteq\mathfrak{g}$ of the above Borel subalgebra $\mathfrak{b}$ under elements g in G. All these conjugates are

7.9.2. A character of the Cartan group H is a homomorphism $\chi : H \to G_m = GL_1$. The characters of H form a group denoted $X^*(H)$.

The differential of a character χ at $e \in H$ is a linear map $d_e\chi:\ T_eH = \mathfrak{h} \to T_eG_m = \Bbbk$, so it is a linear functional $d_e\chi \in\ \mathfrak{h}^*$ on $\mathfrak{h}$.

Lemma. Taking the differential gives an isomorphism of the character group $X^*(H)$ and the group of integral weights $P \subseteq \mathfrak{h}^*$. □

From now on we will identify any integral weight $\lambda \in P$ with the corresponding character of H which we will also denote λ.

The canonical map of Lie algebras $\mathfrak{b} \to \mathfrak{h}$ (zero on $\mathfrak{n}$) gives a canonical map of Lie groups $B \to H$ (with kernel N). So for any $\lambda \in P$ we get a 1-dimensional representation $\Bbbk_\lambda$ of B via $B \twoheadrightarrow H \xrightarrow{\lambda} G_m = Gl(\Bbbk)$.

As in a construction of Verma modules we now induce this to a representation of G. The "induction" is in this case slightly different and it is called coinduction. To a representation $\Bbbk_\lambda$ of the group B we associate G-equivariant line bundle $\mathcal{L}_\lambda$ over the flag variety G/B. This is called the associated bundle

$$\mathcal{L}_\lambda \ \stackrel{\text{def}}{=}\ (G \times \mathcal{L}_\lambda)/B \quad \to \quad (G \times \text{pt})/B =\ \mathcal{B}.$$

Because G acts on the line bundle $\mathcal{L}_\lambda$, it also acts on the space of global section of the line bundle $\mathcal{L}_\lambda$

$$Coind_B^G(\mathcal{L}_\lambda) \ \stackrel{\text{def}}{=}\ \Gamma(\mathcal{B}, \mathcal{L}_\lambda).$$

Theorem. (a) [Borel-Weil] When $\lambda \in P^+$ then $\Gamma(\mathcal{B}, \mathcal{L}_\lambda)$ is an irreducible finite dimensional representation of G and therefore also of the Lie algebra $\mathfrak{g}$.

(b) As a representation of $\mathfrak{g}$ the space $\Gamma(\mathcal{B}, \mathcal{L}_\lambda)$ has highest weight λ.

Remarks. (0) This implies that $\Gamma(\mathcal{B}, \mathcal{L}_\lambda)$ is the irreducible representation $L(\lambda)$ which was defined as the unique irreducible quotient of the Verma module $M(\lambda)$.

(1) If λ is not dominant then $\Gamma(\mathcal{B}, \mathcal{L}_\lambda) = 0$.

(2) Bott's contribution is the calculation of all cohomology groups of line bundles $L(\lambda)$.

7.10. Classification of finite dimensional representations of $\mathfrak{g} = sl_n$.

Theorem. Finite dimensional representations of $\mathfrak{g} = sl_n$ are semisimple. So, each one is isomorphic to a sum $\oplus_{\lambda \in P^+}\ L(\lambda)^{m_\lambda}$ for some multiplicities $m_\lambda \in \mathbb{N}$. □

Again, we will postpone the proof for the general setting of semisimple Lie algebras.

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called Borel subalgebras of $\mathfrak{g}$ and the particular one $\mathfrak{b}$ that we started with can be called the "standard" Borel subalgebra.

Remark. As in sl_2, effectively such decomposition comes from choosing a basis $v_1^\lambda, ..., v_{m_\lambda}^\lambda$ of the spaces V_λ^0 of primitive vectors for each dominant weight λ.

8. **Category $\mathcal{O}$**

Classification of all irreducible modules for sl_n is a *wild* problem, i.e., one can prove that we do not have a way to list all irreducibles. (This is an observation in mathematical logic.) Instead, what is interesting is to classify irreducible representations lying in certain interesting subcategories. One subcategory is the category $Rep^{fd}(\mathfrak{g})$ of all finite dimensional representations.

The next most basic and most influential one is the category $\mathcal{O}$ introduced by Joseph Bernstein, Israel Gelfand and Sergei Gelfand.[18] here "$\mathcal{O}$" stands for *ordinary* (in Russian).

