LIE ALGEBRAS (FALL 2017)

Contents

Date:

0. **Introduction**

0.1. **Topics.**

0.1.1. *The field* $\Bbbk$. We will work in vector spaces (usually finite dimensional) over a field $\Bbbk$ which is either $\mathbb{R}$ or $\mathbb{C}$. In representation theory it will often be convenient for the field to be algebraically closed so that operators have eigenvectors. Therefore in representation theory we will usually consider Lie algebras over $\mathbb{C}$.

0.1.2. *Lie algebras.* In the end we will see that these are just the *infinitesimal groups.*[1] However, in order to make sense of this claims requires some knowledge of the language of manifolds (or algebraic varieties).

We will postpone a recollection of manifolds in order to introduce Lie algebras in a simple way. We base the definition in 0.3.3 on a single example of general linear groups $GL_n(\Bbbk)$ in 0.3.1.[2]

0.2. **Manifolds and Lie groups.** Here we will only recall these ideas on the intuitive level. This will suffice for some examples of Lie algebras.

0.2.1. *Manifolds.* First, the idea of manifolds appears in several types of geometries which are distinguished by the type of functions $f : U \to \Bbbk$ (for U open in $\Bbbk$) that are used.

For $\Bbbk = \mathbb{R}$ this could be

- *continuous* functions $C(U, \mathbb{R})$ (this gives topological manifolds);
- *smooth* functions (i.e., infinitely differentiable functions) $C^\infty(U, \mathbb{R})$ (gives differentiable manifolds);
- *analytic* functions (which ar locally powers series) $C^\omega(U, \mathbb{R})$ (gives analytic manifolds).

For $\Bbbk = \mathbb{C}$ this will be the holomorphic functions $\mathcal{H}(U, \mathbb{C})$ (gives holomorphic manifolds).

In each case we will denote this class of "allowed" functions by $\mathcal{O}(U)$. Notice that when $\mathcal{O}$ is any of these classes of $\Bbbk$-valued functions on U's open in $\Bbbk$, it extends to class of $\Bbbk$-valued functions $\mathcal{O}(U)$ on any U open in any $\Bbbk^n$, and then it further extends to a class of maps $Map_{\mathcal{O}}(U, V)$ for any open $U \subseteq \Bbbk^n$ and V open in $\Bbbk^m$. (we may define it simply $Map(U, V)$ when we think that the class $\mathcal{O}$ is obvious.)

A *manifold* of class $\mathcal{O}$ over $\Bbbk$ is a topological space M with a consistent system of local identifications (called *charts*) with open subsets of $\Bbbk^n$. (Here 'consistent" means that the differences between these trivializations are in the class $\mathcal{O}$ of maps between open subsets of $\Bbbk^n$'s.)

[1] A more precise claim is that there is a notion of infinitesimally small groups and these equivalent to Lie algebras (over fields of characteristic zero).

[2] This only requires calculations in $\Bbbk^N$.

Now, manifolds of a given class $\mathcal{O}$ form a category, i.e., for two $\Bbbk$-manifolds M, N we have the allowed class of maps $Map_{\mathcal{O}}(M, N)$ (the maps that are in $\mathcal{O}$ when rewritten in terms of local charts).

0.2.2. *Lie groups in class $\mathcal{O}$.* These are groups $(G, \cdot)$ with a compatible structure of an $\mathcal{O}$-manifold. The compatibility requirement is that the multiplication $G \times G \dot{\to} G$ is a map of manifolds (i.e., that locally, after identification with parts of $\Bbbk^n$, it is in the class $\mathcal{O}$).

0.3. **Lie algebras: definition and examples.**

0.3.1. *The relation of the group GL_n and the vector space M_n.* M_n happens to be a $\Bbbk$-algebra and motivated by the following lemma on M_n we will consider the commutator $[x, y] \stackrel{\text{def}}{=} xy - yx$ in $M_n(\Bbbk)$.

Lemma. (a) There is a well defined exponential map $M_n(\Bbbk) \to GL_n(\Bbbk)$.

(b) It restricts to an isomorphism between certain neighborhoods of $0 \in M_n(\Bbbk)$ and $1 \in GL_n(\Bbbk)$.

(c) $e^{ux}e^{ux}e^{ux}e^{ux} \;=\; 1 + u^2[x, y] \;+\; O(u^3)$. □

Remark. So, M_n sees GL_n near 1 and the commutator in GL_n for elements near 1/

0.3.2. *Commutator operation in an associative $\Bbbk$-algebra.* We will use the lemma 0.3.1 as a motivation to consider the properties of the commutator operation.

Lemma. If A is an associative $\Bbbk$-algebra then

(a) (Antisymmetry) $[y, x] = -[x, y]$;

(b) (Jacobi identity)

$$[x, [y, z]] \;+\; [y, [z, x]] \;+\; [z, [x, y]] \;=\; 0.$$

□

Remark. The Jacobi identity is an infinitesimal rewriting of associativity. The proof uses only associativity in A.

0.3.3. *Lie algebras definition.* A *Lie algebra* over a $\Bbbk$ is a vector space $\mathfrak{g}$ together with a bilinear operation $[-,-]$ which is antisymmetric and satisfies the Jacobi identity.

(Ex 0.) Any associative algebras $(A,\cdot)$ gives a Lie algebra $(A,[-,-])$ for the commutator operation. [This is actually our motivation for the definition of Lie algebras. A better motivation comes from Lie groups.]

(Ex 1.) A subspace $\mathfrak{h}$ of a Lie subalgebra $\mathfrak{g}$ is (said to be) a *Lie subalgebra* if it is closed under the bracket in $\mathfrak{g}$. Such $\mathfrak{h}$ is naturally a Lie algebra.

Lemma. (Ex 2.) For any associative $\Bbbk$-algebra A the set of derivations of A

$$Der_{\Bbbk}(A) \stackrel{\text{def}}{=} \{\alpha \in \ \mathrm{End}_{\Bbbk}(A);\ \alpha(ab) = \ \alpha(a)b + a\alpha(b)\}$$

is a Lie subalgebra of the associative algebra $\mathrm{End}_{\Bbbk}(A)$.

Example. (Ex 2'.) Let M be a $\Bbbk$-manifold Then $\mathcal{V}(M) \stackrel{\text{def}}{=} \mathrm{Der}_k[\mathcal{O}(M)]$, the derivations of functions on M, is a Lie algebra called *vector fields on M*

Lemma. For $M = \Bbbk^n$ the vector fields are a free module over functions with the basis given by partial derivatives

$$\mathcal{V}(\Bbbk^n) \ = \ \oplus_1^n \mathcal{O}(\Bbbk^n)\frac{\partial}{\partial x_i}.$$

0.3.4. *Actions of groups.* **A. Actions on sets.** An *action of a group* $(G,\cdot)$ on a set X is a map $* : \ G\times X \ \to X$ (we denoted $*(g,x)$ by $g*x$ such that $e*x = x$ and $g*(h*x) = \ (g\cdot h)*x$. [We often denote $g*x$ simply by gx. Another notational possibility is to denote for each $gh \in G$ by $\pi(g)$ the map $X \to X$ given by by the action, i.e., $\pi(g)\ (x) \ = \ g*x$.]

Remark. When G acts on X we say that G is a *symmetry* of X. Interesting groups arise as symmetries of objects.

Lemma. When G acts on X then for each $a \in X$ its *stabilizer* $G_a = \{g \in G;\ g*a = a\}$ is a subgroup of G. □

B. The induced actions on the sets of structures on a given set X. A structured sets is a pair (X,Σ) of a set with some structure Σ on X. For simplicity we will not formally define what we mean by a structure on a set, or a class of structures on sets, but it will be clear from examples.

Lemma. For any class $\mathcal{S}$ of structures on n-tuples of sets $(X_1,...,X_n)$, an action of a group $(G,\cdot)$ on sets X_i extends to an action of G on the set $\mathcal{S}(X_1,...,X_n)$ of structures of type $\mathcal{S}$ on $(X_1,...,X_n)$. □

Remark. This is a part of the *transport of action* principle:

"If G acts on each of the sets X_i then it acts on any set naturally produced from X_i's."

Examples. (0) If G acts on the sets X and Y then it acts on the set $Map(X,Y)$ of maps from X to Y. For a function $\phi : X \to Y$ any $g \in G$ defines a new function by the "conjugation" formula ${}^g\phi(x) \overset{\text{def}}{=} g *_Y f(g^{-1} * x)$.

(1) If G acts on all X_i it acts on $\prod_{i\in I} X_i$ and $\sqcup_{i\in I} X_i$.

(2) If G acts on X it also acts on the set $BO(X)$ of all binary operations on X. (G acts on X, hence on $X\times X$ and then also on $Map(X\times X, X) = BO(X)$ of all binary operations on X. Here, $g \in G$ acts on an operation $\circ : X\times X$ by ${}^{(g}\circ)(x,y) \overset{\text{def}}{=} g*(\circ\,[g^{-1} * {}^{-1}(x,y)])$, i.e., $x({}^g\circ)y \overset{\text{def}}{=} g * [g^{-1} * x \circ g^{-1} * y]$. □

C. Actions on structured sets. An action of a group $(G,\cdot)$ on a *set X with a structure* Σ is an action of G on X which preserves Σ (i.e., if Σ belongs to the class $\mathcal{S}$ of structures on sets then we require that g is in the stabilizer of Σ for the action of G on $\mathcal{S}(X)$).

Examples. [Actions that preserve structures on sets.] (0) If G acts on the sets X and Y then it preserves a map $\phi \in Map(X,Y)$ if ${}^g\phi = \phi$, i.e., we have that $({}^g\phi) = g* \phi(g*_X{}^{-1}x)$ equals $\phi(x)$ for all $x \in X$. When we write $y = g^{-1}x \in X$, the condition becomes that for all $y \in X$ we have $g * \phi(y) = \phi(g * y)$. So, the whole group G preserves the function ϕ iff ϕ is a G-map from X to Y (for the standard notion of G-maps).

(1) If the structure Σ on X is that of a $\Bbbk$-manifold then $g \in G$ preserves Σ iff the map $g*- : X \to X$ is a map of manifolds. (Then $g*-$ is actually an isomorphism of manifolds with the inverse $g^{-1} * -$.)

(2) Similarly, when G acts on a set X then G preserves the operation $\circ$ on X if for all $x, y \in G$ one has $({}^g\circ)(x,y) \overset{\text{def}}{=} g*(\circ\,[g^{-1} * {}^{-1}(x,y)])$, i.e., ${}^g(x\circ y) = g^x \circ g^y]$. □

Lemma. (a) An action of a group G on a vector space V is the same as an action of G on the set V by linear operators.

(b) This is the same as a homomorphism of groups $\pi : G \to GL(V)$.

Proof. (a) We notice that the phrase "vector space V" means a structured set, the set V with operations $+$ of addition and $\cdot$ of multiplication with a scalar. So, an action on the vector space V really means an action on the set V which preserves these operations, i.e., $g*(u+v) = g*u + g*v$ and $g*(c\cdot v) = c\cdot(g*v)$ for $c \in \Bbbk$. If we denote by $\pi(g) : V \to V$ the action $g*-$ of g on V this is exactly the requirement that the action operators $\pi(g)$ are linear.

Then automatically π is a homomorphism, i.e., $\pi(e) = id_V$ (since $\pi(e)v = e*v = v$) and $\pi(gh) = \pi(g)\cdot\pi(h)$ (since $\pi(gh)v = gh*v = g*(h*v)) = \pi(g)\pi(h)v$. In particular, $\pi(g) \in GL(V)$ with the inverse $\pi(g^{-1})$. □

Remark. An action of a group G on a vector space V is called a *representation* of G on V. Usually it is written as a map of groups $\pi : G \to GL(V)$.

D. The "transport of action" principle. It says that

"If G acts on each of the structures sets (X_i, Σ_i) then G acts on any structured set naturally produced from (X_i, Σ_i)'s."

Here are some examples.

Lemma. (a) An action of G on set X gives an action of G on the algebra of functions on X.

(b) An action of G on a $\Bbbk$-manifold M of class $\mathcal{O}$ gives on

- an action of G on the algebra $\mathcal{O}(M)$ of $\mathcal{O}$-functions on M and also
- an action of G on the Lie algebra $\mathcal{V}(M)$ of vector fields on M.

Proof. (b) Group G acts on the set of all functions $f : X \to \Bbbk$ using the given action of G on M and the trivial action of G on $\Bbbk$, i.e., ${}^g f(x) \overset{\text{def}}{=} f(g^{-1} * x)$. The phrase "$\Bbbk$-algebra of functions" means a structured set of function with operations of addition, multiplication with a scalar and the multiplication of functions. Preserving these structures means that

$$ {}^g(\phi + \psi) = {}^g\phi + {}^g\psi, \ {}^g(\phi \cdot \psi) = {}^g\phi \cdot {}^g\psi \quad \text{and} \quad {}^g(c \cdot \psi) = c \cdot {}^g\psi $$

for $c \in \Bbbk$.

(c) Again, if we consider a manifold as a structured set then the phrase "G acts on the manifold M" means that G acts on the set M and that all maps $g * - : M \to M$ are morphisms of manifolds.

Then G acts on the algebras of all function $f : M \to \Bbbk$ as in (b). However, we need that this action ${}^g\phi = g \circ \phi \circ g^{-1}$ preserves the subalgebra of differentiable functions on M. This is so because $g : M \to M$ is differentiable.

Finally, by transporting actions, since G acts on manifold M it also acts on

- the algebra $\mathcal{O}(M)$;
- the associative algebra $\mathrm{End}_{\Bbbk}(M)$, hence also on
- the Lie algebra of $\mathrm{Der}_{\Bbbk}[\mathcal{O}(M)] = \mathcal{V}(M)$ of derivations of M.

□

Remark. In part (c), the action of G on functions f and vector fields ξ are compatible with the action of vector fields on functions, i.e., ${}^g\xi \, {}^g\phi = {}^g(\xi\phi)$. This comes from the definition of the action on vector fields by transport of structure (if you write the proof of (c) in detail). □

E. Invariants of actions. When G acts on a set X, the set of G-invariants in X is

$$X^G \stackrel{\text{def}}{=} \{x \in X;\ g * x = x \ \text{ for all } g \in G\}.$$

Lemma. (a) If G acts on a set X and trivially on the point pt then X^G is the set of G-maps from pt to X.

(b) If G acts on a structured set (X, Σ) then X^G inherits the structure of the same kind as Σ. □

Example. If G acts on a Lie algebra $\mathfrak{h}$ then the invariants $\mathfrak{h}^G \subseteq \mathfrak{h}$ are a Lie subalgebra of $\mathfrak{g}$.

0.3.5. *The Lie algebra $Lie(G) = T_eG$ of a Lie group G.*

Proposition. For any Lie group G its tangent space T_eG (at the neutral element) is canonically a Lie algebra.

Proof. A group G acts on the set G by left translations by $L_g(u) = gu$ and by right translations $R_g(u) = ug^{-1}$. These actions commute so we get an action of $G \times G$ on G by $(g, h)u = \ guh^{-1}$. (Via the diagonal embedding $G \hookrightarrow G \times G$ we also get the diagonal action of G on G by conjugation $C_g(u) = gug^{-1}$.) Therefore G acts on the Lie algebra $\mathcal{V}(G)$ of vector fields on G in three ways L, R, G.

We define the *left invariant vector fields on G* as the invariants $\mathcal{V}(G)^{G \times 1}$ of the left multiplication action. We know that this is a Lie subalgebra of all vector fields. This makes T_eG a Lie algebra via the isomorphism

$$\mathcal{V}(G)^{G \times 1} \ \cong \ T_e(G).$$

We see this from the dual nature of vector fields on a manifold M and on the other they are sections of the tangent vector bundle TG. Now, the left action of G on the Lie group G can be used trivialize the tangent vector bundle TG. For any $g \in G$ we have an isomorphism $L_g : G \xrightarrow{\cong} G$ which takes e to G, so its differential $d_eL_g :\ T_eG \to T_gG$ is an isomorphism.

Therefore, for any $v \in T_eG$ there is precisely one left invariant vector field ξ on G such that its value at e is V. (A vector field ξ is left invariant iff at each g in G we have $\xi(g) = \ (d_eL_g)\ \xi(e)$).)

So, the evaluation at e is an isomorphism of vector spaces. Now we can use it to transport the Lie algebra structure from $\mathcal{V}(G)^{G \times 1}$ to T_eG. □

Remarks. (0) Now one should check that the commutator bracket on $M_n(\Bbbk)$ is the same as the Lie algebra structure on $T_1GL_n(\Bbbk)$ defined as left invariant vector fields on G.

(1) One can also use the right multiplication action of G on G to trivialize TG and this gives another Lie algebra structure on T_eG which is now identified with $\mathcal{V}(G)^{1\times G}$.

We define the *Lie algebra of a Lie group* G (usually denoted $\mathfrak{g}$) to be T_eG with the bracket coming from left invariant vector fields.

1. Representations of Lie algebras

1.1. **Representations of groups.** We have defined representations of a group G on a vector space V as synonymous with an action of G on the vector space V (viewed as a set with the structure of a vector space).

The simplest example of a representation arises from sets – when G acts on a set X then it acts on the algebra $Map(X,\Bbbk)$ of $\Bbbk$-valued functions on X, so $Map(X,\Bbbk)$ is a representation of G. If X has a structure of an $\mathcal{O}$-manifold then the same holds for the interesting vector space of functions $\mathcal{O}(X)$. (Actually any vector space produced from X is a representation of G, such as differential forms $\Omega^p(X)$ or vector fields $\mathcal{V}(X)$ on X.)

1.1.1. *Category $Rep_\Bbbk(G)$ of representations of G over $\Bbbk$.*

Lemma. (a) Representations of G over $\Bbbk$ form a category $Rep_\Bbbk(G)$.

(b) This is an *abelian* category, i.e., it behaves like the category $\mathcal{A}b$ of abelian groups or any of the categories $\mathfrak{m}(A)$ of modules over an associative algebra A.

Proof. (a) For two representations U,V one defines $\mathrm{Hom}_{Rep_\Bbbk(G)}(U,V)$ (denoted simply $\mathrm{Hom}_G(U,V)$) as all linear maps functions $\alpha: U\to V$ such that $\alpha(gv)= g(\alpha v)$ for $v\in V$ and $g\in G$.[3]

The meaning of (b) is that a map of representations $\alpha: U\to V$ has the kernel $\mathrm{Ker}(\alpha)\subseteq U$, image $Im(\alpha)\subseteq V$ which are again representations of G and these satisfy all properties that hold in $\mathfrak{m}(A)$. In particular, there are notions of (short) exact sequences, of a subrepresentation V' of V (a subspace V' of V which is invariant under G is a representation) and the quotient vector space V/V' is again a representation. □

In particular one can say that a representation V of G is irreducible if the only subrepresentations are the trivial (obvious) ones: 0 and V. If a representation V is not irreducible then it has a proper subrepresentation U and then one says that V is an *extension* of the quotient representation V/U by the subrepresentation U. So, the irreducible subrepresentations of G are the basic building blocks of the representation theory of G.

[3] In other words as all functions $\alpha: U\to V$ that are compatible with the structure that is present, that of a vector space and of a G-action.

1.1.2. *The Representation Theory Strategy.* It deals with the problem of "understanding the structure of interesting representations", i.e., how it is built by successive extensions of irreducible representations.

So, the steps in a representation theory of G are roughly

(1) Find the list $Irr(G)$ of all irreducible representations of G,
(2) Study each $V \in Irr(G)$ in detail (dimension, character,...).
(3) Study how "interesting representations" are built from ("well understood") irreducible representations.

The idea is that if (1-2) have been accomplished for G then they can be used as in (3) in any setting where G appears as a symmetry. So, (1-2) can be considered as the abstract part and then the concrete applications appear in step (3) when concrete problems supply interesting representations of G.

1.1.3. *Harmonic Analysis.* The Harmonic Analysis studies spaces X in terms of relevant functions $\mathcal{O}(X)$ on X. If X has G for a symmetry then $\mathcal{O}(X)$ is a representation of G. The the understanding of this representation can be called the "organization of our thinking about X in terms of its symmetry G".

More abstractly , the Representation Theory is the principle that

"Any subject should be organized in terms of its symmetries".

1.1.4. *Example: Fourier transform.* The Fourier transform is the particular case of organization of functions according to symmetries. Here the space on which G acts is G itself, so one organizes functions on G in terms of $Irr(G)$.

We will here consider only the case when G is a commutative group and we will start with the case of the circle group.

The irreducible representations of the circle group $\mathbb{T} = \{s \in \mathbb{C};\ |s| = 1\}$ are the 1-dimensional representations $\chi_n : \mathbb{T} \to \mathbb{C}^* = GL_1$ by $\chi_n(s) = s^n$. It appears as a subrepresentation $\mathbb{C}\chi_n$ of $C^\infty(\mathbb{T}) \subseteq L^2(\mathbb{T})$. The decomposition of L^2 functions on $\mathbb{T}$ into irreducible representations is as a sum of Hilbert spaces

$$L^2(\mathbb{T}) \ \cong\ \oplus_{n\in\mathbb{Z}}\ \mathbb{C}s^n.$$

Remark. Notice here that that $\mathbb{Z}$ appears as $Irr(\mathbb{T})$ and that $Irr(\mathbb{T}) = \mathbb{Z}$ is itself a group. The above decomposition now says that L^2-functions on the group $\mathbb{T}$ are the same as L^2-functions on the group $\mathbb{Z}$.

Theorem. This remains true for all commutative groups A: $Irr(A)$ is a group (called the dual group of A) and one has a canonical isomorphism

$$L^2(A) \ \cong\ L^2(Irr(A)). \quad \square$$

As in the above example this isomorphism is called the *Fourier transform* for the group A.

Example. For a vector space V over $\mathbb{R}$ The dual group $Irr(V)$ is the dual vector space V^* in the sense that any $\alpha \in V^*$ gives a 1-dimensional representation $V \xrightarrow{e_\alpha} \mathbb{C}^* = GL_1(\mathbb{C})$ by $e_a l(v) \stackrel{\text{def}}{=} e^{2\pi i\langle\alpha,v\rangle}$. Then $L^2(V)$ is a continuous(!) sum, i.e., an integral of irreducible representations e_α of V, the meaning of this is that for $f \in L^2(V)$ we have the *Fourier inversion formula*

$$f(v) \;=\; \int_{V^*} c_\alpha \; e^{2\pi i\langle\alpha,v\rangle} \; d\alpha$$

where the coefficients c_α are given by the standard Fourier transform of f.

Functions on an interval. Classically one is rather interested in functions on an interval, say on. $[0, 2\pi]$. This interval maps to $\mathbb{T}$ by $\theta \mapsto e^{2\pi i\theta}$ and then the periodic functions on the interval $[0, 2\pi]$ are the same as functions on $\mathbb{T}$. So, function $\chi^n(s) = s^n$ now becomes the periodic function $e^{2\pi in\theta}$ on $[0, 2\pi]$. Moreover, when one passes to L^2-functions the "periodic function on $[0, 2\pi]$ does not make sense and we get an isomorphism of spaces of all functions $L^2[0, 2\pi] \;\cong\; L^2(\mathbb{T})$.

Now, irreducible representations of $\mathbb{T}$ give the basis $e^{2\pi in\theta},\; n \in \mathbb{Z}$ of $L^2(\mathbb{T}, \mathbb{C})$ of complex valued functions on $[0, 2\pi]$. In classical physics complex numbers were not meaningful so one would talk in terms of the the basis $\cos(2\pi in\theta),\; \cos(2\pi in\theta)$ of $L^2(\mathbb{T}, \mathbb{R})$.

Therefore, the classical analysis on an interval, for instance solving a heat equation, has representation theoretic background which need not always be manifest.

1.1.5. *Harmonic analysis on homogeneous spaces.* The case when symmetry G of X completely controls X itself is when X is a homogeneous space of G, i.e., $X = G/A$. This is the case when the decomposition of $\mathcal{O}(X)$ as a representation of G organizes $\mathcal{O}(X)$ the best.

The sphere S^{n-1} in $\mathbb{R}^n$ is a homogeneous space of the orthogonal group $O(n)$.

A 2d disc is a homogeneous space of the group $SL(2, \mathbb{R})$.

1.2. **Representations of Lie algebras.**

1.2.1. *Representations.* We define a *homomorphism of Lie algebra* from $\mathfrak{g}$ to $\mathfrak{h}$ as a function $\phi : \mathfrak{g} \to \mathfrak{h}$ which preserves all structures, i.e., ϕ is linear and $\phi([x, y]_\mathfrak{g}) = \; [\phi x, \phi y]_\mathfrak{h}$.

Then a *representation of a Lie algebra* $\mathfrak{g}$ on a vector space V is defined as a homomorphism of Lie algebras $\pi : \mathfrak{g} \to gl(V)$.

Remark. To make the analogy with groups more complete we define an action of a Lie algebra $\mathfrak{g}$ on a manifold M as a homomorphism of Lie algebras $\pi : \mathfrak{g} \to \mathcal{V}(M)$. So. Lie algebras act on manifolds by vector fields!

Lemma. For any x in the Lie algebra $\mathfrak{g}$ denote by $ad\ x = [x, -] : \mathfrak{g} \to \mathfrak{g}$ the linear operator $(ad\ x)(y) \stackrel{\text{def}}{=} [x, y]$. Then the map $ad : \mathfrak{g} \to \mathrm{End}_{\Bbbk}(\mathfrak{g})$ is a Lie algebra representation. (We call it the*adjoint* representation of $\mathfrak{g}$.)

1.2.2. *Multilinear algebra of representations.*

Lemma. (a) Any vector space U carries the *trivial* representation of any Lie algebra $\mathfrak{g}$ – we just let all $x \in \mathfrak{g}$ act by the 0 operator.

(b) For a representation (π, V) of a Lie algebra $\mathfrak{g}$ show that on the dual vector space V^* we have a representation π^* of $\mathfrak{g}$ when the action $\pi^*(x)\phi$ for $x \in \mathfrak{g},\ \phi \in V^*$, is defined by

$$\langle \pi^*(x)\phi, v\rangle \ = \ -\langle \phi, \pi(x)v\rangle.$$

(c) For two representations V, U of a Lie algebra $\mathfrak{g}$ show that on the tensor product $U \otimes V$ there is a well defined representation of $\mathfrak{g}$ such that for $u \in U, , v \in V,\ x \in \mathfrak{g}$

$$x(u \otimes v) \ = \ xu \otimes v \ + \ u \otimes xv.$$

(d) For two representations V, U of a Lie algebra $\mathfrak{g}$ show that the space of linear operators $\mathrm{Hom}_{\Bbbk}(U, V)$ is a $\mathfrak{g}$ by such that $x \in \mathfrak{g}$ acts on $A \in \mathrm{Hom}_{\Bbbk}(U, V)$ by the commutator formula

$$(xA)u \ \stackrel{\text{def}}{=} \ x(Au) \ - \ A(xu) \ \text{ for } \ u \in U.$$

□

Remark. The canonical map $V \otimes_{\Bbbk} U^* \xrightarrow{\iota} \mathrm{Hom}_{\Bbbk}(U, V)$ is a $\mathfrak{g}$-map. □

1.2.3. *Schurr lemma.* The following is an expression of the idea that irreducible representations are as simple as possible.

Lemma. Let $\Bbbk = \mathbb{C}$. If U, V is an irreducible finite dimensional $\mathfrak{g}$-modules then $\mathrm{Hom}_{\mathfrak{g}}(U, V) = \mathbb{C}$ if $U \cong V$ and zero otherwise.

Proof. For any $\alpha \in \mathrm{Hom}_{\mathfrak{g}}(U, V)$, $\mathrm{Ker}(\alpha) \subseteq U$ and $Im(\alpha) \subseteq V$ are submodules, so they have to be either 0 or the whole representation. Therefore, if $\alpha \neq 0$ we see that $\mathrm{Ker}(\alpha) = 0$ and $Im(\alpha) = V$. So, if $\alpha \neq 0$ then it is an invertible linear operator hence an isomorphism of $\mathfrak{g}$-modules.

It remains to prove that any $\alpha \in \mathrm{Hom}_{\mathfrak{g}}(V, V)$ is a scalar. Since V is irreducible, it is not 0 hence α has an eigenvalue λ. Then $\alpha - \lambda 1_V$ is in $\mathrm{Hom}_{\mathfrak{g}}(V, V)$ and it is not invertible. So, $\beta - \lambda 1_V = 0$. □

1.2.4. *Extensions in an abelian category.* For any Lie algebra $\mathfrak{g}$ the category $Rep(\mathfrak{g})$ of its representations of is an *abelian category*, meaning that it behaves like the category of modules over an associative algebra.[4]

In particular there are short exact sequences $0 \to U \to V \to W \to 0$ of $\mathfrak{g}$-representations. We say that V is an *extension* of W by U.

A *splitting* of the short exact sequences $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ means any of the following equivalent data

(1) A submodule W' of V complementary to U.
(2) A *section* β' of β, i.e., a $\mathfrak{g}$-map $\beta' : W \to V$ such that $\beta' \circ \beta = id_W$.
(3) A *retraction* α' of α, i.e., a $\mathfrak{g}$-map $\alpha' : V \to U$ such that $\alpha \circ \alpha' = id_U$.

Proof. We check that the data are indeed equivalent. For instance W' gives α' as the composition $V \cong U \oplus W' \xrightarrow{pr_1} U$ and α' gives $W' = \mathrm{Ker}(\alpha')$. □

1.2.5. *How representations are built from irreducibles.*

Lemma. Consider the category $\mathcal{R} = Rep^{fd}(\mathfrak{g})$ of finite dimensional modules for a Lie algebra $f\mathfrak{g}$.

(a) Any object V has a *Jordan-Hoelder* series, meaning a sequence of submodules $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$, such that all V_i/V_{i-1} are irreducible.

(b) The following is equivalent for category $\mathcal{R}$

- (i) all short exact sequences split;
- (ii) all modules are direct sums of irreducible modules;
- (iii) all extensions of irreducible modules split.

If this holds and $L_k,\ k \in K$ is a complete list of representatives of isomorphism classes of irreducible $\mathfrak{g}$-modules, then for any $V \in \mathcal{R}$ the canonical map

$$\oplus_k \mathrm{Hom}_{\mathfrak{g}}(L_k, V) \otimes L_k \xrightarrow{\iota} V, \quad \iota(\sum_k A_k \otimes l_k) \stackrel{\mathrm{def}}{=} \sum_k A_k l_k$$

is an isomorphism of $\mathfrak{g}$-modules.[5]

Proof. (a) is proved by induction on $\dim(V)$.

(b) We will prove that (i) is equivalent to (ii) in (b_1-b_4) below.

[4] Later we will see that $Rep(\mathfrak{g})$ actually is the category of modules over an associative algebra called the *enveloping algebra* $U\mathfrak{g}$ of $\mathfrak{g}$.

[5] Here, the vector spaces $\mathrm{Hom}_{\mathfrak{g}}(L_k, V)$ carry the trivial $\mathfrak{g}$-action, i.e., they are just the multiplicities of representation L_k in V.

(b_1) First we notice that if all SES split then any objects V is a sum irreducibles. The reason is that a JH-series of V, submodule V_{i-1} of V_i has a complement C_i and then $C_i \cong V_i/V_{i-1}$ hence C_i as irreducible and $V = V_n = V_{n-1} \oplus C_n = \cdots = C_1 \oplus \cdots \oplus C_n$.

(b_2) Second, for a module V which is a sum of irreducible submodules $V = \oplus_1^N C_i$ we check the decomposition $V \cong \oplus_k \operatorname{Hom}_{\mathfrak{g}}(L_k, V) \otimes L_k \xrightarrow{\iota} V$. If $L_k,\ k \in K$, are as above, a complete list of non isomorphic irreducible modules in $\mathcal{R}$, then each C_i is isomorphic to $L_{k(i)}$ for unique $k(i) \in K$. Then $\operatorname{Hom}_{\mathfrak{g}}(L_{k(i)}, C_i)$ is 1-dimensional and we can regard this line as the difference between $L_{k(i)}$ and C_i since the canonical map

$$\mathcal{M}_i \otimes L_{k(i)} \to C_i, \quad A \otimes v \mapsto Av;$$

is an isomorphism of vector spaces (and of $\mathfrak{g}$-modules when the line $\mathcal{M}_i$ is regarded as a trivial $\mathfrak{g}$-module.)

Now,

$$\operatorname{Hom}(L_k, V) \ = \ \oplus_{i=1}^N \operatorname{Hom}_{\mathfrak{g}}(L_k, C_i) \ = \ \oplus_{i,\ k(i)=k} \operatorname{Hom}_{\mathfrak{g}}(L_k, C_i) \ = \ \oplus_{i,\ k(i)=k} \mathcal{M}_i;$$

and therefore

$$\begin{aligned} V \ \cong \ \oplus_{i=1}^N C_i \ \cong \ \oplus_{k \in K} \oplus_{i,\ k(i)=k} C_i \ \cong \ \oplus_{k \in K} \oplus_{i,\ k(i)=k} \mathcal{M}_i \otimes L_k \\ \cong \ \oplus_{k \in K} \operatorname{Hom}(L_k, V) \otimes L_k. \end{aligned}$$

(b_3) Assume that all modules in $\mathcal{R}$ are sums of irreducibles, i.e., $V \cong \oplus_k M_k \otimes L_k$ for the multiplicity vector spaces $M_k = \operatorname{Hom}_{\mathfrak{g}}(L_k, V)$. Then all submodules $V' \subseteq V$ are of the form $\oplus_k M'_k \otimes L_k$ for some choice of vector subspaces $M'_k \subseteq M_k$.

The reason is that any submodule V' is also of the form $\oplus_k M'_k \otimes L_k$ for its multiplicity vector spaces $M'_k = \operatorname{Hom}_{\mathfrak{g}}(L_k, V')$. Then the inclusion $V' \subseteq V$ gives $M'_k \subseteq M_k$.

(b_4) Now it is clear that if all modules are sums of irreducibles then any submodule has a complement (just choose complementary vector subspaces for M'_k in M_k). $\square$

2. Representation theory of sl_2

When we consider representation theory then $\Bbbk$ is assumed to be $\mathbb{C}$ (unless otherwise stated).

2.1. Lie algebra sl_2.

2.1.1. *Lie algebras sl_n.* We start with the example $gl(V) \stackrel{\text{def}}{=} \mathrm{End}_{\Bbbk}(V)$ of the Lie algebras of operators on a vector space V with the commutator bracket.

Lemma. (a) For a finite dimensional vector space V the subspace $sl(V) \subseteq gl(V)$ of traceless operators is a Lie subalgebra.

(b) Group $GL(V)$ acts on the Lie algebra $gl(V)$ by conjugation. This action preserves the Lie subalgebra $sl(V)$.

Proof. (b) For any associative $\Bbbk$-algebra A, the group A^* of invertible elements of A acts on the associative algebra $(A, +, \cdot)$ by conjugation. Then by the transport of action A^* also acts on the Lie algebra $(A, [-,-])$ by conjugation. (Because the Lie algebra structure was constructed from the associative algebra structure by $[x, y] \stackrel{\text{def}}{=} xy - yx$.

So, the group $GL(V)$ of invertible elements of the algebra $\mathrm{End}(V)$ acts (by conjugation) on the algebra $\mathrm{End}(V)$ and on the associated Lie algebra $gl(V)$. The conjugation preserves the trace hence it also preserves the Lie subalgebra $sl(V)$. $\square$

It turns out that the Lie algebras sl_n has interesting structure and an interesting representation theory. We will learn these in detail and then we will extend this into a theory of representations of the class of *semisimple* Lie algebras. We will start here with the case of $\mathfrak{g} = sl_2$.

2.1.2. *Lie algebra sl_2.*

Lemma. (a) The following elements of sl_2 form a basis (called the *standard basis*)

$$e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \quad h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

(b) It satisfies

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \quad \square$$

Corollary. [e and f have "symmetric" roles in sl_2.] There is a unique automorphism of the Lie algebra sl_2 that takes e, h, f to $f, -h, e$. $\square$

2.2. Weights and primitive vectors.

2.2.1. *The weights of a representation.* The *weights* λ in a representation V of $\mathfrak{g} = sl_2$ are defined as eigenvalues λ of h in V. One calls the λ-eigenspace

$$V_\lambda \stackrel{\text{def}}{=} \{v \in V;\ hv = \lambda v\}$$

the *weight λ subspace* of V and $\dim(V_\lambda)$ is called the *multiplicity of weight* λ in V.

Lemma. (a) The sum $\sum_\lambda V_\lambda \subseteq V$ is direct.

(b) $eV_\lambda \subseteq V_{\lambda+2}$, $\quad fV_\lambda \subseteq V_{\lambda-2}$, $\quad hV_\lambda \subseteq V_\lambda$.

(c) The sum $\sum_\lambda V_\lambda \subseteq V$ is a $\mathfrak{g}$-submodule.

(d) $0 \neq \dim(V) < \infty$ then $\sum_\lambda V_\lambda \neq 0$. □

2.2.2. *Primitive vectors.* The non-zero vectors in

$$V_\lambda^o \stackrel{\text{def}}{=} \{v \in V_\lambda;\ ev = 0\}$$

are called the *primitive vectors of weight* λ in V (or the *highest weight vectors of weight* λ).

For a primitive vector v of weigh λ wee define the vectors

$$v_n \stackrel{\text{def}}{=} \frac{f^n}{n!} v \in V, \quad n \in \mathbb{N}.$$

So, $v_0 = v$, $v_1 = fv$ etc.

Lemma. (a)

$$fv_k = (k+1)v_{k+1}, \quad hv_k = (\lambda - 2k)v_k \quad k \in \mathbb{N} \quad \text{and} \quad ev_k = (\lambda + 1 - k)v_{k-1}, \quad (k > 0).$$

(b) The nonzero vectors v_n are independent.

(c) The sum $\sum_{n\in\mathbb{N}} \mathbb{C}v_n \subseteq V$ is a $\mathfrak{g}$-submodule.

(d) If $v_n \neq 0$ but $v_{n+1} = 0$ then $\lambda = n$. □

Corollary. If $V \neq$ is finite dimensional then

- (a) V has a primitive vector and the weights of all primitive vectors are natural numbers.
- (c) If v is a primitive vector of weight $n \in \mathbb{N}$ then $v_0, ..., v_n$ are independent and $v_i = 0$ for $i > n$.

Proof. (c) Since the nonzero vectors v_n are independent and V is finite dimensional, there is some $m \in \mathbb{N}$ such that $v_m \neq 0$ and $v_{m+1} = 0$ (hence $v_i = 0$ for $i > m$). By (d) we have $m = n$. Also, since $v_n \neq 0$ we have $v_i \neq 0$ for $i \leq n$. □

2.3. **Classification of irreducible finite dimensional modules for $\mathfrak{g} = sl_2$.**

Lemma. (a) For any $n \in \mathbb{N}$ there is a representation $L(n)$ of sl_2 of dimension $n+1$ given by matrices

$$E = \begin{pmatrix} 0 & n & & & & \\ & 0 & n-1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 2 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \quad H = \begin{pmatrix} n & & & & & \\ & n-2 & & & & \\ & & n-4 & & & \\ & & & \ddots & & \\ & & & & 2-n & \\ & & & & & -n \end{pmatrix}, \quad F = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & n-1 & 0 & \\ & & & & n & 0 \end{pmatrix}.$$

(b) $L(n)$ is irreducible and has a primitive vector of weight n. □

Remark. (c) The standard basis of $\Bbbk^{n+1}$ is of the form $v_0, ..., v_n$ for a primitive vector of weight n.

Theorem. $L(n),\ n \in \mathbb{N}$, is a classification of irreducible finite dimensional representations of sl_2.

Proof. The claim is that all $L(n)$ are irreducible and that each finite dimensional irreducible representation of sl_2 is isomorphic to precisely one of $L(n)$. □

Corollary. (a) Any irreducible finite dimensional sl_2-module V has a unique primitive vector (up to an invertible scalar). Its weight n is the highest weight of V. All weight multiplicities in V are≤ 1.

(b) The dual V^* of any irreducible finite dimensional representation V of sl_2 is isomorphic to V. [6] □

Remark. So, the irreducible modules are classified by their highest weights by $n \mapsto L(n)$. □

2.4. **Semisimplicity theorem for sl_2.** Here we state the theorem and and consider its consequences. The proof is postponed to 2.6 (Weyl's proof) and .... (algebraic proof).

Theorem. Finite dimensional representations of sl_2 are semisimple (i.e., a sum of irreducibles).

Corollary. For any finite dimensional representation V of sl_2:

(a) V is isomorphic to a representation of the form $\oplus_{n\in\mathbb{N}}\ L(n)^{\oplus m_n}$ where m_n is the dimension of V_n^o. More precisely,

$$V \ \cong\ \oplus_{n\in\mathbb{N}}\ L(n)\otimes V_n^0 \quad \text{and} \quad \mathrm{Hom}_{sl_2}[L(n), V] \ \cong\ V_n^o.$$

(b) e, f act on V as nilpotents and h as a semisimple operator with integer eigenvalues (i.e., $V = \oplus_{i\in\mathbb{Z}}\ V_i$).

(c) $V^* \cong V$.

[6] This is a special property of sl_2, it is not true for sl_n.

(d) For any $n \in \mathbb{N}$ the map

$$V_{-n} \xrightarrow{e^n} V_n$$

is an isomorphism.

Proof. (a) We have proved in 1.2.5 that

$$V \cong \oplus_{n\in\mathbb{N}} L(n)\otimes M_n \quad \text{for} \quad M_n = \mathrm{Hom}_{sl_2}[L(n), V].$$

Now, since operators h, e act on V so that they preserve each summand and they act trivially on M_n's, we get that

$$V_p \cong \oplus_{n\in\mathbb{N}} L(n)_p\otimes M_n \quad \text{and} \quad V_p^o \cong \oplus_{n\in\mathbb{N}} L(n)_p^o\otimes M_n \cong M_p$$

because $L(n)_p^o = 0$ for $p \neq 0$ and $L(n)_p^o$ is a line.

Each of the claims (b,c,d) is easily checked for $V = L(n)$ and then one notices that it remains true for sums of copies of $L(n)$'s. □

2.4.1. *Proofs of the semisimplicity theorem.* This theorem holds for the class of so called *semisimple Lie algebras* that we will define later, including all sl_n. One can prove this theorem by using one of the following ideas:

(1) Weyl's unitary trick (it is intuitive but requires relation to Lie groups and the Haar measure on compact Lie groups).
(2) Casimir operator C (this approach gives a proof which is elementary but long, it requires the enveloping algebras of Lie algebras).
(3) Lie algebra cohomology (requires homological algebra).
(4) In the case of sl_2 the representations $L(n)$ are so well understood that one can provide a proof without any of these extra tools. (For instance a proof which follows (2) above but without Casimir.)

2.5. **Crystals: a combinatorial view on representations.** Kashiwara found that finite dimensional representations V of semisimple Lie algebras come with a combinatorial structure $cr(V)$ called the *crystal* of V.

The crystal $cr[L(n)] = c(n)$ of $L(n)$ is the graph

$$-n \xrightarrow{\mathsf{e}} -n+2 \xrightarrow{\mathsf{e}} -n+4 \xrightarrow{\mathsf{e}} \cdot \xrightarrow{\mathsf{e}} n-2 \xrightarrow{\mathsf{e}} n.$$

Its vertices correspond to the standard basis $v_n, v_{n-1}, ..., v_0$ of $L(n)$ (or to the lines $\Bbbk v_i$ through this basis). The vertices are labeled by the weights of vectors in the basis. The arrows are labeled by e symbolizing the action of the operator e on $L(n)$.

Let us denote by $\mathcal{C}[sl_2]$, the crystals of $\mathfrak{g} = sl_2$, the set of all finite disjoint unions of crystals of $L(n)$'s.

Then one can define the crystal $cr(V)$ of any finite dimensional representation V as the disjoint union of crystals of $L(n)$'s corresponding to the decomposition of V into a sum of copies of $L(n)$'s.

Remark. The isomorphism classes of sl_2-crystals are in bijection with isomorphism classes of finite dimensional representations of sl_2. In particular, the connected crystals correspond to irreducible representations.

Example. The basis of primitive vector in V corresponds to vertices in $cr(V)$ which are ends of e-strings.