For us the category $\mathcal{O}$ is the home for objects that we have already encountered in our study of finite dimensional representations – Vermas $M(\lambda)$ and irreducibles $L(\lambda)$. It also gives us an opportunity to notice how the behavior of infinite dimensional $\mathfrak{g}$-modules is more subtle than that of finite dimensional ones.

8.1. **Category $\mathcal{O}$ for $\mathfrak{g} = sl_n$.** This is the subcategory of the category $Rep(\mathfrak{g}) = \mathfrak{m}(U\mathfrak{g}))$ of $\mathfrak{g}$-representations (i.e., $U\mathfrak{g}$-modules) that consists of all $\mathfrak{g}$-modules V such that

(1) V is finitely generated;
(2) $\mathfrak{h}$ acts semisimply on V, i.e., $V = \oplus_{\lambda\in\mathfrak{h}^*} V_\lambda$;
(3) V is *locally finite* for the subalgebra $\mathfrak{n}$. The meaning is that for any vector v in V, the $\mathfrak{n}$-submodule $U(\mathfrak{n})v$ that it generates is finite dimensional.

Lemma. (a) The category $Rep^{fd}(\mathfrak{g})$ of finite dimensional representations lies in $\mathcal{O}$.

(b) If V is in $\mathbb{O}$ then any submodule or quotient of V is also in $\mathcal{O}$.

Theorem. (a) Verma modules $M(\lambda)$ lie in $\mathcal{O}$.

(b) The irreducible representations in $\mathcal{O}$ are precisely all $L(\lambda)$, $\lambda \in \mathfrak{h}^*$.

8.2. **The Kazhdan-Lusztig theory.** It deals with the structure of Verma modules. The basic fact is the following.

Lemma. Any $V \in \mathcal{O}$ has a finite length, i.e., it has a finite filtration $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$ with all graded pieces $Gr_i(V) = V_{i-1}/V_i$ irreducible. □

Such filtration is called a *Jordan-Hoelder* series of V. It is a general fact in algebra that though such filtration need not be unique, the number of times a given irreducible module L appears in the list of $Gr_i(V)$'s for $i = 1, .., n$ is independent of the choice of the filtration. This number is called the *multiplicity* of L in V and it is denoted $[V : L]$. When V is in $\mathcal{O}$ then all subquotients $Gr_i(V)$ are again in $\mathcal{O}$, hence each is of the form $L(\mu)$ for some $\mu \in \mathfrak{h}^*$.

[18] Israel was one of the most important mathematicians in 20$^{\text{th}}$ century. Sergei is his son.

Problem. For any $\lambda, \mu \in \mathfrak{h}^*$ find the multiplicity $[M(\lambda) : L(\mu)]$ of irreducibles $L(\mu)$ in Verma modules $M(\lambda)$.

A conjectural answer to this question was provided by a joint work of Kazhdan and Lusztig (the "Kazhdan-Lusztig" conjecture). The proof was obtained by Beilinson-Bernstein and independently by Brylinski-Kashiwara. It was based on

- the theory of *D-modules* which is the algebraization of the theory of linear partial differential equations;
- the *intersection homology* and *perverse sheaves* in algebraic topology of complex algebraic varieties;
- Deligne's proof of Weil conjectures on the use of positive characteristic geometry for algebraic topology of complex algebraic varieties.

The Beilinson-Bernstein version was very strong and elegant, so it has a become one of basic modes of thinking in representation theory and one of a few origins of the so called *Geometric Representation Theory* (The other two are Drinfeld's *Geometric Langlands program* and Springer's construction of representations of *Weyl groups* such as the symmetric groups S_n).

Now we define the *primitive vectors* of weight $\lambda \in \mathfrak{h}^*$ as non-zero vectors in

$$V_\lambda^o \overset{\text{def}}{=} \{v \in V_\lambda : \ \mathfrak{n}\, v \ = \ 0\}.$$

We will see that these control the structure of the representation.

Part 4. Appendices

Appendix A. Algebraic groups

A.1. **Algebraic geometry.** A priori, it studies solutions of systems of polynomial equations. Since the polynomials are very simple (in comparison with smooth and analytic functions) this is in a sense the simplest type of geometry. Consequently this is in many ways the best understood type of geometry and this makes it widely influential.