2.5.1. *The tensor product of crystals.* Kashiwara defines the operation of the tensor product of crystals

$$\mathcal{C}(\mathfrak{g})\ tim\ \mathcal{C}(\mathfrak{g}) \ \xrightarrow{\otimes}\ \mathcal{C}(\mathfrak{g})$$

with the property that

Theorem. $cr(U)\otimes cr(V) \ \cong\ cr(U\otimes V)$. □

Remark. So, crystals contain the information of how tensor products decompose into irreducibles!

2.6. **Weyl's proof of semisimplicity.**

2.6.1. *Semisimplicity for compact topological groups.*

Theorem. Let U be a compact topological group.

(a) For any continuous representation of G on a Hilbert space $\big(H,(-,-)_0\big)$ there is new inner product $(-,-)$ on H which is U-invariant, i.e.,

$$(ux,uy) \ = \ (x,y) \ \text{ for } \ x,y\in H,\ u\in U.$$

(b) For any finite dimensional continuous representation (π,V) of G, any subrepresentation V' of V has a complementary representation V''.

2.6.2. *Proof I: (a) implies (b).* The point is that a finite dimensional vector space V always has an inner product $(-,-)_0$ and therefore it also has an invariant inner product $(-,-)$. Now, V'' can be chosen as $(V')^{\perp}$ the orthogonal complement with respect to the invariant inner product. □

2.6.3. *Proof II: proof of (a).* This is based on the following fact in analysis.

Theorem. For any compact topological group G :

(a) There is a measure μ on G which is

(1) Strictly positive, i.e., for any open nonempty V $\mu(V) > 0$.
(2) Invariant under left and right translations where these translations are the canonical actions of the group G on measures on G :

$$(L_g\mu)(V) = \mu(L_{g^{-1}}V) = \mu(g^{-1}V) \quad \text{and} \quad (R_g\mu)(V) = \mu(R_{g^{-1}}V) = \mu(Vg).$$

(b) This measure is unique up to a multiplication with some positive number. □

Remarks. (a) The theorem holds for all locally compact groups except that in this case one can only ask for invariance under say the left translations.

(b) These measure are called *Haar measures*

Proof of part (a) in the theorem 2.6.4. We get the invariant inner product as an *average* of translates of any existing inner product

$$(x, y) \stackrel{\text{def}}{=} \int_{u\in U} (ux, uy)_0 \, d\mu(u) \quad \square.$$

2.6.4. *Weyl's proof of the semisimplicity theorem.* Here we finally finish the proof of the semisimplicity theorem for sl_2 from 2.4. It turns out that the theorem and Weyl's proof remain valid for any $\mathfrak{g}$ in the class of *semisimple complex Lie algebras* which contains sl_2. So, we will state and prove the theorem it in this generality. The key is the relation of semisimple Lie algebras and compact Lie groups that we state in 2.7.

Theorem. For any semisimple complex Lie algebra $\mathfrak{g}$ the category $Rep^{fd}(\mathfrak{g})$ of finite dimensional representations of $\mathfrak{g}$ is semisimple.

Proof. By 2.7, the category $Rep^{fd}(\mathfrak{g})$ behaves the same as the category of representations of a compact Lie group U. However, for compact groups we know semisimplicity by averaging inner product. □

2.7. **Appendix. Relation of semisimple Lie algebras and Lie groups.**

Theorem. Let $\mathfrak{g}$ be a semisimple complex Lie algebra.

To a semisimple complex Lie algebra $\mathfrak{g}$ we associate a complex Lie group $G \stackrel{\text{def}}{=} \widetilde{\mathrm{Aut}_{Lie}(\mathfrak{g})}$, the universal cover of the group of automorphisms of the Lie algebra $\mathfrak{g}$. Moreover, it turns out that all *maximal compact subgroups* U of G are essentially the same – more precisely they are conjugated by elements of G. So, we can choose any maximal compact subgroup U of G. Then

Theorem. The following categories are canonically equivalent

(1) $Rep^{fd}(\mathfrak{g})$, the finite dimensional representations of the Lie algebra $\mathfrak{g}$;
(2) $Rep^{fd}_{hol}(G)$, the finite dimensional holomorphic representations of the complex Lie group G;
(3) $Rep^{fd}_{C^\infty}(U)$, the finite dimensional smooth representations of the real Lie group G on complex vector spaces.

Here, the equivalence $Rep^{fd}_{hol}(G) \to Rep^{fd}(\mathfrak{g})$ is given by differentiating a representation π of G on V to a representation π' of $\mathfrak{g}$ on V by

$$\pi'(x) \stackrel{\text{def}}{=} \frac{d}{ds}|_{s=0}\ \pi(e^{sx}).$$

The equivalence $Rep^{fd}_{hol}(G) \to Rep^{C^\infty}(U)$ is simply given by restricting the action of G to the subgroup U.

Example. When G is sl_n then the group $G = SL(n)$ acts on the Lie algebra $\mathfrak{g}$ by conjugation. However, the center $Z(G)$ of G acts trivially on $\mathfrak{g}$. The center consists of all diagonal matrices $s{\cdot}1_n$ which lie in SL_n, i.e., $\det(s1_n) = s^n$ should be 1. This means that $Z(G) \cong \mu_n$ for the group μ_n of all n^{th} roots of unity. Then the map

$$SL_n(\mathbb{C})/\mu_n \ \to\ \mathrm{Aut}_{Lie}(sl_n)$$

is an isomorphism. The group G that is attached to $\mathfrak{g} = \mathfrak{sl}_n$ is the universal cover of $\mathrm{Aut}_{Lie}(sl_n) = SL_n(\mathbb{C})/\mu_n$ and this is again SL_n. So, $G = SL_n(\mathbb{C})$. Finally, a maximal compact subgroup U of G can be chosen as $SU(n)$ where $U(n)$ is the group of *unitary* matrices and $SU(n) \stackrel{\text{def}}{=} SL_n(\mathbb{C}) \cap U(n)$.

3. **Enveloping algebras of Lie algebras**

We will here encode a Lie algebra $\mathfrak{g}$ in terms of an associative algebra called the *enveloping algebra* $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$.

Remark. The enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ resolves the following problem. When considering a representation (π, V) of a Lie algebra one naturally needs to calculate with linear combinations of compositions $\pi(x_1)\cdots\pi(x_n)$ of operators attached to elements x_i of $\mathfrak{g}$.

This is a computation in the associative algebra $\mathrm{End}(V)$. However, some of these computations are done just by using the commutators in the Lie algebra $\mathfrak{g}$. Typical example is the change of order in a product as we can replace the product of length two $\pi(x_{i-1})\pi(x_i)$ with the product in the opposite order $\pi(x_i)\pi(x_{i-1})$ (again of length 2), at the price of adding a term $\pi[x_{i-1}, x_i]$ which is a shorter product (of length 1).

Such computations do not really involve the representation π, rather one is computing in some sense with linear combinations of sequences $x_1, ..., x_n$ of elements of $\mathfrak{g}$. We would like

to have a natural setting for such computations and this is what the enveloping algebra $U\mathfrak{g}$ does.

3.1. **Enveloping algebra** $U\mathfrak{g}$**.** Let us say that a *linearization* of a Lie algebra $\mathfrak{g}$ is a pair (A, ϕ) consisting of an associative algebra A endowed with a map of Lie algebras $\phi : \mathfrak{g}a_1, ..., a_nA$ (i.e., $\phi[x, y] = \phi(x)\phi(y) - \phi(y)\phi(x)]$ for $x, y \in \mathfrak{g}$. For instance any representations π of $\mathfrak{g}$ on a vector space V is a linearization $\pi : \mathfrak{g} \to A = \mathrm{End}(V)$.

We define the *enveloping algebra* $U = U\mathfrak{g}$ as the associative algebra generated by the vector space $\mathfrak{g}$ and the relations $x \cdot_U y - y \cdot_U x = [x, y]_{\mathfrak{g}}$ for $x, y \in \mathfrak{g}$. Formally, this means that $U\mathfrak{g}$ is the quotient of the free algebra generated by $\mathfrak{g}$ (this is the tensor algebra $T\mathfrak{g}$) by the relation $x \otimes y - y \otimes x = [x, y]_{\mathfrak{g}}$ for $x, y \in \mathfrak{g}$. So, $U\mathfrak{g}$ comes with the canonical map of vector spaces $\iota : \mathfrak{g} \to U\mathfrak{g}$ (the composition $\mathfrak{g} \subseteq T\mathfrak{g} \twoheadrightarrow U\mathfrak{g}$).

Lemma. (a) $\iota : \mathfrak{g} \to U\mathfrak{g}$ is a linearization of $\mathfrak{g}$.

(b) This is the universal linearization of $\mathfrak{g}$, i.e., for any linearization (A, ϕ) of $\mathfrak{g}$ there is a unique map of associative algebras $U\mathfrak{g} \xrightarrow{f} A$ such that $\phi = f \circ \iota$.

(c) For an associative algebra A the map

$$\mathrm{Hom}_{AssocAlg}(U\mathfrak{g}, A) \xrightarrow{\phi \mapsto \phi \circ \iota} \mathrm{Hom}_{LieAlg}(\mathfrak{g}, A)$$

is a bijection.

Proof. (b) is just the definition of $U\mathfrak{g}$ and (a) is a part of (b).

(c) is a restatement of (b). □

Corollary. Representations of the Lie algebra $\mathfrak{g}$ are the same as modules for the associative algebra $U\mathfrak{g}$.

Proof. A representation of $\mathfrak{g}$ on a vector space V is the same as a map of Lie algebras $\pi : \mathfrak{g} \to \mathrm{End}(V)$. A structure of an $U\mathfrak{g}$-module on V is the same as a map of associative algebras $\tau : U\mathfrak{g} \to \mathrm{End}(V)$. □

Remark. The correspondence of π's and τ's in the lemma is that τ restricts to $\pi = \tau \circ \iota$ and π extends to τ by $\tau(\iota(x_1) \cdots \iota(x_N)) \stackrel{\mathrm{def}}{=} \pi(x_1) \circ \cdots \circ \pi(x_N)$ for $x_i \in \mathfrak{g}$. □

3.1.1. *Filtered algebra structure on* $U\mathfrak{g}$*.* Let F_p be the span of all $\iota(x_1) \cdots \iota(x_N)$ where $x_i \in \mathfrak{g}$ and $N \leq p$.

Lemma. (a)

Lemma. The pair $(U\mathfrak{g}, F)$ is a filtered associative algebra with an increasing filtration, i.e., we have $F_p \subseteq F_{p+1}$ and $F_0 \ni 1$ as well as $F_p \cdot F_q \subseteq F_{p+q}$.

(b) Construction $\mathfrak{g} \mapsto U\mathfrak{g}$ is a functor from the category Lie of Lie algebras to the category Assoc of associative algebras. In other words, says that any map of Lie algebras $\mathfrak{g}' \xrightarrow{\alpha} \mathfrak{g}$ defines a map of associative algebras $U\mathfrak{g}' \xrightarrow{\widetilde{\alpha}} U\mathfrak{g}$.

(c) Moreover, $\widetilde{\alpha}$ is a map of filtered associative algebras, i.e., $\widetilde{\alpha}(F_p U\mathfrak{g}') \subseteq F_p(U\mathfrak{g})$.

Proof. □

3.2. **Poincare-Birkhoff-Witt theorem.**

3.2.1. *Formulation and the proof of the easy part.*

Theorem. If $b_1, ..., b_n$ is a basis of $\mathfrak{g}$ then the monomials $\iota(b_1)^{p_1} \cdots \iota(b_n)^{p_n}$ with $p \in \mathbb{N}^n$ form a basis of $U\mathfrak{g}$. □

By the "easy part" of the theorem we will mean the claim that the monomials span $U\mathfrak{g}$.

Proof of the easy part of the PBW theorem. Let A_N be the span of all monomials $\iota(b_1)^{e_1} \cdots \iota(b_n)^{e_n}$ of degree $\sum e_i$ less or equal to N. Clearly, $A_N \subseteq F_N$. We will prove by induction that these are the same.

F_p is the span of all $\iota(x_1) \cdots \iota(x_N)$. By writing each x_i as a linear combination of elements b_j of the basis and multiplying out, we see that actually F_p is the span of all $\iota(x_1) \cdots \iota(x_N)$ with $N \leq p$ and all x_i in $\{b_1, ..., b_n\}$.

Such product $\iota(x_1) \cdots \iota(x_N)$ is in A_p if the order $x_1, ..., x_N$ is compatible with the order in $b_1, ..., b_n$ in the basis. If not we have two neighbors x_{i-1}, x_i which are in the wrong order. Then we write $\iota(x_{i-1})\iota(x_i)$ as $\iota(x_i)\iota(x_{i-1}) + \iota[x_{i-1}, x_i)]$. So, the product $\iota(x_1) \cdots \iota(x_N)$ can be written as another product of the same length N with one inversion straightened out, plus a shorter product which lies in F_{p-1} which is by induction assumption A_{p-1}.

By repeating this procedure we can write the product $\iota(x_1) \cdots \iota(x_N)$ as a sum of the same product but in the correct order (this lies in A_p) and some thing in A_{p-1}. So, $\iota(x_1) \cdots \iota(x_N) \in A_p$. □

3.2.2. *Some consequences of PBW theorem.*

3.3. **The Casimir element C of $U\mathfrak{g}$.**

3.3.1. *The center of the enveloping algebra.* Denote by $\mathfrak{z} \stackrel{\text{def}}{=} Z(U\mathfrak{g})$ the center of the algebra $U\mathfrak{g}$.

Sublemma. (a) $\mathfrak{g}$ acts on the vector space $U\mathfrak{g}$ by

- *left multiplication* $L_x\ u \overset{\text{def}}{=} x{\cdot}u$ for $u \in U\mathfrak{g}$ and $x \in \mathfrak{g}$;
- *right multiplication* $R_x\ u \overset{\text{def}}{=} u{\cdot}(-x)$;
- *adjoint action* $ad_x\ u \overset{\text{def}}{=} [x,u] \overset{\text{def}}{=} xu - ux$.

(b) The left and right multiplication actions commute: $L_xR_y = R_yL_x$ and $ad = L + R$.

Lemma. (a) The adjoint action is by derivations of the algebra $U\mathfrak{g}$, i.e.,

$$ad_x(uv) \ = \ (ad_xu)v) \ + \ u(ad_xv).$$

(b) The map $\iota : \mathfrak{g} \to U\mathfrak{g}$ is a map of $\mathfrak{g}$-modules for the adjoint action on $U\mathfrak{g}$.

(c) The center of $U\mathfrak{g}$ is the same as $\mathfrak{g}$-invariants in $U\mathfrak{g}$ for the adjoint action:

$$Z(U\mathfrak{g}) \ \cong \ (U\mathfrak{g})^{\mathfrak{g}}.$$

Proof. By definitions. □

3.3.2. *Casimir elements of bilinear forms on* $\mathfrak{g}^*$. A bilinear form $\sigma = \ (-,-)$ on $\mathfrak{g}^*$ is a linear map $\mathfrak{g}^*{\otimes}fg^* \ \to \ \Bbbk$, so its adjoint is a map $\sigma^* : \ \Bbbk \to \ \mathfrak{g} \to \mathfrak{g}$, i.e., a vector $\sigma^*(1)$ that lies in $\mathfrak{g}{\otimes}\mathfrak{g} = \ T^2\mathfrak{g}{\subseteq}T\mathfrak{g}$. Then $\pi : \ T\mathfrak{g}{\twoheadrightarrow}U\mathfrak{g}$ defines an element $C_{\mathfrak{g}} \overset{\text{def}}{=} \pi(\sigma^*(1_{\Bbbk})$ in the enveloping algebra, called the *Casimir element* of σ.

Lemma. (a) In terms of dual bases x_i of $\mathfrak{g}$ and y^i of $\mathfrak{g}^*$ in $U\mathfrak{g}$ we have

$$C_{\mathfrak{g}} \ = \ \sum_{ij} \ \sigma(y^i, y^j) \ x_i{\cdot}x_j$$

(b) If the bilinear form σ is $\mathfrak{g}$-invariant then $C_{\mathfrak{g}}$ lies in the center $Z(U\mathfrak{g})$.

Proof. (a) The adjoint of $\sigma : \mathfrak{g}^*{\otimes}\mathfrak{g}^* \to \Bbbk$ takes $1_{\Bbbk} \in \Bbbk$ to $\sigma(y^i, y^j)\ x_i{\otimes}x_j$. Its image $C_{\mathfrak{g}}$ in $U(\mathfrak{g})$ is $\sum_{ij}\ \sigma_{ij}x_i{\cdot}x_j$.

(b) A bilinear form σ on a representation V of $\mathfrak{g}$ is said to be $\mathfrak{g}$-*invariant* if $\sigma(x{\cdot}u, v) + \sigma(u, x{\cdot}v) = 0$ for all $u, v \in V$. We know that this is equivalent to the condition that the corresponding vector σ in the $\mathfrak{g}$-module $V^*{\otimes}V^*$ is $\mathfrak{g}$-*invariant* in the sense of being killed by $\mathfrak{g}$.

In our case $V = \mathfrak{g}^*$ hence $\sigma \in V^*{\otimes}V^* = \mathfrak{g}{\otimes}\mathfrak{g}$. Now C_σ is the image of σ under the natural map $\mathfrak{g}{\otimes}\mathfrak{g} \subseteq T\mathfrak{g} \twoheadrightarrow U\mathfrak{g}$. The naturality of this map implies that it is a $\mathfrak{g}$-map for the adjoint action on $U\mathfrak{g}$.

Therefore, it sends $\mathfrak{g}$-invariants to $\mathfrak{g}$-invariants, hence C_σ lies in $(U\mathfrak{g})^{\mathfrak{g}}$ which is the same as $Z(U\mathfrak{g})$. □

Remarks. (0) A symmetric bilinear form σ on V defines a quadratic form q on V by $q(v) = \sigma(v, v)$. One can go in the opposite direction by $\sigma(u, v) = \frac{1}{2}[\sigma(u + v, u + v) - \sigma(u, u) - \sigma(v, v)]$.

(1) A symmetric bilinear form (or equivalently a quadratic form) on a Lie algebra is also called a *level*.

3.3.3. *Casimir elements of nondegenerate bilinear forms on* $\mathfrak{g}$. We will use the following lemma to move a *non-degenerate* bilinear form τ on $\mathfrak{g}$ to a bilinear form σ on $\mathfrak{g}^*$.

Sublemma. (a) A non-degenerate bilinear form τ on a finite dimensional vector space V is the same as a non-degenerate bilinear form σ on V^*.

(b) To a basis v_i of V one can attach a τ-dual basis v^i of V and also the dual basis y^i of V^*. Then the matrices of τ and σ in bases v^i and y^i of V and V^* are the same.

Proof. (a) Any bilinear form τ on $\mathfrak{g}$ can be viewed as a linear operator $\widetilde{\tau} : \mathfrak{g} \to \mathfrak{g}^*$ by $\widetilde{\tau}(x) = \tau(x, -)$. The nondegeneracy of τ is then the same as $\widetilde{\tau}$ being invertible. We can then use it to move τ from V to V^* by

$$\sigma(\lambda, \mu) \stackrel{\text{def}}{=} \tau(\widetilde{\tau}^{-1}\lambda, \widetilde{\tau}^{-1}\mu).$$

(b) follows because the operator $\widetilde{\tau} : V \stackrel{\cong}{\to} V^*$ sends v^i to y^i since $\widetilde{\tau}v^i = \tau(v^i, -)$ equals y^i. $\square$

Now, to any $\mathfrak{g}$-invariant non-degenerate bilinear form τ on $\mathfrak{g}$ we can associate a non-degenerate bilinear form σ on $\mathfrak{g}^*$. We, define the Casimir C^τ of τ as C_σ. If τ is $\mathfrak{g}$-invariant then $\widetilde{\tau} : \mathfrak{g} \to \mathfrak{g}^*$ is a $\mathfrak{g}$-map, and therefore σ is also $\mathfrak{g}$-invariant. Therefore, C^τ lies in $Z(U\mathfrak{g})$.

Lemma. When τ is non-degenerate for any basis x_i of $\mathfrak{g}$ there is a τ-dual basis x^i of $\mathfrak{g}$, i.e., $\tau(x^i, x_j) = \delta_{ij}$. Then

$$C^\tau = \sum x_i x^i.$$

Proof. The map $\widetilde{\tau} : \mathfrak{g} \to \mathfrak{g}^*$ is given by $\widetilde{\tau}(x^i) = \tau(x^i, -) = y^i$. Therefore, $C^\tau \stackrel{\text{def}}{=} C_\sigma$ is

$$\sum \sigma(y^i, y^j)\, x_i \otimes x_j = \sum \tau(x^i, x^j)\, x_i \otimes x_j = \sum_i x_i \otimes (\sum_j \tau(x^i, x^j)\, x_j) = \sum_i x_i \otimes x^i. \quad \square$$

3.3.4. *Examples.* These will come from finite dimensional representations. Any finite dimensional representation V of $\mathfrak{g}$ defines a bilinear form $\kappa_V^{\mathfrak{g}} = \kappa_V$ on $\mathfrak{g}$ by

$$\kappa_V(x, y) \stackrel{\text{def}}{=} Tr_V(xy).$$

In the particular case when the representation V of $\mathfrak{g}$ is the adjoint representation $V = \mathfrak{g}$, the form $\kappa_{\mathfrak{g}}$ is called the *Killing form* of the Lie algebra $\mathfrak{g}$

Lemma. (a) Form κ_V is always symmetric and invariant.

(b) Let

For $\mathfrak{g} = sl_n$ and $V = \mathbb{k}^n$, the corresponding invariant symmetric bilinear form $\beta = \kappa_V$ on $\mathfrak{g}$ is non-degenerate. It is called the *basic level* β.

(c) For $\mathfrak{g} = sl_n$ the Killing form $\kappa_{\mathfrak{g}}$ is non-degenerate.

Proof. (a) Symmetry comes from $Tr(xy) = Tr(yx)$. The invariance property $\kappa_V([[x.u], v]) + \kappa_V([u, [x, v]])$ means the vanishing of

$$Tr\big([[x.u], v]+[u, [xv]]\big) \;=\; Tr\big(xuv-uxv+uxv-uvx[[x.u], v]+[u, [xv]]\big) \;=\; Tr\big(x(uv)-(uv)x\big).$$

(b) We can regard $V = \mathbb{k}^n$ as a representation of gl_n. Then we have $\beta^{gl_n}(E_{ij}, E_{pq}) = Tr[E_{ij}E_{pq} = \delta_{jp}\delta_{qi}$. So, we have dual basis in sl_n given by $E'_{ij}s$ and E_{ji}'s and the form is non-degenerate.

Notice that $sl_n \subseteq gl_n$ is orthogonal to scalars $k{\cdot}1_n$ as $\kappa_V(x, 1_n) = Tr(x)$. So, $\beta = \beta^{sl_n}$ which is the restriction of the non-degenerate form β^{gl_n} is again nondegenerate. $\square$

Question. The Killing form $\kappa \stackrel{\text{def}}{=} \kappa_{sl_n}$ is a positive integral multiple of β, $\kappa = ?{\cdot}\beta$.

Remark. As observed by Filip Dul, the Casimir for sl_2 appears in quantum mechanics as the *angular momentum* operator.

3.3.5. *More on invariant bilinear forms.* This should be skipped until we need it later.

The *radical* $rad(\kappa)$ of a symmetric bilinear form κ on a vector space W is the set of all $w \in \mathbb{W}$ such that $\kappa(w, W) = 0$, i.e., $rad(\kappa) = W^{\perp}$.

Lemma. (a) A symmetric bilinear form κ on a vector space W descends to the space $W/rad(\kappa)$.

(b) The center $Z(\mathfrak{g})$ lies in the radical of the Killing form $\kappa_{\mathfrak{g}}$

(c) If κ is an invariant symmetric bilinear form on a Lie algebra $\mathfrak{g}$ then for any ideal $\mathfrak{a}$ in $\mathfrak{g}$, $\mathfrak{a}^{\perp}$ is also an ideal. In particular the radical of κ is an ideal in $\mathfrak{g}$.

(d) For operators on a vector space V the expression $Tr([x, y]z)$ is cyclically invariant, i.e.,

$$Tr([x, y]z) \;=\; Tr([y, z]x) \;=\; Tr([z, x]y).$$

Proof. (a) is the claim that $\overline{\kappa}(u + rad(\kappa), u + rad(\kappa)) \stackrel{\text{def}}{=} \kappa(u, v)$ is well defined.

(b) If $z \in Z(\mathfrak{g})$ then $ad(z) = 0$ hence $\kappa_{\mathfrak{g}}(z, x) = Tr_{\mathfrak{g}}[ad(z)ad(x)] = 0$.

(c) If $z \in \mathfrak{a}^{\perp}$ and $x \in \mathfrak{g}$ we want $[x, z]$ to be in $\mathfrak{a}^{\perp}$ again. But, for any $y \in \mathfrak{a}$, the invariance of κ gives

$$\kappa([x, z], y) \;=\; -\kappa(z, [x, y]).$$

This is zero since $[x, y] \in \mathfrak{a}$ and $z \in \mathfrak{a}^{\perp}$. The last claim follows from $rad(\kappa) = fg^{\perp}$.

(d) $Tr([x, y]z) = Tr(x(yz)) - Tr(y(xz)) = Tr((yz)x) - Tr((xz)y) = Tr([y, z]x)$. $\square$ $\square$

Proposition. (a) For a linear operator $A \in \mathrm{End}(V)$, if $V' \subseteq V$ is a subspace such that $AV \subseteq V'$ then $Tr_V(A) = Tr_{V'}(A|_{V'})$.

(b) If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ then the Killing form on $\mathfrak{a}$ is the restriction of the Killing form on $\mathfrak{g}$.

Proof. (a) Just consider the matrix of A in a basis of V that contains a basis of V'.

(b) follows, for $x, y \in \mathfrak{a}$, $\kappa_{\mathfrak{g}}(x, y) = Tr_{\mathfrak{g}}(ad(x)ad(y))$ which is by part(a) just $Tr_{\mathfrak{a}}(ad(x)ad(y)) = \kappa_{\mathfrak{a}}(x, y)$. $\square$

4. Finite dimensional representations of sl_n

4.0. **Summary.** sl_n is the next example of the class of *semisimple* Lie algebras (which we will define later). A key feature of Lie algebras $\mathfrak{g}$ in this class is that "everything" is captured by combinatorial data called the *system of roots of* $\mathfrak{g}$.

The combinatorial data come from consider certain maximal abelian Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ called *Cartan subalgebra*. By considering $\mathfrak{g}$ as a representation over $\mathfrak{h}$ we find the finite set $\Delta \subseteq \mathfrak{h}^*$ of *roots* of the Lie algebra $\mathfrak{g}$, defined as the nonzero *weights* (i.e., eigen-functionals) of the action of $\mathfrak{h}$ on $\mathfrak{g}$. The structure of the root system on the set Δ essentially refers to *angles* between the roots and to lengths of roots. Here roots are considered as vectors in the vector space $\mathfrak{h}^*$ endowed with a certain inner product.

Each root α gives a copy $\mathfrak{s}_\alpha$ of sl_2 that lies inside $\mathfrak{g}$. The subalgebras $\mathfrak{s}_\alpha$ generate $\mathfrak{g}$, so the Lie algebra structure of $\mathfrak{g}$ is captured by the relation of Lie subalgebras $\mathfrak{s}_\alpha$. These are in turn determined by the *angles* and *lengths* for the corresponding roots.

4.1. Lie algebra sl_n.

4.1.1. *Cartan and Borel subalgebras.* Lie algebra $\mathfrak{g} = sl_n$ lies inside a larger Lie algebra $\mathfrak{g}_0 = gl_n$. The following subspaces of $\mathfrak{g}_0$ have special names.

- Diagonal matrices $\mathfrak{h}_0 \stackrel{\text{def}}{=} \begin{pmatrix} * & & & & \\ & * & & & \\ & & \ddots & & \\ & & & * & \\ & & & & * \end{pmatrix}$ ("Cartan subalgebra");
- Upper triangular matrices $\mathfrak{b}_0 \stackrel{\text{def}}{=} \begin{pmatrix} * & * & \cdots & * \\ & * & & \\ & & \ddots & \vdots \\ & & * & * \\ & & & * \end{pmatrix}$ ("Borel subalgebra");
- Strictly upper triangular matrices $\mathfrak{n}_0 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & * & \cdots & * \\ & 0 & & \\ & & \ddots & \vdots \\ & & 0 & * \\ & & & 0 \end{pmatrix}$ ("the nilpotent radical of the Borel subalgebra $\mathfrak{b}_0$").

Notice that $\mathfrak{b}_0 = \mathfrak{h}_0 \oplus \mathfrak{n}_0$.

The same terminology is used for intersections with $\mathfrak{g} = sl_n$: $\mathfrak{h} = \mathfrak{h}_0 \cap \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{b} = \mathfrak{b}_0 \cap \mathfrak{g}$ is a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{n} = \mathfrak{n}_0 \cap \mathfrak{g} = \mathfrak{n}_0$ is the nilpotent radical of $\mathfrak{b}$, and we have $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$.

Lemma. (a) $\mathfrak{g}_0, \mathfrak{b}_0, \mathfrak{h}_0, \mathfrak{n}_0$ are associative algebras.

(b) $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{n}$ are Lie algebras.

Proof. (a) Consider the lines $L_i = \Bbbk e_i$ and subspaces $F_i = L_1 \oplus \cdots \oplus L_i$ of $V = \Bbbk^n$. Then

- $\mathfrak{h}_0$ consists of all $A \in M_n(\Bbbk)$ such that $AL_i \subseteq L_i$;

- $\mathfrak{b}_0$ consists of all $A \in M_n(\Bbbk)$ such that $AF_i \subseteq F_i$;
- $\mathfrak{h}_0$ consists of all $A \in M_n(\Bbbk)$ such that $AF_i \subseteq F_{i-1}$.

(b) Now $\mathfrak{g}, \mathfrak{b}_0, \mathfrak{h}_0, \mathfrak{n}_0$ are all known to be Lie subalgebras of $\mathfrak{g}_0$. So the intersections $\mathfrak{b}, \mathfrak{h}, \mathfrak{n}$ are also Lie subalgebras. $\square$

Remark. There are symmetrically subalgebras $\mathfrak{b}_0^-, \mathfrak{b}^-$ of lower triangular matrices and $\mathfrak{n}_0^- = \mathfrak{n}^-$ of strictly lower triangular matrices

Example. In sl_2 we have $\mathfrak{g} = \{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\}$ with $\mathfrak{h} = \Bbbk h$, $\mathfrak{n} = \Bbbk e$, $\mathfrak{h}^- = \Bbbk f$. So, the Lie subalgebras $\mathfrak{h}, \mathfrak{n}, \mathfrak{n}^-$ of $\mathfrak{g}$ will play the role analogous to that of the basis h, e, f of sl_2.

4.1.2. *Weights.* Notice that the Lie algebra $\mathfrak{h}$ is *abelian*, i.e., $[x, y] = 0$ for $x, y \in \mathfrak{h}$. This makes it reasonable to look for joint eigenvectors for $\mathfrak{h}$.

For a representation V of $\mathfrak{g}$ and $\lambda \in \mathfrak{h}^*$, the λ-weight space in V is

$$V_\lambda \stackrel{\text{def}}{=} \{v \in V : \ hv = \ \langle \lambda, h \rangle \cdot v \ \text{ for } \ \text{all } h \in \mathfrak{h}\}.$$

Then we say that λ is a *weight* of V if $V_\lambda \neq$. Let $\mathcal{W}(V)$ be the set of weights in V.

The *primitive vectors* of weight λ are the nonzero vectors in

$$V_\lambda^o \stackrel{\text{def}}{=} \{v \in V_\lambda : \ \mathfrak{n}\ v \ = \ 0\}.$$

Remark. A basis of $\mathfrak{h}_0^*$ is given by linear functionals ε_i^o such that $\langle \varepsilon_i^o, diag(a_1, ..., a_n) \rangle \ = \ a_i$. We denote by ε_i the restriction of ε_i^o to $\mathfrak{h} \subseteq \mathfrak{h}_0$, so $\sum \ \varepsilon_i = 0$ (since $\sum \ \varepsilon_i^o$ is the trace on $\mathfrak{h}_0$) and $\varepsilon_1, ..., \varepsilon_{n-1}$ is a basis of $\mathfrak{h}^*$.

4.1.3. *Roots of* $\mathfrak{g} = sl_n$. For any Lie algebra fg we have one obvious representation, the adjoint representation of $\mathfrak{g}$ on the vector space $\mathfrak{g}$ (1.2.1). The study of this representation is the study of the structure of the Lie algebra $\mathfrak{g}$.

We define the set Δ of *roots* of $\mathfrak{g} = \mathfrak{sl}_n$ as the nonzero weights in the adjoint representation:

$$\Delta \stackrel{\text{def}}{=} \mathcal{W}(\mathfrak{g}) - \{0\}.$$

Lemma. (a) The roots of $\mathfrak{sl}_n$ are all linear functionals $\alpha_{ij} \stackrel{\text{def}}{=} \ \varepsilon_i - \varepsilon_j$ for $1 \leq i, j \leq n$ and the corresponding weight spaces (now called *root spaces*) are $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \ \Bbbk E_{ij}$.

(b) $\mathcal{W}(\mathfrak{sl}_n) = \ \Delta(sl_n) \sqcup \{0\}$ and $\mathfrak{g}_0 = \mathfrak{h}$.

(c) We have $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta} \ \mathfrak{g}_\alpha$.

(d) The roots in $\mathfrak{n}$ (i.e. the roots α such that $\mathfrak{g}_\alpha \subseteq \mathfrak{n}$) are all $\varepsilon_i - \varepsilon_j$ with $i < j$. We denote these by Δ^+ and call them the *positive roots*. Then $\mathfrak{n} = \ \oplus_{\alpha \in \Delta^+} \ \mathfrak{g}_\alpha$.

(e) For a representation V of $\mathfrak{g}$ and $\lambda \in \mathcal{W}(V)$, $\alpha \in \mathcal{W}(\mathfrak{g}) = \Delta \sqcup 0$ we have

$$\mathfrak{g}_\alpha \ V_\lambda \subseteq V_{\lambda + \alpha}.$$

In particular $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

Proposition. (a) If $\alpha, \beta, \alpha+\beta$ are roots then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(b) If $\alpha, \beta \in \Delta$ but $\alpha+\beta \notin \Delta$ and $\beta \neq -\alpha$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.

(c) For $\alpha \in \Delta$, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a one dimensional subspace of $\mathfrak{h}$.

Proof. It is easy to see that $E_{ij}E_{pq} = \delta_{j,p}E_{iq}$, hence $[E_{ij}, E_{pq}] = \delta_{j,p}E_{iq} - \delta_{i,q}E_{pj}$,

(a) We know that $\alpha = \varepsilon_i - \varepsilon_j$ and $\beta = \varepsilon_p - \varepsilon_q$ with $i \neq j$ and $p \neq q$. Then $\alpha+\beta$ is a root (i.e., of the form $\varepsilon_r - \varepsilon_s$ with $r \neq s$), iff $j = p$ and $i \neq q$ or (symmetrically) $q = i$ and $j \neq p$. In the first case $\alpha+\beta = \alpha_{ij} + \alpha_{iq} = \varepsilon_i - \varepsilon_q = \alpha_{iq}$ and $[E_{ij}, E_{jq}] = E_{iq}$. In the second case $\alpha+\beta = \alpha_{ij} + \alpha_{pi} = \varepsilon_p - \varepsilon_j = \alpha_{pj}$ and $[E_{ij}, E_{pi}] = -E_{pj}$. So, in both cases $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(b) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ but the conditions are that $\alpha+\beta \notin \Delta \sqcup 0 = \mathcal{W}(\mathfrak{g})$ hence $\mathfrak{g}_{\alpha+\beta} = 0$

(c) $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$. □

4.1.4. *The sl_2 subalgebras $\mathfrak{s}_\alpha \subseteq \mathfrak{g}$ associated to roots α.* For a root $\alpha \in \Delta$ let

$$\mathfrak{s}_\alpha \overset{\text{def}}{=} \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}.$$

Lemma. (a) $\mathfrak{s}_\alpha$ is a Lie subalgebra.

(b) There is a Lie algebra map $\psi : \mathfrak{sl}_2 \to \mathfrak{g}$ such that $0 \neq \psi(e) \in \mathfrak{g}_\alpha,\ \psi(f) \in \mathfrak{g}_{-\alpha}$. Any such ψ gives an isomorphism $\psi : \mathfrak{sl}_2 \to \mathfrak{s}_\alpha$.

(c) The image $\psi(h)$ is independent of the choice of ψ. We denote it $\check{\alpha}$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathfrak{s}_\alpha \cap \mathfrak{h}$ has basis $\check{\alpha}$.

Proof. (a) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_{\alpha+-\alpha} = \mathfrak{h}$ and $\mathfrak{h}$ preserves each $\mathfrak{g}_\phi$. Anyway, (a) follows from (b).

(b) A root $\alpha = \alpha_{ij}$, i.e., a choice of indices $i \neq j$, gives an embedding of of Lie algebras $\phi : sl_2 \hookrightarrow sl_n$ by $\phi(e) = E_{ij},\ \phi(f) = E_{ji},\ \phi(h) = E_{ii} - E_{jj}$.

For another choice ψ, $\psi e \in \mathfrak{g}_\alpha = \Bbbk \phi e$ we have $\psi(e) = a\phi e,\ \psi(f) = b\psi f$ for some scalars a, b. Then $\psi h = ab\phi(h)$ as

$$\psi h = \psi[e, f] = [\psi e, \psi f] = [a\phi e, b\phi f] = ab\phi[e, f]h = ab\phi(h).$$

So it remains to notice that $ab = 1$ since

$$2\psi e = \psi[h, e] = [\psi h, \psi e] = [ab\,\phi h, a\phi e] = a^2 b[\phi h, \phi e] = a^2 b\,\phi[h, e] = a^2 b\, 2\phi e = ab\, 2\psi e.$$

Finally, $\mathfrak{s}_\alpha \cap \mathfrak{h} = (\mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha})\mathfrak{h} = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \Bbbk[\phi e, \phi f] = \Bbbk \phi h = \Bbbk \check{\alpha}$ for $\alpha = \alpha_{ij}$. □

Remark. $\check{\alpha_{ij}} = E_{ii} - E_{jj}$ was noticed in the proof of the lemma.

4.2. **Root systems.**

4.2.1. *Reflections.* For a real vector space V a pair of a vector $v \in V$ and a "covector" $u \in V^*$ such that $\langle u, v\rangle = 2$ defines a liner map $s_{v,u} : V \to V$ by $s_{v,u}x \overset{\text{def}}{=} x - \langle u, sx\rangle v$.

Lemma. (a) $s_{u,v}$ is identity on the hyperplane $u^\perp \subseteq V$ and $s_{v,u} = -1$ on $\Bbbk v$.

(b) $s_{v,u}^2 = id$.

Proof. (a) $s_{u,v}v = v - \langle u, v\rangle v = v - 2v = -v$. (b) follows. □

Remarks. (0) We say that $s_{v,u}$ is a *reflection* in the hyperplane $u^\perp$.

(1) If V has an inner product then a non-zero vector $\alpha \in V$ defines a pair (v, u) with vector $v = \alpha$ and the linear functional $u = (\check{\alpha}, -)$ where we denote

$$\check{\alpha} \overset{\text{def}}{=} \frac{2}{(\alpha, \alpha)}\alpha \ \in V.$$

Clearly, $\langle u, v\rangle = 2$ and the corresponding reflection $s_{v,u}$ only depends on vector α so we denote it

$$s_\alpha x \ = \ x - \langle(\check{\alpha}, -), x\rangle\alpha \ = \ x - (\check{\alpha}, x)\alpha \ = \ x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha.$$

This is a reflection in the hyperplane H_α of all vectors orthogonal to α. The reflection s_α is *orthogonal*, meaning that it preserves the inner product on V. (Because $V = \mathbb{R}\alpha \oplus H_\alpha$ is an orthogonal decomposition and s_α is ± 1 on summands.)

(2) We will for now work with in an Euclidean vector space V and use the orthogonal reflections s_α. However, eventually one finds it advantageous to work withe pairs of a vector space V which contains α and the dual vector space V^* which contains $\check{\alpha}$.

4.2.2. *Root systems.* A *root system* in a real vector space V with an inner product is a finite subset $\Sigma \subseteq V - 0$ such that

- For each $\alpha \in \Sigma$, reflection s_α preserves Σ.
- For $\alpha, \beta \in \Sigma$, $\langle \alpha, \check{\beta}\rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.
- Σ spans V.

We say that a root system is *reduced* if $\alpha \in \Sigma$ implies that $2\alpha \notin \Sigma$. The non-reduced root systems appear in more complicated representation theories. When we say *root system* we will mean a reduced root system.

Our first example will be the roots Δ of sl_n as a root system in an Euclidean vector space $\mathfrak{h}^*_{\mathbb{R}}$ which we define next.

4.2.3. *Real form $\mathfrak{h}^*_{\mathbb{R}}$ of $\mathfrak{h}^*$, root lattice Q and the positive cone Q_+.* Recall that $\mathfrak{h}^*_0$ has a basis ε_i dual to the basis E_{ii} of $\mathfrak{h}_0$ $(1 \le i \le n)$. Their restrictions to $\mathfrak{h}$ are linear

functionals $\varepsilon_i = \varepsilon_i^o|_{\mathfrak{h}}$ on $\mathfrak{h}$ with $\sum_1^n \varepsilon_i = 0$. While ε_i for $1 \le i < n$ is a basis of $\mathfrak{h}^*$, we will actually use another basis of $\mathfrak{h}^*$ given by the *simple* roots

$$\Pi \stackrel{\text{def}}{=} \{\alpha_i \stackrel{\text{def}}{=} \alpha_{i,i+1} = \varepsilon_i - \varepsilon_{i+1};\ i = 1, ...n-1\}.$$

Lemma. Π is a basis of $\mathfrak{h}^*$.

Proof. We use the relation $-\varepsilon_n = \sum_{i<n} \varepsilon_i = \sum_{i<n} \alpha_{in} + \varepsilon_n$. Solving for ε_n we get that ε_n lies in $\mathrm{span}_{\mathbb{Q}}\Delta$.

However, $\mathrm{span}_{\mathbb{Q}}\Delta = \mathrm{span}_{\mathbb{Q}}\Pi$ since $\Delta^+ \subseteq \mathrm{span}_{\mathbb{N}}\Pi$ (for $i < j$ one has $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$), hence $\Delta \subseteq \mathrm{span}_{\mathbb{Z}}\Pi$. Now $\mathrm{span}_{\mathbb{Q}} + \Pi = \mathrm{span}_{\mathbb{Q}}\Delta$ contains ε_n, hence also all $\varepsilon_i = \varepsilon_n + \alpha_{ni}$. □

Now, inside $\mathfrak{h}^*$ we define

- the real vector subspace $\mathfrak{h}^*_{\mathbb{R}} \stackrel{\text{def}}{=} \oplus_1^{n-1} \mathbb{R}\alpha_i$ generated by simple roots,
- the subgroup $Q \stackrel{\text{def}}{=} \oplus_1^{n-1} \mathbb{Z}\alpha_i$ generated by simple roots,
- the semigroup $Q_+ \stackrel{\text{def}}{=} \oplus_1^{n-1} \mathbb{N}\alpha_i$ generated by simple roots.

We have $\mathfrak{h}^* \supseteq \mathfrak{h}^*_{\mathbb{R}} \supseteq Q \supseteq Q_+$. By the preceding proof we know that $Q = \mathrm{span}_{\mathbb{Z}}\Delta$ so we call it the *root lattice* and $Q_+ = \mathrm{span}_{\mathbb{N}}\Delta^+$ so we call it the *positive cone.*

Lemma. For $\lambda, \mu \in \mathfrak{h}^*$ let $\lambda \le \mu$ mean that $\mu - \lambda \in Q_+$. This is a partial order on $\mathfrak{h}^*$. □

4.2.4. *The inner product on* $\mathfrak{h}^*_{\mathbb{R}}$. We will first define it by a formula and then we will deduce it from an obvious inner product on $\mathfrak{h}^*_{0,\mathbb{R}}$.

Lemma. (a) On $\mathfrak{h}^*_{\mathbb{R}}$ there is a unique inner product such that (α_i, α_j) is

- 2 if $i = j$,
- -1 if i, j are neighbors, i.e., $|j - i| = 1$,
- 0 otherwise.