History of algebraic geometry follows a pattern that when calculations about one class of spaces hit a difficulty, we invent a more general class of spaces which makes these calculations easier:

- Numbers
- Affine spaces
- Affine algebraic varieties
- Projective algebraic varieties
- Algebraic varieties
- Schemes
- Stacks
- Derived stacks

So, one of successes of algebraic geometry is that it keeps advancing the notion of *space*.

A.1.1. *Algebraic varieties.* Each class of spaces X has its own class of functions $\mathcal{O}(X)$. On affine spaces $\mathbb{A}^n_{\Bbbk}$ for a field $\Bbbk$, these are the polynomials $\mathcal{O}(\mathbb{A}^n_{\Bbbk}) = \Bbbk[x_1, ..., x_n]$.

Affine algebraic varieties X defined over $\Bbbk$ lie in affine spaces $\mathbb{A}^n_{\Bbbk}$ and are described by a system of polynomial equations $P_1 = \cdots = P_k = 0$.

One can alternatively think of affine algebraic varieties X over a field $\Bbbk$ as factories that produce sets $X(\Bbbk')$ from rings $\Bbbk'$ that contain $\Bbbk$: $X(\Bbbk')$ is the set of solutions of of the system $P_1 = \cdots = P_k = 0$ in $(\Bbbk')^n$. (Actually, it is sufficient to have a homomorphism of rings $\Bbbk \to \Bbbk'$.)

Algebraic variety X is characterized by the ring $\mathcal{O}(X)$ of natural functions that we consider on X. These are restrictions of polynomials $\mathcal{O}(X) = \{f|_X;\ f \in \Bbbk[x_1, ..., x_n]\}$. So, if we know the ring $\mathcal{O}(X)$ we know X even if we do not specify an embedding of X into some $\mathbb{A}^n_{\Bbbk}$. So, functions provide a more invariant way of thinking about affine algebraic varieties.

All algebraic varieties X are glued from finitely many open pieces U_i which are affine algebraic varieties. Then $\mathcal{O}(X)$ consists of all functions f such that the restrictions $f|_{U_i}$ are in $\mathcal{O}(U_i)$. So, the condition is that f is locally a restriction of a polynomial.

A.1.2. *Schemes.* Schemes were invented by Grothendieck. The idea was that for each commutative ring A there is a geometric object X (called the spectrum $\mathrm{Spec}(X)$ of X), such that the ring of functions $\mathcal{O}(X)$ on X is A. Such X is said to be an *affine scheme* and all schemes are obtained by gluing open pieces which are affine schemes. This allowed a geometric point of view on commutative rings such integers $\mathbb{Z}$ etc.

A.1.3. *Fibered products, Cartesian squares and Base Change.* When X, Y map to the same space Z by $X \xrightarrow{f} Z \xleftarrow{g} Y$, we define the *fibered product* $X \times_Z Y$ of X and Y over Z as a subspace of $X \times Y$ coexisting all pairs $(x, y) \in X \times Y$ such that $f(x) = g(y)$. Then the projections give maps $X \xleftarrow{F} Z \xrightarrow{G} Y$ and therefore a commutative square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{F} & Y \\ {\scriptstyle G}\downarrow & & {\scriptstyle g}\downarrow \\ X & \xrightarrow{f} & Z \end{array} .$$

Such commutative squares are said to be *Cartesian*. Geometrically we indicate it by drawing a small square inside this diagram.

One also says that a Cartesian square represents a *Base Change* of the space Y over Z, i.e., that the passage from Y to $X \times_Z Y$ is the produced by changing the base Z of Y to base X of the new space.

Example. (a) For a map $\pi : Y \to X$ and $X' \subseteq X$, the fibered product $Y \times_X X'$ is the inverse $\pi^{-1} X' \subseteq Y$ of X' under the map π. fiber $Y_x \stackrel{\text{def}}{=} \pi^{-1} x$ of the map at x. For instance, a point $k : x \in X$, the fibered product $Y \times_X x$ is the fiber $Y_x \stackrel{\text{def}}{=} \pi^{-1} x$ of the map at x.

(b) If $X, Y \subseteq Z$ then $X \times_Z Y$ is the intersection $X \cap Y$.

Appendix B. **Sheaves**

This is a quick summary of theory of constructible sheaves. A nice resource for this is the book *Cohomology of sheaves* by Iverson.

We use constructible sheaves in the Springer construction of irreducible representations of S_n in section 3.

B.1. **(Constructible) sheaves.** The constructible sheaves behave well Here we restrict to locally compact topological spaces. This allows us to double our direct/inverse image functoriality f_*, f^* by adding functors $f_!, f^!$ of direct/inverse image with *compact support.* The combination of these functors makes a powerful machinery, i.e., allows analyzing and calculating many things.

B.1.1. *Category $Sh(X, \mathcal{C})$ of sheaves on X with values in $\mathcal{C}$.* For any topological space X and "any" category $\mathcal{C}$ one can consider the category $Sh(X, \mathcal{C})$ of sheaves on X with values in $\mathcal{C}$. For us $\mathcal{C}$ will be usually be an abelian category and then the category $Sh(X, \mathcal{C})$ will again be be abelian.

Example. (i) When $\mathcal{C}$ is the category of modules $fm(R)$ over some ring R we denote $Sh(X, \mathfrak{m}(R))$ simply by $Sh(X, R)$. $\square$

(ii) $Sh(\mathrm{pt}, \mathcal{C}) = \mathcal{C}$. $\square$

For a sheaf $\mathcal{F}$ and any open $U \subseteq X$ we call the elements of the R-module $\mathcal{F}(U)$ the *sections* of $\mathcal{F}$ on U.

B.1.2. *Functoriality for sheaves.* Any map of topological spaces $X \xrightarrow{f} Y$ produces two functors $Sh(X, R) \xrightarrow{f_*} Sh(Y, R) \xrightarrow{f^*} D[Sh(X, R)$ called the direct and inverse image (push-forward and pull back).

Example. For a subspace $i : Y \hookrightarrow X$, $i^*\mathcal{F}$ is called the *restriction* $\mathcal{F}|Y$ of the sheaf from X to Y. For a point $i : a \in X$ the restriction $i^*\mathcal{F} = \mathcal{F}|a$ is called the *stalk* $\mathcal{F}_a$ of $\mathcal{F}$ at a.

Example. Notice that $Sh(\mathrm{pt}, R)$ is just the category $\mathfrak{m}(R)$ of R-modules. For any space X a module $M \in \mathfrak{m}(R) = Sh(\mathrm{pt}, R)$ gives the sheaf $M_X \overset{\text{def}}{=} (X \to \mathrm{pt})^* M$ which is said to be the *constant sheaf* on X given by M.

B.1.3. *Constructible sheaves.* A sheaf $\mathcal{F}$ on X is *constructible* if there is a stratification $X = \sqcup_1^N S_i$ such that all restrictions $\mathcal{F}|S_i$ are *local systems*, i.e., these sheaves are locally isomorphic to constant sheaves.

B.1.4. *Triangulated categories.* For an abelian category $\mathcal{A}$ let $C(\mathcal{A})$ be the category of complexes of objects in $\mathcal{C}$. It is again abelian but "too large", we want to "identify" complexes which "carry the same information". The basic information that a complex $A = (\cdots A_{-1} \xrightarrow{d} A_0 \xrightarrow{d} A_1 \xrightarrow{d} \cdots)$ carries are the*cohomology groups*

$$\mathrm{H}^n(A) \overset{\text{def}}{=} \mathrm{Ker}(A^n \xrightarrow{d} A^{n+1})/Im(A^{-1} \xrightarrow{d} A^n) \ \in \mathcal{A}.$$

A map of complexes $f : A \to B$ is a *quasi-isomorphism* if it an isomorphism on cohomology group, i.e., $\mathrm{H}^n(f) : \mathrm{H}^n(A) \xrightarrow{\cong} \mathrm{H}^n(B)$.

The *derived category* $D(\mathcal{A})$ is obtained from $C(\mathcal{A})$ by inverting all quasi-isomorphisms (objects are the same but there are more morphisms). This $D(\mathcal{A})$ belongs to the class of *triangulated categories* which approximately means: alike abelian categories but in the world of homological algebra.

In triangulated categories one has no kernels and images any more. Important feature of such categories $\mathcal{T}$ start with the shift functors $A \mapsto A[n]$, say for a complex A we get a complex $A[n], n \in \mathbb{Z}$; with $(A[n])^p = A^{p+n}$. Then a triangle is a diagram where all compositions are zero. One has no kernels or images but there is a replacement for the class of short exact sequences, is a certain class $\mathcal{E}$ of triangles that are called *exact* triangles.