(b) The inner products of roots are $(\alpha_{ij}, \alpha_{pq}) = \delta_{i,p} - \delta_{i,q} - \delta_{j,p} + \delta_{j,q}$. In more details

- $(\alpha_{ij}, \alpha_{pq}) = 0$ when $\{i.j\}$ and $\{p, q\}$ are disjoint.
- $(\alpha_{ij}, \alpha_{iq}) = 1$ when $q \notin \{i, j\}$;
- $(\alpha_{ij}, \alpha_{ij}) = 2$.

Proof. (a) We can embed the vector space $\mathfrak{h}^*_{\mathbb{R}}$ into $\mathfrak{h}^*_{0,\mathbb{R}} \stackrel{\text{def}}{=} \oplus_1^n \mathbb{R}\varepsilon_i^o$ so that α_i goes to $\varepsilon_i^o - \varepsilon_{i+1}^o$. Then point is that on $\mathfrak{h}^*_{0,\mathbb{R}}$ we have an obvious inner product $(-,-)$ with orthonormal basis ε_i^o. It restricts to an inner product on $\mathfrak{h}^*_{\mathbb{R}}$ such that

$$(\alpha_{ij}, \alpha_{pq}) = \delta_{i,p} - \delta_{i,q} - \delta_{j,p} + \delta_{j,q}.$$

Now all formulas in (b) are clear. □

Corollary. (a) All roots $\alpha \in \Delta(sl_n)$ have the same length ($= \sqrt{2}$).

(b) All possibilities for the angle θ between two roots α, β in $\Delta(sl_n)$ are

(1) $\theta = 2\pi/3$ iff $\alpha + \beta$ is a root;
(2) $\theta = \pi/3$ iff $\alpha - \beta$ is a root;
(3) $\theta = \pi/2$ iff neither of $\alpha \pm \beta$ is a root and $\beta \neq \pm\alpha$.
(4) $\theta = 0$ iff $\beta = \alpha$;
(5) $\theta = \pi$ iff $\beta = -\alpha$.

(c) For $\beta \neq \pm\alpha$ the following are equivalent (i) $\theta = 2\pi/3$; (ii) $(\alpha, \beta) = -1$; (iii) $\alpha + \beta$ is a root; (iv) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$; (v) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. The cosine of the angle between α, β is $\frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}} = \frac{1}{2}(\alpha, \beta)$.

(1) If $\alpha + \beta$ is a root then the pair α, β (or β, α) equals α_{ij}, α_{jk} for some distinct i, j, k. Then $(\alpha, \beta) = -1$ and the cosine is $-\frac{1}{2}$.

(2) If $\alpha - \beta$ is a root then α, β (or $-\alpha, -\beta$) are of the form α_{ij}, α_{kj} for distinct i, j, k. Then $(\alpha, \beta) = 1$ and the cosine is $\frac{1}{2}$.

(3) If neither of $\alpha \pm \beta$ is a root and $\beta \neq \pm\alpha$ then our roots are of the form α_{ij}, α_{pq} for disjoint i, j and p, q. Then $(\alpha, \beta) = 0$.

(4-5) Clearly if $\beta = \alpha$ then $\theta = 0$ and $\beta = -\alpha$ gives $\theta = \pi$.

By now we have proved implications in (1-5) (from RHS to LHS). This implies equivalences. □

Remark. One can derive the formula for the inner product naturally by using the relation to $gl_n(\mathbb{C})$. I will skip it here.[7]

4.2.5. *Roots of sl_n form a root system.*

Lemma. (a) Roots of sl_n form a root system in $\mathfrak{h}^*_{\mathbb{R}}$.

(b) $\check{\alpha} = \alpha$ for each root.

Proof. Most of the properties are clear form the list of roots α_{ij} : there are finitely many roots and none is zero. Already the roots α_i span $\mathfrak{h}^*_{\mathbb{R}}$.

For each $\alpha \in \Sigma$, we have $\check{\alpha} = \frac{2}{(\alpha,\alpha)}\alpha = \alpha$. Therefore, for $\alpha, \beta \in \Delta$ we have $(\alpha, \check{\beta}) = (\alpha, \beta)$ which is one of $0, \pm 1$ so it is an integer.

Finally, to see that reflections s_α preserve Σ we consider $s_\alpha\beta = \beta - (\check{\alpha}, \beta)\alpha = \beta - (\alpha, \beta)\alpha$. If $\beta \perp \alpha$ this is $\beta \in \Delta$. If $\alpha \pm \beta$ is a root then $(\alpha, \beta) = \mp$, hence $s_\alpha\beta = \beta - \mp\alpha = \beta \pm \alpha \in \Delta$. □

[7] I did it in class but it was more confusing then illuminating.

4.2.6. *Positive roots.* For a root system Σ a subset $\Sigma^+ \subseteq \Sigma$ is called a *system of positive rots* if $\Sigma = \Sigma^+ \sqcup -\Sigma^+$ and Σ^+ is closed under addition within Σ, i.e.,

- If $\alpha, \beta \in \Sigma^+$ and $\alpha + \beta \in \Sigma$ then $\alpha + \beta \in \Sigma^+$.

We often write "$\alpha > 0$" for "$\alpha \in \Sigma^+$".

The set of *simple roots* for a system of positive roots Δ^+ is a subset $\Pi \subseteq \Delta^+$ such that

- (i) It generates all positive roots under addition, i.e., $\Delta^+ \subseteq \mathrm{span}_{\mathbb{N}}(\Pi)$ and
- (ii) Π is the smallest subset of Δ^+ with the property (i).

Lemma. (a) The roots that lie in the Borel subalgebra $\Delta(\mathfrak{b}) = \Delta(\mathfrak{n})$ form a system of positive roots Δ^+ in the root system $\Delta(\mathfrak{g})$.

(b) The subset $\Pi = \{\alpha_1, ..., \alpha_{n-1}$ is the set of simple roots for Δ^+.

Proof. The roots in either $\mathfrak{n}$ or $\mathfrak{b}$ are all α_{ij} with $i < j$. Since $-\alpha_{ij} = \alpha_{ji}$, $-\Delta(\mathfrak{n})$ is given by the condition $j < i$, This makes $\Sigma = \Sigma^+ \sqcup -\Sigma^+$ clear. If $\alpha, \beta, \alpha + \beta \in \Delta$ then (after possibly exchanging the order of α and β), we have $\alpha = \alpha_{ij}$, $\beta = \alpha_{jk}$. If $\alpha, \beta \in \Sigma^+$ then $i < j$ and $j < k$, hence $\alpha + \beta = \alpha_{ik}$ with $i < k$. $\square$

Corollary. $\mathfrak{g}_\alpha$ with $\alpha \in \Pi$ generate the Lie subalgebra $\mathfrak{n}$.

4.3. **Finite dimensional representations of sl_n (announcement).**

4.3.1. *The coroot lattice $\check{Q} \subseteq \mathfrak{h}$.* Recall that to any root $\alpha \in \Delta$ we have associated an element $\check{\alpha}$ of $\mathfrak{h}$. We call such elements of $\mathfrak{h}$ the *coroots*. Inside of the set of coroots $\check{\Delta} \stackrel{\text{def}}{=} \{\check{\alpha};\ \alpha \in \Delta\}$ we have the subset $\check{\Delta}^+ \stackrel{\text{def}}{=} \{\check{\alpha};\ \alpha \in \Delta^+\}$ of *positive coroots* and the subset $\check{\Pi} \stackrel{\text{def}}{=} \{\check{\alpha};\ \alpha \in \Pi\}$ of *simple coroots*.

We define the *coroot lattice* $\check{Q} \subseteq \mathfrak{h}$ to be the subgroup generated by by all coroots $\check{\alpha}$, $\alpha \in \Delta$.

Lemma. $\check{Q} = \oplus_{\alpha \in \Pi} \mathbb{Z}\check{\alpha}$ and $\check{Q}_+ = \oplus_{\alpha \in \Pi} \mathbb{N}\check{\alpha}$.

Proof. $\square$

4.3.2. *The weight lattice $P \subseteq \mathfrak{h}^*$.* We define the subgroup $P \subseteq \mathfrak{h}^*$ of *integral weights* to consist of all $\lambda \in \mathfrak{h}^*$ that are integral with coroots, i.e.,

$$P \stackrel{\text{def}}{=} \{\lambda \in \mathfrak{h}^*;\ \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \text{ for } \alpha \in \Delta.$$

We will often omit the word "integral", so we will call P the *weight lattice*.

We will also use the sub semigroup P^+ of *dominant weights*, these are required to be non-negative on each positive coroot.

$$P^+ \stackrel{\text{def}}{=} \{\lambda \in \mathfrak{h}^*;\ \langle \lambda, \check{\alpha} \rangle \in \mathbb{N} \text{ for } \alpha \in \Delta^+\}.$$

An example will be the *fundamental weights* $\omega_1, ..., \omega_{n-1}$ defined as the basis of $\mathfrak{h}^*$ dual to the *simple coroots basis* $\check{\alpha}_1, ..., \check{\alpha}_p$ of $\mathfrak{h}$.

Lemma. (a) $\omega_i = \varepsilon_1 + \cdots + \varepsilon_p$.

(b) $P = \oplus_1^{n-1} \mathbb{Z}\omega_i$ and $P^+ = \oplus_1^{n-1} \mathbb{N}\omega_i$.

Proof. □

4.3.3. *Primitive vectors.* For a representation V of $\mathfrak{g}$ a vector $v \neq 0$ is *primitive* if it lies in V_λ for some $\lambda \in \mathfrak{h}^*$ and $\mathfrak{n}v = 0$.

Lemma. (a) A vector v is killed by $\mathfrak{n}$ iff it is killed by all simple root spaces $\mathfrak{g}_\alpha,\ \alpha \in \Pi$.

(b) For a primitive vector $v \in V_\lambda$ the $\mathfrak{g}$-submodule generated by v is the subspace $\sum_{n\in\mathbb{N}} (\mathfrak{n}_-)^n v$ of V.

Proof. (a) is clear since we know that the simple root spaces generate the Lie algebra $\mathfrak{n}$.

(b) We start by listing all weights of $\mathfrak{g}$ as a sequence $\beta_1, ..., \beta_M, 0, \gamma_1, ..\gamma_M$ so that β_i's are negative roots and γ_i's are positive roots.

By the next proposition the $\mathfrak{g}$-submodule generated by v is the sum of subspaces

$$\mathfrak{g}_{\beta_1}^{p_1} \cdots \mathfrak{g}_{\beta_M}^{p_M}\ \mathfrak{g}_0^r\ \mathfrak{g}_{\gamma_1}^{q_1} \cdots \mathfrak{g}_{\gamma_M}^{p_M} v$$

over all choices of of powers $p_i, r, q_i \in \mathbb{N}$. Now, if one of q_i is > 0 then the whole expression is zero since positive roots kill a primitive vector. So, we only need to consider $\mathfrak{g}_{\beta_1}^{p_1} \cdots \mathfrak{g}_{\beta_M}^{p_M}\ \mathfrak{g}_0^r v$. Since $\mathfrak{g}_0$ preserves the line through v we can assume that $r = 0$. □

Remark. The weight of a primitive vector $v \in V_\lambda$ is said to be a *highest weight*. Here, "highest" refers to the partial order on $\mathfrak{h}^*$ defined by positive roots. Then the precise meaning is that λ is the highest weight in the submodule generated by v. (This follows from the part (b) of the lemma.

4.3.4. *Finite dimensional representations of sl_n.* The following theorem describes the basic structure of finite dimensional representations. Because sl_n is the sum $\sum_{\alpha\in\Delta} \mathfrak{s}_\alpha$ of subalgebras isomorphic to sl_2, the theorem will follow from results for sl_2.

Theorem. V be a finite dimensional representation of $\mathfrak{g} = sl_n$.

(a) Any $h \in \mathfrak{h}$ acts semisimply on V, equivalently $\oplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ is all of V. More precisely, all weights in V are integral, hence

$$V \ = \ \oplus_{\lambda\in P} V_\lambda.$$

(b) Any $x \in \mathfrak{n}$ acts nilpotently on V. Moreover, for $p >> 0$ we have $\mathfrak{n}^p V = 0$.

(c) If $V \neq 0$ then V has a primitive vector.

(d) The weight of any primitive vector is dominant.

Proof. (a) We can restrict the action of $\mathfrak{g}$ on V to any root subalgebras $\mathfrak{s}_\alpha$, $\alpha \in \Delta$. Then, via $sl_2 \xrightarrow{\cong} \mathfrak{s}_\alpha$ our V becomes a representation of sl_2. However, we know that $h \in sl_2$ acts semisimply in any finite dimensional representation of sl_2. Now, because the standard isomorphisms $sl_2 \xrightarrow{\cong} \mathfrak{s}_\alpha$ takes h to $\check{\alpha}$, we know that $\check{\alpha}$ acts semisimply on V.

Finally, $\mathfrak{h}$ has a basis $\check{\alpha}_i = E_{ii} - E_{i+1,i+1}$ of simple coroots. The actions of these on V form a family of commuting semisimple operators, so they have a simultaneous diagonalization. This proves that $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$.

To see that any weight λ in V is integral notice that for any root $\alpha \in \Delta$, the number $\langle \lambda, \check{\alpha} \rangle$ is an eigenvalue of the action of $\check{\alpha}$ on V. These are integers because all eigenvalues of the action of $h \in sl_2$ in any finite dimensional representation of sl_2 are known to be integers.

(b) By $\mathfrak{n}^p V$ we mean the subspace of V spanned by all $x_1 \cdots x_p v$ for $x_i \in \mathfrak{n}$ and $v \in V$. So, the subspace $\mathfrak{n}^p V \subseteq V$ is a sum of subspaces $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} V_\lambda$ over all choices of $\phi_i \in \Delta^+$ and all weights λ in V.

The set $\mathcal{W}(V)$ of weights in V is finite because the $\dim(V) < \infty$.

Recall that $\mathfrak{g}_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$, hence $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} V_\lambda \subseteq V_{\lambda + \sum_1^p \phi_i}$. However, the set $\mathcal{W}(V)$ of weights in V is finite because the $\dim(V) < \infty$. So, for $p >> 0$, and any $\lambda \in \mathcal{W}(V)$ we have that $\lambda + \sum_1^p \phi_i$ is not in $\mathcal{W}(V)$, hence $V_{\lambda + \sum_1^p \phi_i} = 0$.

For such p we have $\mathfrak{n}^p V = 0$, hence in particular for $x \in\in \mathfrak{n}$ we have $x^p V = 0$.

(c) Again, we use the fact that the set $\mathcal{W}(V)$ of weights in V is finite. Therefore, it contains a maximal element λ for the partial order on $\mathfrak{h}^*$ defined by $\lambda \leq \mu$ if $\mu - \lambda \in Q_+ = \mathrm{span}_{\mathbb{N}}(\Delta^+) = \oplus_1^{n-1} \mathbb{N}\alpha_i$. For such λ we have $\lambda + \alpha \notin \mathcal{W}(V)$ for any $\alpha > 0$, hence $\mathfrak{g}_\alpha V_\lambda \subseteq V_{\lambda+\alpha} = 0$. So, $\mathfrak{n} V_\lambda = 0$ and therefore any vector in V_λ is primitive.

(d) Let λ be the weight of some primitive vector v in V. Then for any positive root α we have $\mathfrak{g}_\alpha v \subseteq \mathfrak{n} v = 0$. Since the standard isomorphisms $sl_2 \xrightarrow{\cong} \mathfrak{s}_\alpha$ take e to $\mathfrak{g}_\alpha$, we see that for the action of sl_2 on V via $sl_2 \xrightarrow{\cong} \mathfrak{s}_\alpha \subseteq \mathfrak{g}$ we have $ev = 0$ and $hv = \langle \lambda, \check{\alpha} \rangle \cdot v$. So, v is also a primitive vector for the action of sl_2 on V, so its sl_2-weight $\langle \lambda, \check{\alpha} \rangle$ must be in $\mathbb{N}$. □

Remark. If u and v are primitive vectors of weights λ and μ in representations U and V then $u \otimes v$ is a primitive vector of weight $\lambda + \mu$ in $U \otimes V$. □

4.4. **Classification of irreducible finite dimensional representations of $\mathfrak{g} = sl_n$.** We know that any irreducible finite dimensional representation V of $\mathfrak{g}$ has a primitive vector with a dominant weight.

Theorem. (a) For each dominant weight $\lambda \in P^+$ there is exactly one (up to isomorphism) irreducible finite dimensional representation with a primitive vector of weight λ. We denote it $L(\lambda)$.

(b) $L(\lambda),\ \lambda \in\ P^+$ is the complete list of irreducible finite dimensional representations of $\mathfrak{g}$.

Proof. (b) follows from (a) since we know that any irreducible finite dimensional representation V of $\mathfrak{g}$ has a primitive vector with a dominant weight.

Claim (a) consists of two parts

- *Existence*: for $\lambda \in P^+$ there exists an irreducible finite dimensional representation with a primitive vector of weight λ.
- *Uniqueness*: If L, L' are two irreducible finite dimensional representation with a primitive vector of weight λ then $L' \cong L$.

We will next prove existence and postpone the proof of uniqueness until the general setting of semisimple Lie algebras.

4.4.1. *Questions.* For the sl-module $V = \Bbbk^n$ what are the highest weights of irreducible modules (i) V, (ii) $\wedge^p\ V$, (iii) the adjoint representations $\mathfrak{g}$, (iv) $S^p\ V$?

4.5. **The $\mathfrak{g}$-submodule generated by a vector.** The proof of the following proposition will later be a motivatxxx

Lemma. Let v be a primitive vector of weight λ in a$\mathfrak{g}$-module V. Denote by S the $\mathfrak{g}$-submodule generated by v.

(a) $S =\ U\mathfrak{n}^-{\cdot}v$.

(b) The weights of S lie in $\lambda - Q^+$, i.e., for any $\mu \in \mathcal{W}(S)$ one has $\mu \leq \lambda$. Moreover, $V_\lambda = \Bbbk v$. (So, λ is the highest weight of S.)

(c) S has a unique irreducible quotient L. One has $\dim(L_\lambda) = 1$.

Proof. (a) *The first proof.* The $\mathfrak{g}$-submodule generated by any vector v is the the subspace of V spanned by all $x_1 \cdots x_p v$ for $x_i \in \mathfrak{g}$. This is the same as $U\mathfrak{g}{\cdot}v$.

We know that the multiplication $U\mathfrak{n}^- \otimes U\mathfrak{b}\ \to U\mathfrak{g}$ is surjective, so $S\ =\ \ U\mathfrak{g}{\cdot}v\ =\ U\mathfrak{n}^-{\cdot}(U\mathfrak{b}{\cdot}v) =\ U\mathfrak{n}^-{\cdot}\Bbbk v =\ U\mathfrak{n}^-{\cdot}v$.

(b) follows from (a). For this we choose a basis $x_1, ..., x_N$ of $\mathfrak{n}^-$ so that x_i lies in $\mathfrak{g}_{\phi_i}$, where $\phi_1, ..., \phi_N$ is any ordering of roots in $\Delta(\mathfrak{n}^-) =\ -\Delta(\mathfrak{n}) = -\Delta^+$. Then the monomials $x_1^{e_1} \cdots x_N^{e_N}$ span $U\mathfrak{n}_-$ and $x_1^{e_1} \cdots x_N^{e_N}$ lies in $(U\mathfrak{n}_-)_{\sum\ e_i\phi_i}$. Moreover, $\sum\ e_i\phi_i$ is 0 iff all e_i are 0.

(c) Quotients Q of S correspond to submodules S' of S. A quotient Q is irreducible iff the submodule S' is a maximal proper submodule. Therefore, an equivalent formulation is that

- (i) S has exactly one maximal proper submodule $\mathcal{S}$ and that

- (ii) $\mathcal{S}$ does not have weight λ (so that for $L = S/\mathcal{S}$ we have $L_\lambda = S_\lambda/\mathcal{S}_\lambda$ is the line S_λ.

For this we notice that

$(*)$ *For any proper submodule $S' \subseteq S$, S' does not contain weight λ.*

Clearly, if $S'_\lambda \subseteq S_\lambda = \Bbbk v$ would be nonzero then S' would contain v and then S' would contain all of S.

Now it is clear that there exists the largest proper submodule $\mathcal{S}$ of S – this is just the sum of all proper submodules S'. This $\mathcal{S}$ is proper since $\mathcal{S}_\lambda = \sum S'_\lambda = 0$. $\square$

We say that λ is the *highest weight* of S in the sense that it is the largest weight in S for the partial order defined by Q_+. (For this reason we also call primitive vectors the *highest weight vectors.*)

Corollary. (a) An irreducible finite dimensional $\mathfrak{g}$-module L has precisely one primitive vector. Its weight λ is the highest weights of L.

(b) If for a given $\lambda \in P^+$ there exists a finite dimensional representation with a primitive vector of weight λ, then there exists an irreducible finite dimensional representation with the highest weight λ.

If for a given $\lambda \in P^+$ there exists a finite dimensional representation U with a primitive vector of weight λ, then there exists an irreducible finite dimensional representation V with a primitive vector of weight λ.

Proof. (a) Since L is irreducible it has a primitive vector v. The submodule S generated by v is not zero so it is all of L. Therefore λ is the highest weight in $S = L$. So, any primitive vector lies in L_λ for the highest weight λ in L. However, L_λ is one dimensional by the lemma.

(b) If v is a primitive vector of weight λ in a finite dimensional representation V then we get an irreducible representation of highest weight λ by taking the unique irreducible quotient of the submodule generated by v. $\square$

4.5.1. *The second proof of the part (a) of the theorem.* This can be skipped – we write the same proof but without introducing the enveloping algebra. So, this version can be viewed as a motivation for introducing the enveloping algebras in the first place.

Proposition. Let us write all weights of $\mathfrak{g}$ as a sequence $\beta_1, ..., \beta_N$. Then for any representation V the $\mathfrak{g}$-submodule generated by a given vector v is the sum of subspaces $\mathfrak{g}_{\beta_1}^{p_1} \cdots \mathfrak{g}_{\beta_N}^{p_N} v$ over all choices of of powers $p_i \in \mathbb{N}$.