Finally, any $a \in \mathcal{A}$ defines a complex which has a in degree 0 and other terms are zero. This gives an embedding of categories $\mathcal{A} \hookrightarrow D(\mathcal{A})$.

B.2. **Functoriality for the derived categories $D_c(X,R)$ of constructible sheaves.** Choose a ring R and associate to each topological space X the triangulated category $D[Sh(X,R)]$ of sheaves X with values in $\Bbbk$-modules. We will consider a triangulated subcategory $D(X,R) \overset{\text{def}}{=} D_c[Sh(X,R)]$ of *constructible complexes of sheaves.* Here, $\mathcal{F} \in D[Sh(X,R)]$ is said to be , *constructible* if for all integers n, the cohomology $\mathrm{H}^n(\mathcal{F}) \in Sh(X,R)$ is a constructible sheaf.[19]

Category $D_c(X,R)$ has the *Verdier duality* functor $\mathbb{D}_X : D_c(X,R)^o \xrightarrow{\cong} D_c(X,R)$ with $\mathbb{D}^2 = id$, which is a generalization of duality of vector spaces. Moreover, any map of topological spaces $X \xrightarrow{f} Y$ produces four functors

$$D_c(X,R) \xrightarrow{f_!, f_*} D_c(Y,R) \xrightarrow{f^*, f^!} D[Sh(X,\Bbbk)].$$

Remark. We previously had direct image f_*^{sh} for sheaves and now we have a direct image f_*^{dsh} for the derived category of sheaves. These are related but not the same. If $\mathcal{F}$ is sheaf, i.e., $\mathcal{F} \in Sh_c(X,R) \subseteq D_c[Sh(X,R)] = D_c(X,R)$ then $f_*^{sh}\mathcal{F}$ is a sheaf but $f_*^{dsh}\mathcal{F}$ is a complex of sheaves, the derived version carries more information since $f_*^{sh}\mathcal{F} = \mathrm{H}^0[f_*^{dsh}\mathcal{F}]$. From now on we denote $f_* \overset{\text{def}}{=} f_*^{dsh}$ and we think of $f_*^{sh}\mathcal{F}$ as $\mathrm{H}^0(f_*\mathcal{F})$.

The same also works for f^* and $f_!$.[20] The sheaf version $\mathrm{H}^0(f_!)$ of $f_!$ is understandable, it is the *direct image with compact support.* However, $f^!$ is something new, i.e., it does not have a natural incarnation without the derived category.

B.2.1. *Properties of functoriality of the construction $D_c(-,R)$.*

Theorem.

(1) For $X \xrightarrow{f} Y \xrightarrow{g} Z$ one has $(g \circ f)_? = g_? \circ f_?$ and $(g \circ f)^? = f^? \circ g^?$ for $? \in \{!, *\}$. Also, $(id_X)_? = id_{D_c(X,R)} = (id_X)^?$.
(2) $\mathbb{D}_X^2 \cong id_{D_c(X,R)}$ and $f_! = {}^{\mathbb{D}}(f_*) \overset{\text{def}}{=} \mathbb{D}_Y \circ f_* \circ \mathbb{D}_X$, $f^! = {}^{\mathbb{D}}(f^*) \overset{\text{def}}{=} \mathbb{D}_X \circ f_* \circ \mathbb{D}_Y$. (One is really conjugating with Verdier duality.)
(3) $(f_!, f^!)$ and (f^*, f_*) are adjoint pairs of functors.

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19 In some hidden way we use here the much larger category $D[Sh(X,R)]$ of all sheaves of R-modules on X. This is actually, a much smarter construction than the derived category $D[Sh_c(X,R)]$ of the abelian category of constructible sheaves.

20 Actually, f^*_{sh} is exact, hence which means that f^*_{dsh} is just f^*_{sh} applied to complexes.

(4) [Base Change.] For any Cartesian square of spaces (see A.1.3)

$$\begin{array}{ccc} \Sigma & \xrightarrow{F} & Y \\ {\scriptstyle G}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ X & \xrightarrow{f} & Z \end{array}$$

one can go from X to Y in two compatible ways

$$f^! g_* \;=\; G_! F^*.$$

(5) [Gluing or excision.]
(6) When f is smooth then $f^!\mathcal{G} \;\cong\; f^*\mathcal{G}[\dim_{\mathbb{R}}(f)]\otimes\, or_f$ where $\dim_{\mathbb{R}}(f) \stackrel{\text{def}}{=} \dim_{\mathbb{R}}(X) - \dim_{\mathbb{R}}(Y)$ and or_f is the local system of orientations of fibers of f.
(7) There is a canonical map of functors $f_! \to f_*$ and when f is proper this is an isomorphism.