Proof. The $\mathfrak{g}$-submodule generated by v is the the subspace of V spanned by all $x_1 \cdots x_p v$ for $x_i \in \mathfrak{g}$. We can think of it as the sum of subspaces $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} v$ over all choices of $\phi_i \in \mathcal{W}(\mathfrak{g}) = \Delta \sqcup 0$. Let $\mathcal{V}_q$ be the sum of all such subspaces $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} v$ with $p \leq q$. It contains $\mathcal{U}_q$ which is the sum of subspaces $\mathfrak{g}_{\beta_1}^{p_1} \cdots \mathfrak{g}_{\beta_N}^{p_N} v$ for all choices of of powers p_i such that $\sum p_i \leq q$. We will prove by induction in q that $\mathcal{U}_q \subseteq \mathcal{V}_p$ is equality.

If the sequence $\phi_1, ..., \phi_p$ is compatible with the chosen order on $\mathcal{W}(\mathfrak{g})$ then $\mathfrak{g}_{\phi_1} \cdots \mathfrak{g}_{\phi_p} v$ is of the above form $\mathfrak{g}_{\beta_1}^{p_1} \cdots \mathfrak{g}_{\beta_N}^{p_N} v$. If not then there are some neighbors ϕ_{i-1}, ϕ_i which are in the wrong order. However, for x and y in $\mathfrak{g}_{\phi_{i-1}}$ and $\mathfrak{g}_{\phi_i}$,

$$\pi(x)\pi(y) = \pi(y)\pi(x) + [\pi(x), \pi(y)] = \pi(y)\pi(x) + \pi[x, y].$$

So, we can replace the product of length two $\mathfrak{g}_{\phi_{i-1}}\mathfrak{g}_{\phi_i}$ with the product in the opposite order $\mathfrak{g}_{\phi_i}\mathfrak{g}_{\phi_{i-1}}$ (again of length 2), at the price of adding a term which is a product of length 1. □

4.6. **Existence of irreducible representations.** Here we prove the first part of the theorem 4.4.

Lemma. For each dominant weight $\lambda \in P^+$ there exists an irreducible finite dimensional representation L with a primitive vector of weight λ. (Then λ is the highest weight in L.)

Proof. From homeworks we know that when λ is one of the fundamental ω_i then such representation is given by $\wedge^i \Bbbk^n$. Denote by v_{ω_i} its primitive vector.

Now for any dominant weight $\lambda \in P^+$ we have $\lambda = \sum_1^{n-1} \lambda_i \omega_i$ with $\lambda_i \in \mathbb{N}$. Then in $\otimes_1^{n-1} (\wedge^i \Bbbk^n)^{\otimes \lambda_i}$ the vector $\otimes_1^{n-1} v_i^{\otimes \lambda_i}$ is primitive of weight λ (see the remark in 4.3.4). □

4.7. **Uniqueness of irreducible representation with a given highest weight.**

4.7.1. *Irreducible finite dimensional representations of* $\mathfrak{b}$.

Lemma. (a) $\mathfrak{n}$ is an ideal in $\mathfrak{b}$.

(b) $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$ and $\mathfrak{b}^{\mathrm{ab}} \cong \mathfrak{h}$.

Proof. $\mathfrak{h}$ is commutative, i.e., $[\mathfrak{h}, \mathfrak{h}] = 0$. Also for a root α we have $[\mathfrak{h}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ since $\mathfrak{h}$ acts on $\mathfrak{g}_\alpha$ by $\alpha \in \mathfrak{h}^*$ which is not zero. This implies that $[\mathfrak{h}, \mathfrak{b}] = \mathfrak{n}$. Together with $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{n}$ ($\mathfrak{n}$ is a subalgebra) this implies that $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$.

Now $\mathfrak{b}^{\mathrm{ab}} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$. □

Remark. Using $\mathfrak{b} \twoheadrightarrow \mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$ we get $\mathfrak{h}^* \hookrightarrow \mathfrak{b}^*$. The meaning is that a linear functional λ on$\mathfrak{h}$ extends to $\mathfrak{b}$ by zero on $\mathfrak{n}$.

Proposition. (a) Any $\lambda \in \mathfrak{h}^*$ gives a 1-dimensional representation $\Bbbk_\lambda^{\mathfrak{b}}$ of $\mathfrak{b}$. The vector space is $\Bbbk$ and $\mathfrak{b}$ acts on it by λ viewed as a functional on $\mathfrak{b}$, i.e., $x \cdot 1_{\Bbbk} = \langle \lambda, x \rangle \, 1_{\Bbbk}$.

(b) This is the complete classification of 1 dimensional representations of $\mathfrak{b}$.

Proof. (b) is a case of lemma 6.2.1 since $\mathfrak{b}^{\mathrm{ab}} = \mathfrak{h}$. □

4.7.2. *Verma modules for* $\mathfrak{g} = sl_n$. The *Verma module* with the highest weight λ is defined as the induced module[8]

$$M(\lambda) \stackrel{\text{def}}{=} Ind_{\mathfrak{b}}^{\mathfrak{g}} \Bbbk_\lambda^{\mathfrak{b}} \stackrel{\text{def}}{=} U\mathfrak{g} \otimes_{U\mathfrak{b}} \Bbbk_\lambda^{\mathfrak{b}}.$$

The most obvious vector in $M(\lambda)$ is $v_\lambda = 1_{U\mathfrak{g}} \otimes 1_{\Bbbk}$.

[8] The notation we initially used used in class was M_λ.

Lemma. (a) v_λ is a primitive vector with weight λ.

(b) v_λ generates M_λ.

4.7.3. *Corollary.* (1) $\mathfrak{h}$ acts semisimply on $M(\lambda)$ and the weights $\mathcal{W}(M(\lambda))$ lie in $\lambda - Q_+$ (i.e., weights are $\leq \lambda$). Moreover the λ weight space $M(\lambda)_\lambda$ is $\Bbbk v_\lambda$.

(2) $M(\lambda)$ has the largest proper submodule $M(\lambda)^+$. Equivalently, it has a unique irreducible quotient, we denote it $L(\lambda)$. $L(\lambda)$ is also generated by a primitive vector of weight λ (the image of v_λ which we again denote v_λ) and $L(\lambda)_\lambda = \Bbbk v_\lambda$.

4.7.4. *The universal property of Verma modules.* The categorical formulation of the following lemma is that the object $M(\lambda) \in \mathfrak{m}(\mathfrak{g})$ represents the functor $-^o_\lambda : \ \mathfrak{m}(fg) \to Vec_\Bbbk$ of taking the primitive vectors.

Lemma. For any $\mathfrak{g}$-module V there is a canonical isomorphism

$$\mathrm{Hom}_\mathfrak{g}[M(\lambda), V] \ \underset{\cong}{\overset{\iota}{\to}} \ V^o_\lambda.$$

Here, $\iota(\phi) \ = \ \phi(v_\lambda) \in V^o_\lambda$.

Proof. We use the Frobenius reciprocity, i.e., the fact that the induction $U\mathfrak{g} \otimes_\mathfrak{b} -$ is the left adjoint of the forgetful functor $\mathcal{F}^\mathfrak{b}_\mathfrak{g})$:

$$\mathrm{Hom}_\mathfrak{g}[M(\lambda), V] = \ \mathrm{Hom}_{U\mathfrak{g}}[U\mathfrak{g} \otimes_{U\mathfrak{b}} \Bbbk^\mathfrak{b}_\lambda), V] \ \cong \ \mathrm{Hom}_{U\mathfrak{b}}(\Bbbk^\mathfrak{b}_\lambda, V).$$

A linear map $\psi : \Bbbk^\mathfrak{b}_\lambda \to V$ is the same as a choice of a vector $v = \psi(1_\Bbbk)$ in V. Now, ψ is an $\mathfrak{h}$-map iff $\mathfrak{h}$ acts on v by λ, and ψ is an $\mathfrak{n}$-map iff $\mathfrak{n}$ kills v. So, ψ is an $\mathfrak{b}$-map iff $v \in V^o_\lambda$,

Now one checks that the isomorphism $\iota : \ \mathrm{Hom}_\mathfrak{g}[M(\lambda), V] \overset{\cong}{\to} V^o_\lambda$ that we have constructed acts by the formula in the lemma. □

Corollary. For any $\lambda \in \mathfrak{h}^*$ there is a unique irreducible $\mathfrak{g}$-module L which has a primitive vector of weight λ. This L is the unique irreducible quotient $L(\lambda)$ of the Verma $M(\lambda)$.

Proof. For existence of L we note that the above $L(\lambda)$ satisfies the properties. We will also see that any irreducible $\mathfrak{g}$-module L which has a primitive vector v of weight λ is isomorphic to $L(\lambda)$.

First, to a primitive vector v in L there corresponds some homomorphism $\phi : \mathbb{M}(\lambda) \to L$. Since $v \neq 0$ we have $\phi \neq 0$. Then $0 \neq Im(\phi)$ is a submodule of L, since L is irreducible we have $Im(\phi) = L$, i.e., L is an irreducible quotient of $M(\lambda)$. But there is only on irreducible quotient of $M(\lambda)$ and it is $L(\lambda)$. □

4.8. **Proof of the classification of irreducible finite dimensional representations of sl_n (theorem 4.4). A.** The first claim in this theorem is that for each dominant weight $\lambda \in P^+$ there is exactly one irreducible finite dimensional representation$L$ with a primitive vector of weight λ.

The existence of L was proved in 4.6. The uniqueness is a special case of the corollary 4.7.4. This corollary also says that such L is the representation $L(\lambda)$ constructed as the unique irreducible quotient of $M(\lambda)$.

B. The second claim in the theorem is that $L(\lambda)$ for $\lambda \in P^+$ is the classification of irreducible finite dimensional representations of $\mathfrak{g}$, i.e., that

- (i) any irreducible finite dimensional representation L is isomorphic to one of $L(\lambda)$'s and
- (ii) there are no repetitions in the list, i.e., the only way that $L(\lambda) \cong L(\mu)$ is when $\lambda = \mu$.

For (i) notice that since L is irreducible we have $L \neq 0$. Then we know that since L is finite dimensional and $\neq 0$ it has a primitive vector of some weight $\lambda \in P^+$. Then the corollary 4.7.4 guarantees that L is $L(\lambda)$.

For (ii), recall that λ is the highest weight in $L(\lambda)$, so if $L(\lambda) \cong L(\mu)$ then they have the same highest weight hence $\lambda = \mu$. □

Remark. Our proof of this classification is actually not yet complete. The one thing that we have used without proof is the semisimplicity theorem for sl_2. We will prove this theorem for all sl_n's in 4.10 below.

4.9. **Classification of finite dimensional representations of $\mathfrak{g} = sl_n$.**

Theorem. Finite dimensional representations of $\mathfrak{g} = sl_n$ are semisimple. So, each one is isomorphic to a sum $\oplus_{\lambda \in P^+} L(\lambda)^{m_\lambda}$ for some multiplicities $m_\lambda \in \mathbb{N}$. □

Again, we will postpone the proof for the general setting of semisimple Lie algebras.

Remark. As in sl_2, effectively such decomposition comes from choosing a basis $v_1^\lambda, ..., v_{m_\lambda}^\lambda$ of the spaces V_λ^0 of primitive vectors for each dominant weight λ.

4.10. **Proof of the semisimplicity theorem 4.9.** In order to show that the category $Rep^{fd}(\mathfrak{g})$ is semisimple it suffices to show that any extension

$$0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0$$

splits. The following proof is the most elementary one but it is not obviously enlightening.

4.10.1. *A modification of extensions.*

Lemma. For any extension of $\mathfrak{g}$-modules

$$(\mathcal{E}) \quad 0 \to W \xrightarrow{\alpha} V \to V/W \to 0$$

consider the $\mathfrak{g}$-modules $\mathcal{H} \supseteq \mathcal{H}_1 \subseteq \mathcal{H}_2$ where

- $\mathcal{H} = \mathrm{Hom}_{\Bbbk}(V, W)$,
- $\mathcal{H}_1 = \{A \in \mathrm{Hom}_{\Bbbk}(V, W);\ A|_W \text{ is a scalar } c1_W\}$,
- $\mathcal{H}_2 = \{A \in \mathrm{Hom}_{\Bbbk}(V, W);\ A|_W = 0|]$.

Then we have an extension of $\mathfrak{g}$-modules

$$(\mathcal{E}') \quad 0 \to \mathcal{H}_2 \underset{\subseteq}{\xrightarrow{\alpha}} \mathcal{H}_1 \xrightarrow{\beta} \Bbbk \to 0,$$

where the action of $\mathfrak{g}$ on $\Bbbk$ is trivial and $\beta(A) = c$ when $A|_W = c1_W$.

4.10.2. *Reduction to extensions with $E'' = \Bbbk$.*

Lemma. It suffices to know that the extensions split when the last term is the trivial module $\Bbbk$.

Proof. Then the SES $\mathcal{E}'$ splits, i.e., that there is a $\mathfrak{g}$-invariant line L in $\mathcal{H}_1$ such that $\beta|_L : L \to \Bbbk$ is an isomorphism. Let A be the element of L such that $\beta(A) = 1$, i.e., $A|_W = id_W$. Since the action of $\mathfrak{g}$ on $\Bbbk$ is trivial, the action of $\mathfrak{g}$ on L is also trivial, i.e., for $x \in \mathfrak{g}$ we have that $xA = x \circ A - A \circ x$ is zero. In other words, $A : V \to W$ is a $\mathfrak{g}$-map.

Now, $A \circ \alpha = id_W$, i.e., A is a retraction for the original SES $\mathcal{E}$. So, $\mathcal{E}$ itself splits. $\square$

4.10.3. *Reduction to extensions with $E'' = \Bbbk$ and E' irreducible.* We will here assume that all SES of the form $0 \to I \to U \to \Bbbk \to 0$ with I irreducible do split. Then we will prove (by induction in dimension of W) that *any* SES of the form $0 \to W \to V \to \Bbbk \to 0$ splits.

If W is not irreducible then it has a proper submodule W' with $W' = 0$ and $W' \neq W$. Since W is finite dimensional there exists a maximal such W'. Now we consider SES

$$0 \to \frac{W}{W'} \to \frac{V}{W'} \xrightarrow{\gamma} \frac{V}{W'} / \frac{W}{W'} \to 0.$$

The last terms is isomorphic to $V/W \cong \Bbbk$. However, W/W' is irreducible since W' is a maximal submodule of W. So, our assumption implies that this SES splits, i.e., there is a $\mathfrak{g}$-invariant line $L \subseteq V/W'$ complementary to W/W'.

Now we use something specific to the Lie algebra $\mathfrak{sl}_n$, by 6.2.1 the only 1-dimensional representation L of $\mathfrak{g}$ is the trivial representation $\Bbbk$, hence $L \cong \Bbbk$.

We can write $L \subseteq V/W'$ as W''/W' for some submodule W'' of V that contains W'. In the corresponding SES

$$0 \to W' \to W'' \to L \to 0$$

we have $L \cong \Bbbk$. and also $\dim(W'') < \dim(W)$. So, by the induction assumption we know that this SES splits. This means that W'' (which lies in W hence in V) contains some $\mathfrak{g}$-invariant line L'' complementary to W'.

By now,we can rewrite V/W' using $V/V' \cong W/W' \oplus L$ and $W'' = W' \oplus L''$, as

$$\frac{V}{W'} \cong \frac{W}{W'} \oplus L \cong \frac{W}{W'} \oplus \frac{W''}{W'} \cong \frac{W}{W'} \oplus \frac{W' \oplus L''}{W'} \cong \frac{W \oplus L''}{W'}.$$

Therefore, $V \cong W \oplus L'$.

4.10.4. *The case with $E'' = \Bbbk = E'$.* Here, E is two dimensional with a basis e_1, e_2 with $e_1 \in E'$. Therefore for the action π of $\mathfrak{g}$ on E the matrix of $x \in \mathfrak{g}$ in this basis is of the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. Therefore, π is a morphism of Lie algebras from $\mathfrak{g}$ to $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ which is one dimensional hence abelian. Now $\pi(\mathfrak{g})$ is an abelian quotient of $\mathfrak{g}$. We know that since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ is equality for $\mathfrak{g} = sl_n$, so $\mathfrak{g}$ has no abelian quotients. Therefore, $\pi(\mathfrak{g}) = 0$, i.e., the action of $\mathfrak{g}$ on E is trivial so any line complementary to E' is a $\mathfrak{g}$ submodule.

4.10.5. *The case with $E'' = \Bbbk$ and E' is irreducible but not $\Bbbk$.* In this case we will produce the splitting as a $\mathfrak{g}$-map $C : E \to E$ which preserves $\mathcal{E}'$ and has different eigenvalues α, β on E' and $E/E' \cong E''$. Then the kernel of $C - \beta$ will be a submodule of E complementary to E'.

A supply of operators that act on each $\mathfrak{g}$-module E and preserve each $\mathfrak{g}$-submodule E' is given by elements C of $U\mathfrak{g}$. In order that the operator $C : E \to E$ given by $C \in U\mathfrak{g}$, to be a $\mathfrak{g}$-map we need C to be in the center $Z(U\mathfrak{g})$. In fact we will choose C as the Casimir operator $C^\beta \in Z(U\mathfrak{g})$ from the appendix 4.11. □

4.11. **Appendix. Casimir elements $C \in Z(U\mathfrak{g})$.** By the Casimir element for sl_n we mean the the Casimir element $C = C^\beta$ of Let β be the bilinear form $\beta(x, y) \stackrel{\text{def}}{=} Tr_{\Bbbk^n}(xy)$ on $\mathfrak{g}$.

Lemma. (a) β is nondegenerate.

(b) $\mathfrak{h}^\perp = \mathfrak{n} \oplus \mathfrak{n}^-$ and $\mathfrak{b}^\perp = \mathfrak{n}$.

(c) The restriction of β to $\mathfrak{h}_\mathbb{R} \subseteq \mathfrak{h} \subseteq \mathfrak{g}$ is the standard inner product **??**.

Proof. In the proof of the lemma 3.3.4.c. we have noted that the extension β_0 of β to $\mathfrak{g}_0 = gl_n$ satisfies $\beta_0(E_{ij}, E_{pq}) = \delta_{ij,qp}$. So, the restriction to the Cartan $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is the standard inner product, i.e., it has orthonormal basis E_{ii}, $1 \le i \le n$.

The restriction β of β_0 to sl_n is also nondegenerate because $sl_n \perp \Bbbk 1_n$ for β_0. Moreover, the restriction of β to $\mathfrak{h}_\mathbb{R}$ is the restriction of β_0 from $\mathfrak{h}_0$, and this is the standard inner product $(-, -)$ on $\mathfrak{h}_\mathbb{R}$.

Claim (b) is also obvious from the formula for β_0. □

Denote $\rho = \sum_{\alpha>0}\ \alpha \in \mathfrak{h}^*$ and $\check{\rho} = \sum_{\alpha>0}\ \check{\alpha} \in \mathfrak{h}$.

Corollary. (a) For the standard inner product on $\mathfrak{h}_{\mathbb{R}}$ let k_i be the basis of $\mathfrak{h}$ dual to the basis $\check{\alpha}_i$ $(1 \le i < n)$, For a root $\alpha = \alpha_{ij}$ denote $E_\alpha = E_{ij} \in \mathfrak{g}_\alpha$. Then the Casimir element is

$$C^\beta \ = \ 2\sum_{\alpha>0} F_\alpha E_\alpha \ + \ \sum_1^{n-1}\ k_i\check{\alpha}_i \ + \ 2\check{\rho}.$$

(b) For $\mathfrak{g} = sl_n$ the Casimir element $C = C^\beta$ acts on the representations $L(\lambda)$ and $M(\lambda)$ by the scalar

$$(\alpha, \lambda + 2\rho) \ = \ (\lambda+\rho)^2 - \rho^2.$$

(c) Casimir distinguishes the trivial representation $L(0) = \Bbbk$ from all other irreducibles $L(\lambda)$.

Proof. (a) We use the basis of $\mathfrak{h}$ given by $E_\alpha,\ \alpha \in \Delta$ and $\check{\alpha}_i,\ 1 \le i < n$. Its β-dual basis $E_{-\alpha},\ \alpha \in \Delta$ and $k_i,\ 1 \le i < n$. So,

$$C^\beta \ = \ \sum_{\alpha\in\Delta} E_{-\alpha}E_\alpha + \ sum_{i<n}\ k_i\check{\alpha}_i.$$

It remains to notice that $E_\alpha E_{-\alpha} = E_\alpha E_\alpha + \check{\alpha}$ gives

$$\sum_{\alpha\in\Delta} E_{-\alpha}E_\alpha \ = \ \sum_{\alpha\in\Delta^+} E_{-\alpha}E_\alpha \ + \ \sum_{\alpha\in\Delta^+} E_\alpha E_{-\alpha} \ = \ \sum_{\alpha\in\Delta^+} 2E_{-\alpha}E_\alpha \ + \ \sum_{\alpha\in\Delta^+} \check{\alpha}.$$

(b) On the primitive vector v of highest weight λ, C^β acts by $\sum_1^{n-1}\ k_i\check{\alpha}_i \ + \ \sum_{\le>0}\ \check{\alpha}$, i.e., by

$$\sum_1^{n-1} \langle\lambda, k_i\rangle\ \langle\lambda, \check{\alpha}_i\rangle + \ + \sum_{\le>0} \ < \lambda, \check{\alpha} > .$$

The first term is (λ, λ) and the second is $\sum_{\le>0} (\lambda, \alpha) \ = \ (\lambda, 2\rho)$.

(c) The only $L(\lambda)$ on which C acts by 0 is $L(0)$. The point is that $(\lambda, \lambda + 2\rho) = \lambda^2 + (\lambda, 2\rho) \ge \lambda^2$ since

$$(\lambda, 2\rho) = \sum_{\alpha>0} (\lambda, \alpha) = \sum_{\alpha>0} \langle\lambda, \check{\alpha}\rangle \ \ge 0$$

since each term is ≥ 0 for dominant λ. $\square$

Remarks. (1) Casimir alone does not distinguish all irreducible finite dimensional representations $L(\lambda)$. However, the whole center $Z(U\mathfrak{g})$ does. One has Harish-Chandra's isomorphism

$$Z(U\mathfrak{g}) \ \cong \ (S\mathfrak{h})^W = \mathcal{O}(\mathfrak{h}^*//W)$$

which describes the center as invariants of the Weyl group W in the symmetric algebra $S\mathfrak{h}$ of the Cartan.

Here, $S\mathfrak{h} = \mathcal{O}(\mathfrak{h}^*)$ are the polynomial functions on $\mathfrak{h}^*$ and the W-invariant functions are functions on the "invariant theory quotient" $\mathfrak{h}^*//W$.