Remarks. For the Base Change property it is necessary that one uses different kinds of functors ! and $*$) for direct and inverse images. So, Base Change is the place where having both ! and $*$ functorialities becomes essential for calculations.)

B.2.2. *Functoriality of constructible sheaves contains (co)homological constructions of algebraic topology.* For any space X we have a map $a = a_X : X \to \text{pt}$. For any R-module M, the pull back a^*M from the point is the constant sheaf M_X on X. The pull-back $a^!M$ is called the *dualizing sheaf on X with values in M*. For $M = R$ we denote the dualizing sheaf $a^!R$ by ω_X (or ω_X^R).

Lemma. (a) The dualizing sheaf of a real manifold X of dimension n is $\omega_X \cong\; or_X[\dim_{\mathbb{R}}(X)]$ for the orientation sheaf or_X on X.

If X is a complex manifold then or_X is canonically trivialized so $\omega_X \cong\; R_X[2\dim_{\mathbb{C}}(X)]$.

(b) The sheaf theory refines the cohomology $H^*(X, M)$ of X with coefficients in M to a complex $a_*a^*M = a_*M_X$ in $D_c(\text{pt}, R) = D(\mathfrak{m}(R))$ whose cohomology is $\mathrm{H}^*[a_*a^*M] = H^*(X, M)$. The same works for (compactly supported) cohomology and homology:[21]

(1) $H^*(X, M) \;=\; \mathrm{H}^*[a_*a^*M]$,
(2) $H_c^*(X, M) \;=\; \mathrm{H}^*[a_!a^*M]$,
(3) $H_*(X, M) \;=\; \mathrm{H}^*[a_!a^!M]$,
(4) $H_*^c(X, M) \;=\; \mathrm{H}^*[a_*a^!M]$.

[21] The compactly supported homology is usually called the *Borel-Moore homology.*

Remark. What is usually called the i^{th} homology $H_i^{standard}(X,M)$ is here in degree $-i$, hence it is denoted $H_{-i}(X,M)$. This shift of homology to negative degrees is a consequence of considering homology and cohomology on the same footing.[(22)]

B.3. **Appendix. Functoriality of functions.** The functoriality properties of sheaves are a version of such obvious properties for functions. Here, we consider only functions on finite sets.[(23)]

Appendix C. **Categories**

Appendix D. **Representations of finite groups as a Topological Field Theory [David Ben Zvi and David Nadler]**

In general, 2-dimensional oriented TFTs can be constructed from *symmetric Frobenius algebras.*

We will consider an oriented 2-dimensional TFT Z_Γ constructed from the algebra $\mathbb{Z}[\Gamma]$. It encodes all of the familiar structures in the complex representation theory of Γ. TFT Z_Γ has values in the 2-category of algebras $Alg_{\Bbbk}$ called *Morita theory*

D.0.1. *Symmetric monoidal 2-category $Alg_{\Bbbk}$.*

- objects of $Alg_{\Bbbk}$ are associative algebras A over $\Bbbk$;
- 1-morphisms $M\in \mathrm{Hom}^1_{Alg_{\Bbbk}}(A,B)$ in $\mathcal{M}$ are the $(B.A)$-bimodules that are flat over the source A;
- 2-morphisms in $Alg_{\Bbbk}$ are morphisms of bimodules $M,N\in \mathrm{Hom}^1_{Alg_{\Bbbk}}(A,B)$, i.e.,

$$f\in \mathrm{Hom}^2_{Alg_{\Bbbk}}(M,N)=\mathrm{Hom}_{B,A}(M,N).$$

$Alg_{\Bbbk}$ is a symmetric monoidal category, its tensor product is just the tensor product $\otimes_{\Bbbk}$ of $\Bbbk$-algebras and the unit in this monoidal category is just the algebra $\Bbbk$.