(2) ρ is dominant, actually $\rho = \sum_1^{n-1} \omega^i$.

(3) Our proof uses standard inner products on $\mathfrak{h}$ and $\mathfrak{h}^*$ and their relation. I should rewrite these to make everything more clear.

4.11.1.

5. Category $\mathcal{O}$

Classification of all irreducible modules for sl_n is a *wild* problem, i.e., one can prove that we do not have a way to list all irreducibles. (This is an observation in mathematical logic.) Instead, what is interesting is to classify irreducible representations lying in certain interesting subcategories. One subcategory is the category $Rep^{fd}(\mathfrak{g})$ of all finite dimensional representations.

The next most basic and most influential one is the category $\mathcal{O}$ introduced by Joseph Bernstein, Israel Gelfand and Sergei Gelfand.[9] here "$\mathcal{O}$" stands for *ordinary* (in Russian).

For us the category $\mathcal{O}$ is the home for objects that we have already encountered in our study of finite dimensional representations – Vermas $M(\lambda)$ and irreducibles $L(\lambda)$. It also gives us an opportunity to notice how the behavior of infinite dimensional $\mathfrak{g}$-modules is more subtle than that of finite dimensional ones.

5.1. **Category $\mathcal{O}$ for $\mathfrak{g} = sl_n$.** This is the subcategory of the category $Rep(\mathfrak{g}) = \mathfrak{m}(U\mathfrak{g}))$ of $\mathfrak{g}$-representations (i.e., $U\mathfrak{g}$-modules) that consists of all $\mathfrak{g}$-modules V such that

(1) V is finitely generated;
(2) $\mathfrak{h}$ acts semisimply on V, i.e., $V = \oplus_{\lambda\in\mathfrak{h}^*} V_\lambda$;
(3) V is *locally finite* for the subalgebra $\mathfrak{n}$. The meaning is that for any vector v in V, the $\mathfrak{n}$-submodule $U(\mathfrak{n})v$ that it generates is finite dimensional.

Lemma. (a) The category $Rep^{fd}(\mathfrak{g})$ of finite dimensional representations lies in $\mathcal{O}$.

(b) If V is in $\mathcal{O}$ then any submodule or quotient of V is also in $\mathcal{O}$.

Theorem. (a) Verma modules $M(\lambda)$ lie in $\mathcal{O}$.

(b) The irreducible representations in $\mathcal{O}$ are precisely all $L(\lambda)$, $\lambda \in \mathfrak{h}^*$.

5.2. **The Kazhdan-Lusztig theory.** It deals with the structure of Verma modules. The basic fact is the following.

Lemma. Any $V \in \mathcal{O}$ has a finite length, i.e., it has a finite filtration $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$ with all graded pieces $Gr_i(V) = V_{i-1}/V_i$ irreducible. $\square$

Such filtration is called a *Jordan-Hoelder* series of V. It is a general fact in algebra that though such filtration need not be unique, the number of times a given irreducible module L appears in the list of $Gr_i(V)$'s for $i = 1, .., n$ is independent of the choice of the filtration. This number is called the *multiplicity* of L in V and it is denoted $[V : L]$. When V is in $\mathcal{O}$ then all subquotients $Gr_i(V)$ are again in $\mathcal{O}$, hence each is of the form $L(\mu)$ for some $\mu \in \mathfrak{h}^*$.

[9] Israel was one of the most important mathematicians in 20th century. Sergei is his son.

Problem. For any $\lambda, \mu \in \mathfrak{h}^*$ find the multiplicity $[M(\lambda) : L(\mu)]$ of irreducibles $L(\mu)$ in Verma modules $M(\lambda)$.

A conjectural answer to this question was provided by a joint work of Kazhdan and Lusztig (the "Kazhdan-Lusztig" conjecture). The proof was obtained by Beilinson-Bernstein and independently by Brylinski-Kashiwara. It was based on

- the theory of *D-modules* which is the algebraization of the theory of linear partial differential equations;
- the *intersection homology* and *perverse sheaves* in algebraic topology of complex algebraic varieties;
- Deligne's proof of Weil conjectures on the use of positive characteristic geometry for algebraic topology of complex algebraic varieties.

The Beilinson-Bernstein version was very strong and elegant, so it has a become one of basic modes of thinking in representation theory and one of a few origins of the so called *Geometric Representation Theory* (The other two are Drinfeld's *Geometric Langlands program* and Springer's construction of representations of *Weyl groups* such as the symmetric groups S_n).

6. **Lie algebras**

6.1. **Solvable and nilpotent Lie algebras.**

6.1.1. *Ideals.* A subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is an *ideal* if $[\mathfrak{g},\mathfrak{a}]\subseteq\mathfrak{a}$. Then it is clearly a subalgebra.

Lemma. (a) If $\mathfrak{a},\mathfrak{a}$ are ideals, so is $[\mathfrak{a},\mathfrak{b}]$.

(b) Kernels of morphisms of Lie algebras are ideals.

(c) Consider a subspace $\mathfrak{a}\subseteq\mathfrak{g}$. The quotient vector space $\mathfrak{g}/\mathfrak{a}$ has a Lie algebra structure such that the quotient map $\mathfrak{g}\twoheadrightarrow\mathfrak{g}/\mathfrak{a}$ is a Lie algebra map, if and only if $\mathfrak{a}$ is an ideal. Then this structure is unique and $[x+\mathfrak{a},y+\mathfrak{a}]\stackrel{\text{def}}{=}[x,y]+\mathfrak{a}$.

Examples. (a) $[\mathfrak{g},\mathfrak{g}]$ is the smallest ideal in $\mathfrak{g}$ such that the quotient Lie algebra is abelian. Equivalently, $\mathfrak{g}^{ab}\stackrel{\text{def}}{=}\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is the largest quotient Lie algebra of $\mathfrak{g}$ that is commutative.

(b) The center $Z(\mathfrak{g})=\{a\in\mathfrak{g};\ [a,\mathfrak{g}]=0\}$ is an ideal.

(c) The normalizer of the subalgebra $\mathfrak{a}\subseteq\mathfrak{g}$ is $N_{\mathfrak{g}}(\mathfrak{a})\stackrel{\text{def}}{=}\{x\in\mathfrak{g};\ [x,\mathfrak{a}]\subseteq\mathfrak{a}\}$. This is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{a}$ is an ideal in its normalizer.

(d) In sl_n, $\mathfrak{n}$ is an ideal in $\mathfrak{b}$. Actually $\mathfrak{b}$ is the normalizer of the subalgebra $\mathfrak{n}$ in $\mathfrak{g}$.

6.1.2. *The derived and lower central series of ideals.*

Lemma. (s_1) [The *derived series* of $\mathfrak{g}$.] The following subspaces are ideals

$$\mathfrak{g}^{(0)}\stackrel{\text{def}}{=}\mathfrak{g},\quad \mathfrak{g}^{(n+1)}\stackrel{\text{def}}{=}[\mathfrak{g}^{(n)},\mathfrak{g}^{(n)}],....$$

(s_2) The derived series is a decreasing sequence: $\mathfrak{g}=\mathfrak{g}^{(0)}\supseteq\mathfrak{g}^{(1)}\supseteq\cdots$. Also, $\mathfrak{g}=\mathfrak{g}^{(0)}\supseteq\mathfrak{g}^{(1)}\supseteq\cdots\mathfrak{g}^{(n)}$ is the smallest sequence of ideals $\mathfrak{a}=\mathfrak{a}_0\supseteq\mathfrak{a}_1\supseteq\cdots\mathfrak{a}_n$ of $\mathfrak{g}$ such that all $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ are abelian for $i=1,...n$.

Lemma. (n_1) [The *lower central series* of $\mathfrak{g}$.] The following subspaces are ideals

$$\mathfrak{g}^{0}\stackrel{\text{def}}{=}\mathfrak{g},\quad \mathfrak{g}^{n+1}\stackrel{\text{def}}{=}[\mathfrak{g},\mathfrak{g}^{n}],....$$

(n_2) The lower central series is a decreasing sequence: $\mathfrak{g}=\mathfrak{g}^0\supseteq\mathfrak{g}^1\supseteq\cdots$. Also, $\mathfrak{g}=\mathfrak{g}^0\supseteq\mathfrak{g}^1\supseteq\cdots\mathfrak{g}^n$ is the smallest sequence of ideals $\mathfrak{a}=\mathfrak{a}_0\supseteq\mathfrak{a}_1\supseteq\cdots\mathfrak{a}_n$ of $\mathfrak{g}$ such that all the action of $\mathfrak{g}$ on $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ is trivial for $i=1,...n$.

Proof. By definitions. □

Example. $\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}^1$.

6.1.3. *Solvable and nilpotent Lie algebras.* We say that the Lie algebra $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(n)} = 0$ for some n. We say that the Lie algebra $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}^n = 0$ for some n.

Remember that $\mathfrak{g}$ is abelian if $[\mathfrak{g}, \mathfrak{g}] = 0$.

Remark. We have $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$, hence

$$\textit{abelian} \ \Rightarrow \textit{nilpotent} \ \Rightarrow \textit{solvable.}$$

Lemma. (s) $\mathfrak{g}$ is solvable iff there exists a sequence of ideals $\mathfrak{a} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \mathfrak{a}_n = 0$ of $\mathfrak{g}$ such that all $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ are abelian for $i = 1, ...n$.

(n) $\mathfrak{g}$ is nilpotent iff there exists a sequence of ideals $\mathfrak{a} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \mathfrak{a}_n = 0$ of $\mathfrak{g}$ such that $\mathfrak{a}$ acts on all $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ by zero.

Corollary. The classes of solvable and nilpotent Lie algebras are both closed under taking subalgebras and quotient algebras. The solvable ones are also closed under extensions (nilpotent are not).

Example. Let $\mathfrak{g} = sl_n$.

(a) Its Borel subalgebra $\mathfrak{b}$ is a solvable Lie algebra and $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$.

(b) The nilpotent radical $\mathfrak{n}$ of $\mathfrak{b}$ is a nilpotent Lie algebra.

(c) $\mathfrak{g} = sl_n$ is not solvable.

Proof. Define the height of an element $\phi \in Q$, $\phi = \sum_1^{n-1} \phi_i \alpha_i$, as $ht(\phi) \sum \phi_i$. For instance the simple roots have height one etc and positive roots have positive heights.

For $i \in \mathbb{N}$ let $\mathfrak{b}_i \subseteq \mathfrak{b}$ be the sum of all $\mathfrak{g}_\alpha$ over $\alpha \in \Delta \sqcup 0 = \mathcal{W}(\mathfrak{g})$, with $ht(\alpha) \geq i$. Then $\mathfrak{b}_0 = \mathfrak{n}$ and $\mathfrak{b}_1 = \mathfrak{n}$. The conditions we impose in each $\mathfrak{b}_i$ is the vanishing of coefficients on certain diagonals, for instance $f_n = 0$.

This gives a *filtration* on the Lie algebra $\mathfrak{b}$, meaning that $[\mathfrak{b}_i, \mathfrak{b}_j] \subseteq \mathfrak{b}_{i+j}$. In particular, each $\mathfrak{b}_i$ is an ideal in $\mathfrak{b}$.

(a) $\mathfrak{b}$ is solvable since each $\mathfrak{b}_i/\mathfrak{b}_{i-1}$ is abelian.

(b) $\mathfrak{n}$ is nilpotent since for $i > 0$ we have $[\mathfrak{n}, \mathfrak{b}_i] = [\mathfrak{b}_1, \mathfrak{b}_i] \subseteq \mathfrak{b}_{i+1}$, hence $\mathfrak{b}$ acts trivially on $\mathfrak{b}_i/\mathfrak{b}_{i+1}$. $\square$

Proposition. (a) If $\mathfrak{g}$ is solvable then so is any subalgebra or quotient algebra.

(b) If $\mathfrak{a}$ is solvable ideal in $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{a}$ is also solvable then $\mathfrak{g}$ is also solvable.

(c) If I, J are solvable ideals in $\mathfrak{g}$ then $I + J$ is too.

Proof. (a) "$\mathfrak{g}$ is solvable" means that there exists a sequence of ideals $\mathfrak{g} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \mathfrak{a}_n = 0$ of $\mathfrak{g}$ such that all $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ are abelian for $i = 1, ...n$.

For a subalgebra $\mathfrak{k}\subseteq\mathfrak{g}$ this induces a system of ideals $\mathfrak{k}_i = \mathfrak{a}_i \cap \mathfrak{k}$ with the same property that all $\mathfrak{k}_i/\mathfrak{k}_{i+1}$ are abelian for $i = 1, ...n$. (Since $\frac{\mathfrak{a}_i\cap\mathfrak{k}}{\mathfrak{a}_{i+1}\cap\mathfrak{k}}$ embeds into the abelian Lie algebra $\frac{\mathfrak{a}_i}{\mathfrak{a}_{i+1}}$).

For a quotient algebra $\pi : \mathfrak{g}\twoheadrightarrow\mathfrak{p}$, we use ideals $\pi(\mathfrak{a}_i)$ in $\pi(\mathfrak{g}) = \mathfrak{p}$.

(b) Suppose that we have sequences of ideals $\mathfrak{a} = \mathfrak{a}_0\supseteq\mathfrak{a}_1\supseteq\cdots\mathfrak{a}_p = 0$ in $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a} = \mathfrak{q}_0\supseteq\mathfrak{q}_1\supseteq\cdots\mathfrak{q}_q = 0$ in $\mathfrak{g}/\mathfrak{a}$ with abelian graded pieces. Then for $j = 1, ..., q$ one can define an ideal $\widetilde{\mathfrak{q}}_j$ in $\mathfrak{g}$ (such that $\mathfrak{a}\subseteq\widetilde{\mathfrak{q}}_j\subseteq\mathfrak{g}$), as the inverse of $\mathfrak{q}_j\subseteq\mathfrak{g}/\mathfrak{a}$ under the quotient map $\mathfrak{g}\twoheadrightarrow\to\mathfrak{g}/\mathfrak{a}$. In the sequence

$$\mathfrak{g} = \widetilde{\mathfrak{q}}_0\supseteq\cdots\supseteq\mathfrak{q}_q = \mathfrak{a} = \mathfrak{a}_0\supseteq\cdots\supseteq\mathfrak{a}_p = 0$$

successive quotients are abelian (since $\widetilde{\mathfrak{q}}_i/\widetilde{\mathfrak{q}}_{i+1} \cong \mathfrak{q}_i/\mathfrak{q}_{i+1}$. are abelian for $i = 1, ...n$.

(c) In the Lie algebra $I+J$ we have an ideal I which is solvable and the quotient $\frac{I+J}{I} \cong \frac{J}{J/\cap I}$ is a subalgebra of J, so it is again solvable. □

Corollary. Any finite dimensional Lie algebra $\mathfrak{g}$ has the largest solvable ideal.

Proof. This is just the sum of all solvable ideals. □

6.2. **Semisimple Lie algebras.** The largest solvable ideal of $\mathfrak{g}$ is called the *radical* $Rad(\mathfrak{g})$.

We say that a Lie algebra $\mathfrak{g}$ is

- *simple* if it has no ideals;
- *semisimple* if it has no solvable ideals
- *reductive* if $Rad(\mathfrak{g}) = Z(\mathfrak{g})$.

Lemma. (a) $\mathfrak{g}$ is simple iff its adjoint representation is irreducible.

(b) "simple" implies "semisimple".

(c) $\mathfrak{g}$ is reductive iff $\mathfrak{g}/Z(\mathfrak{g})$ is semisimple.

(d) $\mathfrak{g}$ is solvable iff $\mathfrak{g}$ has no abelian ideals.

Proof. (a) Ideals in $\mathfrak{g}$ are$\mathfrak{g}$-submodules of the $\mathfrak{g}$-modules $\mathfrak{g}$. Now (c) follows since we checked in homework that for sl_n the adjoint representation is irreducible.

(b) just says that if there no ideals then there are no solvable ones.

(c) The ideals J in $\mathfrak{g}/Z(\mathfrak{g})$ are in bijection with the ideals I in $\mathfrak{g}$ that contain $Z(\mathfrak{g})$. Moreover, I is solvable iff J is solvable. Therefore, $Rad[\mathfrak{g}/Z(\mathfrak{g})] = Rad(\mathfrak{g})/Z(\mathfrak{g})$.

(d) For one direction we notice that abelian ideals are solvable. Conversely, let $\mathfrak{s} \neq 0$ be a solvable solvable ideal in $\mathfrak{g}$. For any ideal $\mathfrak{a}$ in $\mathfrak{g}$, $[\mathfrak{a}, \mathfrak{a}]$ is again ideal in $\mathfrak{g}$. So, the terms in the derived series $\mathfrak{s}^{(i)}$ of $\mathfrak{s}$ are all ideals in $\mathfrak{g}$. If $\mathfrak{s}^{(k)}$ is the last nonzero term then it is abelian since $0 = \mathfrak{s}^{(k+1)} = [\mathfrak{s}^{(k)}, \mathfrak{s}^{(k)}]$.

□

Example. sl_n is simple.

6.2.1. *1-dimensional representations.*

Lemma. The isomorphism classes of 1-dimensional representations of $\mathfrak{g}$ are parameterized by $(\mathfrak{g}^{\mathrm{ab}})^*$. To $\lambda \in (\mathfrak{g}^{\mathrm{ab}})^*$ we associate the representation $\Bbbk^{\mathfrak{g}}_{\lambda}$ on $V = \Bbbk$ where $x \in \mathfrak{g}$ acts by the scalar $\langle \lambda, x \rangle$.

Proof. □

Corollary. For $\mathfrak{g} = sl(n)$ we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ hence $\mathfrak{g}^{\mathrm{ab}} = 0$ and the only one dimensional representation is the trivial representation $\mathfrak{k} = \mathfrak{k}^{\mathfrak{g}}_{\lambda=0}$.

Proof. For roots α one has $[\mathfrak{h}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$, hence $[\mathfrak{g}, \mathfrak{g}]$ contains $\mathfrak{n} \oplus \mathfrak{n}^-$. It also contains $\mathfrak{h}$ since it is spanned by vectors $\check{\alpha}$ for α a (simple) root and $\check{\alpha} \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. □

For $\lambda \in \mathfrak{g}^*$ and any representation V of $\mathfrak{g}$ denote by

$$V^{\mathfrak{g}}_{\lambda} = \{v \in V;\ x \cdot v = \ \langle \lambda, x \rangle v \ \text{ for } \ x \in \mathfrak{g}\}$$

the corresponding $\mathfrak{g}$-eigenspace in V. This is subrepresentation isomorphic to a multiple of the representation $\mathfrak{k}^{\mathfrak{g}}_{\lambda}$.

6.3. **Lie's theorem: solvable algebras are upper triangular.** The precise meaning will be that in any finite dimensional representation one can choose a basis such that the that all elements of the Lie algebra act by upper triangular matrices.

Lemma. A solvable Lie algebra $\mathfrak{g} \neq 0$ has an ideal of codimension one.

Proof. If $\mathfrak{g}$ is solvable then the inclusion $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ is proper (otherwise $\mathfrak{g}^{(n)} = \mathfrak{g}$ for all n !). since the Lie algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, any subspace $\mathfrak{a}$ of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is an ideal. Then any subspace $\widetilde{\mathfrak{a}}$ of $\mathfrak{g}$ that contains $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$ (because it is the inverse of some subspace $\mathfrak{a}$ of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ under the map $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. □

Proposition. If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ then for any finite dimensional representation V of $\mathfrak{g}$, and any $\lambda \in \mathfrak{a}^*$, the $\mathfrak{a}$-eigenspace $V^{\mathfrak{a}}_{\lambda}$ is $\mathfrak{g}$-invariant.

Proof. The meaning is that for $w \in W$ and $x \in \mathfrak{g}$ one has $xw \in V^{\mathfrak{a}}_{\lambda}$, i.e., that for each $y \in \mathfrak{a}$

$$y(xw) = \ xyw + [y, x]w = \ \langle \lambda, y \rangle \ xw \ + \ \langle \lambda, [y, x] \rangle \ w$$

equals $\langle \lambda, y \rangle \cdot xw$, i.e., that $\lambda \perp [\mathfrak{a}, x]$.

To check that for each $y \in \mathfrak{a}$ $\langle \lambda, [y, x] \rangle$ is zero, we will show that its multiple is the trace of $[y, x]$ on some subspace U of V.

Notice that $\mathfrak{b} = \mathfrak{a} + \Bbbk x$ is a subalgebra of $\mathfrak{g}$. Now we pick any nonzero vector w in W and choose U to be the $\mathfrak{b}$-submodule of V generated by w.

We first define the subspaces W_i of V spanned by $w, xw, ..., x^{i-1}w$. So, we have $W_0 = 0 \subseteq W_1 = \Bbbk w \subseteq \cdots$ and $xW_i \subseteq W_{i+1}$. This sequence has to stabilize at since V is finite dimensional. Actually, it stabilizes at the first integer n such that $W_n = W_{n+1}$ (this implies that W_n is x invariant and then clearly $W_k = W_n$ for $k \geq n$.

We will see that the $\mathfrak{b}$-submodule U generated by w is W_n. Clearly, x preserves W_n so it remains to prove that

A. Each W_i is $\mathfrak{a}$-invariant. This will follow from an inductive proof of a more technical claim that for each j :

(a) For $y \in \mathfrak{a}$ we have

$$y\ x^j w \ \overset{W_j}{\cong}\ \langle \lambda, y \rangle \cdot x^j w.$$

(b) W_{j+1} is $\mathfrak{a}$-invariant.

Proof. So let us assume that both claims hold for any power $j \leq i$. To prove (a) for $i+1$ we of course, "commutate one x away":

$$y(x^i w) \ = \ (xy + [y,x])x^{i-1}w \ = \ x\ yx^{i-1}w \ + \ [y,x]x^{i-1}w.$$

Here, $x^{i-1}w \in W_i$ and $[x,y] \in [\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$, so $[y,x]x^{i-1}w \in W_i$ by the part (b) of the induction assumption.

Also, by the part (a) of the induction assumption. there is a vector $w' \in W_{i-1}$ such that

$$y\ x^{i-1}w = \ \langle \lambda, y \rangle \cdot x^{i-1}w + w' \quad \text{hence} \quad x(y\ x^{i-1}w) = \ \langle \lambda, y \rangle \cdot x^i w + xw'$$

and we have $xw' \in xW_{i-1} \subseteq W_i$.