D.0.2. *Realization of the Morita 2-category $Alg_{\Bbbk}$ in $\Bbbk$-linear abelian categories.* Let $\mathcal{A}b_{\Bbbk}$ be the 2-category of small $\Bbbk$-linear abelian categories, where 1-morphisms are exact functors and 2-morphisms are natural transformations. of such functors.

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[22] Traditionally homology is calculated from "complexes of chains", i.e., with the differential going down: $d: C_i\to C_{i-1}$, while cohomology is calculated from "complexes of cochains", i.e., with the differential going up: $d: C^i\to C^{i+1}$,

[23] The direct image of functions involves integration of functions over fibers of the map, hence one needs elements of measure theory. We avoid such analytic questions by restricting to finite sets – when fibers are finite the integral over a fiber is just a sum of values at points in the fiber.

There are other settings in which functoriality exists without analysis: the *D-modules* (linear differential equations) and quasicoherent sheaves (algebraic geometry). These theories are both parallel to the functoriality for constructible sheaves.

Lemma. We can identify the Morita 2-category $Alg_{\Bbbk}$ with a full subcategory of the 2-category $\mathcal{A}b_{\Bbbk}$ by assigning to each algebra A its category $Perf(A)$ of perfect A-modules (these are the summands of finite colimits of free modules),

D.0.3. *The field theory Z_Γ.* It assigns to each cobordism C a linearization of the space of "Γ-gauge fields" on C, or in other words, the orbifold of principal Γ-bundles (also called Γ-torsors) over C.

In particular, it assigns the following to closed 0, 1 and 2-manifolds:

- To a point, Z_Γ assigns the group algebra:
$$Z_\Gamma(\mathrm{pt}) = \ \Bbbk[\Gamma] \in Alg_{\Bbbk}.$$
In the categorical terms we would say that Z_Γ of a point is the category of finite-dimensional representations of Γ:
$$Z_\Gamma(\mathrm{pt}) \stackrel{\mathrm{def}}{=} Rep^{fd}(\Gamma) \ \in \ \mathcal{A}b_{\Bbbk}.$$
Notice that this is the category of finite-dimensional algebraic vector bundles on the orbifold $Bun_\Gamma(\mathrm{pt}) = \mathbb{B}(\Gamma)$ of Γ-torsors on a point.
- To a circle, Z_Γ assigns the $\Bbbk$-modules of class functions on Γ:
$$Z_\Gamma(S^1) \ \stackrel{\mathrm{def}}{=} \ \Bbbk[\Gamma]^\Gamma \ = \ \Bbbk[\frac{\Gamma}{\Gamma}] \ \in \ Vec^{fd}_{\Bbbk} \ = \ 1_{Hom_{\mathcal{A}b_{\Bbbk}}(Vec_{\Bbbk}, Vec_{\Bbbk})}.$$
This is the vector space of functions on the orbifold $Bun_\Gamma(S^1)$ of Γ-bundles on the circle.
- To a closed surface, Z_Γ associates an rational number by counting Γ-bundles on Σ
$$Z_\Gamma(\Sigma) \stackrel{\mathrm{def}}{=} \ |Hom(\pi_1(\Sigma), Ga)/Ga| \ \in \ \Bbbk \ = \ \mathrm{Hom}^2_{\mathcal{A}b_{\Bbbk}}(\Bbbk, \Bbbk).$$
Here, as usual, a bundle P is weighted by $1/|Aut(P)|$ which is the information contained in P. The value $Z_\Gamma(\Sigma)$ is the volume of the *orbifold* $Bun_\Gamma(S)$ of Γ-bundles on the surface Σ.

D.0.4. *Cobordism Hypothesis.* From the point of view of the Cobordism Hypothesis, we only have to specify that we assign the group algebra $\Bbbk[\Gamma]$ to a point. This determines the rest of the TFT structure.

Lemma. (a) Any object of $Alg_{\Bbbk}$ is 1-dualizable, with dual given by the opposite algebra.

(b) 2-dualizable objects of $Alg_{\Bbbk}$ are precisely separable algebras, i.e., algebras for which A is projective as an A-bimodule.

Remarks. (1) Over $\mathbb{C}$ separable algebras are precisely finite-dimensional semi-simple algebras.

(2) Invariance under SO(2) amounts to the data of a non-degenerate trace, or in other words, the structure of a symmetric Frobenius algebra.