This proves (a). Now (b) follows from (a) since

$$yx^i w \in \ \langle \lambda, y \rangle \cdot x^i w \ + \ W_i \subseteq W_{i+1}. \quad \square$$

B. Trace considerations. Since W_i's stabilize at $i = n$ we know that $w, xw, ..., x^{n-1}w$ is a basis of W_n. (Otherwise for some $j < n$ vector $x^j w$ would be a linear combination of the preceding vectors and then the stabilization would occur at W_j,)

We can use this basis to calculate the trace of $y \in \mathfrak{a}$ in W_n. By the claim (a) above, $Tr(y, W_n) = n\langle \lambda, y \rangle$.

For any $z]in\mathfrak{a}$ we have $[z,x] \in \mathfrak{a}$ hence $Tr([z,x], W_n) = n\langle \lambda, [z,x] \rangle$. However, since both z and x are operators on W_n the trace of their commutator is zero. So, $n\langle \lambda, [z,x] \rangle = 0$. Since $n \neq 0$ (W_j's do not stabilize at $W_0 = 0$), we get $\langle \lambda, [z,x] \rangle = 0$. $\square$

Theorem. [Lie's theorem.] If $\mathfrak{g}$ is solvable then any finite dimensional representation V of $\mathfrak{g}$ has an invariant flag.

Proof. We use induction on $\dim(\mathfrak{g})$. To start with, if $\dim(\mathfrak{g}) = 0$ then all subspaces are invariant. So let $\mathfrak{g} \neq 0$. Now we can choose (by the lemma) an ideal $\mathfrak{a} \subseteq \mathfrak{g}$ of codimension one. Let $x \in \mathfrak{g}$ be a vector such that $\mathfrak{a} \oplus \Bbbk x = \mathfrak{g}$.

Recall that the subalgebra $\mathfrak{a}$ is again solvable. So, by induction assumption $\mathfrak{a}$ has an invariant line. Then $\mathfrak{a}$ acts on it by some $\lambda \in \mathfrak{a}^*$, hence $W = V_\lambda^{\mathfrak{a}}$ is not zero.

Now notice that the $\mathfrak{a}$-eigenspace $V_\lambda^{\mathfrak{a}}$ is a $\mathfrak{g}$-invariant because $\mathfrak{a}$ is an ideal in $\mathfrak{g}$.

Then any x-eigenvector v in V_λ gives a $\mathfrak{g}$-invariant line $V_1 = \Bbbk v$ in V.

Now one has an invariant line in V/V_1, hence an invariant plane V_2 in V ... $\square$

Corollary. Let $\mathfrak{g}$ be solvable.

(a) $\mathfrak{g}$ has a flag of ideals.

(b) Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

(c) All elements x of the Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ are *ad-nilpotent* in the sense that the operator $ad_x \in \mathrm{End}(\mathfrak{g})$ is nilpotent.

Proof. (a) Use $\mathfrak{g}$-module $\mathfrak{g}$.

(b) Let $\mathfrak{g} = \mathfrak{g}_0 \supseteq \cdots \supseteq \mathfrak{g}_n = 0$ be a flag of ideals. The action of $\mathfrak{g}$ on $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ induces an action of $\mathfrak{g}' \stackrel{\text{def}}{=} [\mathfrak{g}, \mathfrak{g}]$ whose image lies in the derived subalgebra of $\mathfrak{gl}(\mathfrak{g}_i/\mathfrak{g}_{i+1})$ which is 0.

Now $\mathfrak{g}'$ has a decreasing sequence of ideals $\mathfrak{g}'_i \stackrel{\text{def}}{=} \mathfrak{g}' \cap \mathfrak{g}_i$ with $\mathfrak{g}'_0 = \mathfrak{g}'$ and $\mathfrak{g}'_n = 0$, and such that $\mathfrak{g}'$ acts trivially on the graded pieces $\mathfrak{g}'_i/\mathfrak{g}'_{i+1} \subseteq \mathfrak{g}_i/\mathfrak{g}_{i+1})$. This means that $\mathfrak{g}'$ is nilpotent

(c) has also been proved: since for $x \in \mathfrak{g}'$ we have $ad(x) = 0$ on $\mathfrak{g}_i/\mathfrak{g}_{i+1}$, i.e., $ad(x)(\mathfrak{g}_i) \subseteq \mathfrak{g}_{i+1}$ we see that $ad(x)^n = 0$. $\square$

6.4. **Engel theorem: Lie algebras of nilpotent operators are strictly upper triangular (hence nilpotent).** Here we notice a strong relation between nilpotent Lie algebras and nilpotent linear operators.

6.4.1. *Subalgebras, quotients and extensions.*

Lemma. (a) If $\mathfrak{g}$ is nilpotent then so are all subalgebras and quotients.

(b) If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent then so is $\mathfrak{g}$.

(c) If $\mathfrak{g} \neq 0$ is nilpotent then $Z(\mathfrak{g}) \neq 0$.

(d) $\mathfrak{g}$ is nilpotent if there exists some n such that all $[ad(\mathfrak{g})]^n = 0$, i.e., all products $ad(x_n) \circ \cdots \circ ad(x_1)$ of length n vanish.

In particular, all elements of $\mathfrak{g}$ are ad-nilpotent.

Proof. (a) is proved the same as in the case of solvable algebras. Also, the proof of (b) proceeds the same as for the corresponding claim for solvable algebras.

(c) is a clear from the definition of nilpotent Lie algebras: Since $\mathfrak{g} \neq 0$, we can choose i as the last index such that there is the term $\mathfrak{g}^i$ of the lower central series is $\neq 0$. Since $\mathfrak{g}$ acts by zero on $\mathfrak{g}^i/\mathfrak{g}^{i+1} = \mathfrak{g}^i$, it lies in the center.

(d) Since the lower central series is given by $\mathfrak{g}^i \stackrel{\text{def}}{=} [ad(\mathfrak{g})]^i\mathfrak{g}$, the condition $\mathfrak{g}^n = 0$ is equivalent to $[ad(\mathfrak{g})]^n = 0$. □

Remark. Notice that (b) is weaker then the corresponding statement for solvable Lie algebras which just said that solvable algebras are closed under extensions.

6.4.2. *ad-nilpotent elements.* We say that an element x of a Lie algebra $\mathfrak{g}$ is *ad-nilpotent* if the operator $ad(x)$ on $\mathfrak{g}$ is nilpotent. Similarly for *ad-semisimple.*

Lemma. If $x \in \mathrm{End}(V)$ is semisimple or nilpotent then the same is true for the operator $ad(x)$ on $\mathrm{End}(V)$.

Proof. If x is semisimple then $V = \oplus\, V_\alpha$ for the α-eigenspace V_α of x. Now, on $\mathrm{Hom}(V_\alpha, V_\beta) \subseteq \mathrm{End}(V)$ operator $ad(x)$ acts by $\beta - \alpha$. so it is again semisimple.

If x is nilpotent then so are the multiplication operators L_x, R_x on $\mathrm{End}(V)$. Since they commute, $ad(x) = L_x + R_x$ is also nilpotent. □

6.4.3. *Linear Lie algebras with nilpotent elements.*

Proposition. Let $\mathfrak{g}$ be a Lie subalgebra of $gl(V)$ with $0 < \dim(V) < \infty$. If all elements of $\mathfrak{g}$ are nilpotent operators then

(a) The common kernel $V_0^{\mathfrak{g}}$ of operators in $\mathfrak{g}$ is $\neq 0$.

(b) There exists an invariant flag $0 = V_0 \subseteq \cdots \subseteq V_n = V$ such that $\mathfrak{g} \cdot V_i \subseteq V_{i-1}$ (i.e., $\mathfrak{g}$ acts trivially on the graded pieces).

(c) $\mathfrak{g}$ is nilpotent.

Proof. (a) We will use induction in $\dim(\mathfrak{g})$. The case when dimension is zero is obvious.

A. Any maximal proper subalgebra $\mathfrak{m} \subseteq \mathfrak{g}$ is an ideal in $\mathfrak{g}$ of codimension one: $\mathfrak{g} = \mathfrak{m} \oplus \Bbbk x$. If $\mathfrak{m} \subseteq \mathfrak{g}$ is a proper Lie subalgebra we consider $\mathfrak{m}$-action on $\mathfrak{g}/\mathfrak{m}$. Since it is still by nilpotent operators, by induction assumption there is a line $\Bbbk(x + \mathfrak{m})$ in $\mathfrak{g}/\mathfrak{m}$ on which $\mathfrak{m}$ acts by zero. This means that $[\mathfrak{m}, x] \subseteq \mathfrak{m}$. Now, $\mathfrak{m} \oplus \Bbbk x$ is a subalgebra of $\mathfrak{g}$ which normalizes $\mathfrak{m}$. Since $\mathfrak{m}$ is maximal we find that $\mathfrak{m} \oplus \Bbbk x$ equals $\mathfrak{g}$, so $\mathfrak{g}$ normalizes $\mathfrak{m}$.

B. Applying induction to the $\mathfrak{m}$-module V. The common kernel $V_0^{\mathfrak{m}}$ in V of operators in $\mathfrak{m}$ is non-zero because $\dim(\mathfrak{m}) < \dim(\mathfrak{g})$. Since $\mathfrak{m}$ is an ideal in $\mathfrak{g}$ we find that $V_0^{\mathfrak{m}}$ is

$\mathfrak{g}$-invariant. Now, since the restriction of the operator x to $V_0^{\mathfrak{m}}$ is nilpotent, the kernel of x in $V_0^{\mathfrak{m}}$ is $\neq 0$. However, on this subspace all of $\mathfrak{g}$ acts by zero. $\square$

(b) follows from (a). Once we have a line $V_1 \subseteq V$ on which $\mathfrak{g}$ acts by zero, we consider the action of $\mathfrak{g}$ on V/V_1 etc.

(c) follows from (b) since we can choose a basis $v_1, ..., v_n$ such that $v_1, ..., v_i$ is a basis of V. This identifies $gl(V)$ with gl_n and then $\mathfrak{g}$ lies in the subalgebra $\mathfrak{n}$ of strictly upper triangular matrices which is nilpotent. Therefore, $\mathfrak{n}$ is nilpotent. $\square$

6.4.4. *Engel's theorem.* If Lie algebra $\mathfrak{g}$ is nilpotent it does not imply that its elements act nilpotently in finite dimensional representations. (For instance for an abelian Lie algebra $\mathfrak{a}$, any $\lambda \in \mathfrak{a}^*$ defines a 1-dimensional representation of $\mathfrak{a}$ in which $x \in \mathfrak{a}$ acts by the scalar $\langle \lambda, x \rangle$.)

However, this is true in the adjoint representation – if $\mathfrak{g}$ is nilpotent then all operators $ad(x)$ are known to be nilpotent (lemma 6.4.1.d). The converse is also true:

Theorem. [Engel] If $\dim(\mathfrak{g}) < \infty$ then the Lie algebra $\mathfrak{g}$ is nilpotent iff all elements of $\mathfrak{g}$ are ad-nilpotent.

Proof. By the above remark, we just need to prove know that if all elements of a Lie algebra $\mathfrak{g}$ are ad-nilpotent then $\mathfrak{g}$ itself is a nilpotent Lie algebra.

A. If $\mathfrak{g} \neq 0$ then $Z(\mathfrak{g}) \neq 0$. Now, since $ad(\mathfrak{g})$ is a Lie subalgebra of operators on $\mathfrak{g}$ and all its elements $ad(x)$ are nilpotent operators on $\mathfrak{g}$, the preceding proposition 6.4.3 says that the $ad(\mathfrak{g})$-module $\mathfrak{g}$ has an invariant vector $0 \neq v \in \mathfrak{g}$. The $ad(\mathfrak{g})$-invariance of v means that $v \in Z(\mathfrak{g})$.

B. Induction in $\dim(\mathfrak{g})$. The Lie algebra $\mathfrak{g}/Z(\mathfrak{g})$ again has the property that its elements are ad-nilpotent. So, by induction assumption we know that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. According to the lemma 6.4.1.c this implies that $\mathfrak{g}$ itself is nilpotent. $\square$

6.5. **Jordan decomposition.**

6.5.1. *Jordan decomposition in linear algebra.* We recall

Theorem. Let V be a finite dimensional vector space over a closed field $\Bbbk$. For each linear operator x

(a) There is a unique pair s, n of a semisimple operator s and a nilpotent operator s, such that $x = s + n$ and s, n commute.

(b) Actually, one can choose polynomials S, N without constant coefficients, so that $s = S(x)$ and $n = N(x)$. In particular, s, n commute with the centralizer $Z_{\mathrm{End}(V)}(x)$ and they preserve any subspace that x preserves.

Proof. **1. Generalized eigenspaces of x.** Since $\Bbbk$ is closed, we can factor the characteristic polynomial $\chi(T) = \det(T - x)$ as a product $\prod_{\alpha\in\Sigma} \chi_\alpha$ of linear factors where Σ is the set of zeros of χ and $\chi_\alpha = (T - \alpha)^{m_\alpha}$.

Denote the generalized eigenspaces by $V_\alpha \stackrel{\text{def}}{=} \mathrm{Ker}(\chi_\alpha(x)),\ \alpha \in \Sigma$. Then $V = \oplus_\Sigma V_\alpha$. Consider the polynomials $\chi^\beta \stackrel{\text{def}}{=} \prod_{\alpha\neq\beta} \chi_\alpha$ for $\beta \in \Sigma$. is 1. The Chinese Remainder Theorem says that the greatest common divisor of polynomials χ^β can be written as a sum $\sum_{\beta\in\Sigma} P_\beta\chi^\beta$ for some polynomials P_β. So, $1 = \sum_{\beta\in\Sigma} P_\beta\chi^\beta$. Now, plugin $T = x$ to get that

$$1_V = \sum_{\beta\in\Sigma} P_\beta(x)\chi^\beta(x).$$

The image of the operator $P_\beta(x)\chi^\beta(x)$ is in $V_\beta = \mathrm{Ker}$ since $\chi_\beta(x)\circ P_\beta(x)\chi^\beta(x)v = P_\beta(x)\chi(x)$ and $\chi(x) = 0$.

So, any vector v has a decomposition $v = \sum_{\beta\in\Sigma} v_\beta$ for $v_\beta \stackrel{\text{def}}{=} P_\beta(x)\chi^\beta(x)v \in V_\beta$.

2. Choice of the polynomial S. At this point we can can choose any finite subset $\Sigma'\subseteq\Bbbk$ that contains Σ and for $\alpha \in \Sigma' - \Sigma$ we choose $\chi_\alpha = (T - \alpha)^{m_\alpha}$ for some positive power m_α. Then the system of congruences $S \stackrel{\chi_\alpha}{\cong} \alpha$ for each $\alpha \in \Sigma'$ has a solution because the polynomials $\chi_\alpha,\ \alpha \in \Sigma'$, are pairwise relatively simple. We will actually choose Σ' so that it contains 0, so that S is a multiple of T.

3. Construction of s and n. Let $s = S(x)$ and $n = N(x)$ for $N = T - S$. So, $s + n = x$ and s, n certainly commute with $Z(x)$.

We see that for each $\alpha \in \Sigma$ we have $s - \alpha = (S - \alpha)(x)$ which is a multiple of $\chi_\alpha(x)$. So, $s = \alpha$ on V_α and therefore s is semisimple. Now, on the generalized α-eigenspace V_α operator n is $x - \alpha$ so it is nilpotent.

4. Uniqueness in (a). If $x = s' + n'$ for semisimple s' and nilpotent n' and s', n' commute, then $s', n' \in Z(x)$ and so they commute with the s, n that we have constructed above. Now $s' + n' = x = s + n$ gives $s' - s = n - n'$. Since s, s' are semisimple and they commute $s' - s$ is semisimple. Since n, n' are nilpotent and they commute operator $n - n'$ is nilpotent. Then $s' - s = n - n'$ is both semisimple and nilpotent, hence it is zero. □

Corollary. If $A\subseteq B\subseteq V$ are vector subspaces such that $xB\subseteq A$ then $sB\subseteq A$ and $nB\subseteq A$.

Proof. This follows from the claim (b) in the lemma. □

6.5.2. *Jordan decomposition in Lie algebras.* We can define a *Jordan decomposition* of an element x in any *Lie algebra* $\mathfrak{g}$ to be a pair $s, n \in \mathfrak{g}$ such that $x = s + n$, $[s, n] = 0$ and $ad(s)$ is a semisimple operator while $ad(n)$ is a nilpotent operator on $\mathfrak{g}$.

In general it need not exist nor does it have to be unique.

Lemma. (a) The map $ad : \mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ preserves Jordan decompositions that exist in $\mathfrak{g}$.

(b) If $Z(\mathfrak{g}) = 0$ then Jordan decompositions in $\mathfrak{g}$ are unique.

Proof. (a) If $x = s + n$ is a Jordan decomposition in $\mathfrak{g}$ $ad(x) = ad(s) + ad(n)$ is standard Jordan decomposition in linear algebra since $[ad(s), ad(n)] = ad[s, n] = 0$ and $ad(s)$ is semisimple while $ad(n)$ is nilpotent.

(b) follows since in this case $ad : \mathfrak{g} \to gl(\mathfrak{g})$ is injective. □

However, we will see that both properties hold in the class of *semisimple Lie algebras* $\mathfrak{g}$. Here, we check this for $\mathfrak{g} = sl_n$.

6.5.3. *Jordan decomposition in the Lie algebra sl_n.*

Lemma. For $x \in gl_n$ the Jordan decomposition of the linear operator $ad(x)$ on gl_n is $ad(s) + ad(n)$ where $x = s + n$ is the Jordan decomposition of the linear operator x on $\Bbbk^n$. (Also, if $x \in sl_n$ then $s, n \in sl_n$.)

Proof. We have $ad(s) + ad(n) = ad(s + n) = ad(x)$ and $[ad(s), ad(n)] = ad([s, n]) = 0$. So it remains to show that $ad(s)$ is semisimple and $and(n)$ is nilpotent. □

Proposition. If $\mathfrak{g} = sl_n$ over a closed field $\Bbbk$ then any element $x \in \mathfrak{g}$ has a unique Jordan decomposition. It coincides with the Jordan decomposition in linear operators over $\Bbbk^n$.

Proof. The requirement that $[ad(s), ad(n)] = ad([s, n])$ be zero is equivalent to $[s, n] = 0$ since $\mathrm{Ker}(sl_n) = Z(sl_n) = 0$. Now the conditions on s, n are equivalent to asking that $ad(s), ad(n)$ are the Jordan decomposition of the linear operator $ad(x)$ on sl_n. □

6.5.4. *Jordan decomposition in Lie algebras of derivations.* Here we establish Jordan decomposition in a special class of Lie algebras.

Lemma. Let A be a finite dimensional associative algebra over a closed field $\Bbbk$ and let $Der(A) \subseteq \mathrm{End}_{\Bbbk}(A)$ be the Lie algebra of its derivatives.

(a) If $x \in Der(A)$ has Jordan decomposition $x = s + n$ then both s, n lie in $Der(A)$.

(b) (b) In Lie algebras of form $Der(A)$ there is a Jordan decomposition.

Proof. (a) Let $A_{(\alpha)}$ be the generalized eigenspace of x with eigenvalue $\alpha \in \Bbbk$. Then

$$A_{(\alpha)} \cdot A_{(\beta)} \subseteq A_{(\alpha+\beta)}.$$

The reason is that for $u, v \in A$

$$(x - (\alpha + \beta))^n (uv) \;=\; \sum_{0}^{n} \binom{n}{p} (x - (\alpha + \beta))^p u \cdot (x - \beta)^{n-p} v.$$

Now, s acts on the subspace $A_{(\alpha)}$ of A by scalar α. So, for $u \in A_{(\alpha)}$ and $v \in A_{(\beta)}$ we have

$$s(u)\cdot v \ + \ u\cdot s(v) = \ (\alpha+\beta)uv \ = \ s(uv).$$

So, s is a derivative on A. Since s, x commute so is $n = x - s$.

(b) We know that $[s, n] = 0$. Also, since s is a semisimple operator on A, $ad(s)$ is a semisimple operator on $\mathrm{End}(A)$ and then its restriction $ad_{Der(A)}(s)$ to $Der(A)\subseteq\mathrm{End}(A)$ is also semisimple. Similarly, $ad_{Der(A)}(n)$ is nilpotent. □

6.6. **Cartan's criterion for solvability of Lie algebras.**

6.6.1. *Linear algebra relating nilpotency and traces.*

Lemma. For two subspaces $A\subseteq B$ of $gl(V)$, let $\mathcal{N}$ consist of all $x \in \mathrm{End}(V)$ such that $xB\subseteq A$. Then the intersection $\mathcal{N}\cap\mathcal{N}^{\perp}$ (for the bilinear form $\kappa(x, y) = Tr_V(xy)$) consists of nilpotent operators.

Proof. Longish. □

6.6.2. *Cartan criterion.* The first version uses a faithful representations of $\mathfrak{g}$, i.e., $\mathfrak{g}\subseteq gl(V)$ and the corresponding form $\kappa_{\mathbb{V}}$. The second version is completely general since it does not require any particular representation (it uses the adjoint representation and the Killing form).[10]

Theorem. [Cartan] **(1)** A Lie subalgebra $\mathfrak{g}$ of $gl(V)$ is solvable iff it is orthogonal to its derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ for the bilinear form $\kappa_V(x, y) = Tr_V(xy)$ on $\mathfrak{g}$.

(2) A Lie algebra $\mathfrak{g}$ is solvable iff the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is self-orthogonal for the Killing form $\kappa_{\mathfrak{g}}(x, y) \stackrel{\text{def}}{=} Tr_{\mathfrak{g}}(ad(x)ad(y))$ on $\mathfrak{g}$.

Proof. **1a**. If $\mathfrak{g}$ is solvable then its action on V can be represented by upper-triangular matrices, i.e., in terms of some basis of V we have identification $gl(V) \cong gl_n$ such that $\mathfrak{g}$ lies in the Borel subalgebra $\mathfrak{b}$ of upper triangular matrices. Therefore, the action of $\mathfrak{g}'\subseteq\mathfrak{b}' = \mathfrak{n}$ (the strictly upper-triangular matrices. Now recall that we have proved that $\mathfrak{b} \perp \mathfrak{n}$ (actually $\mathfrak{b}^{\perp} = \mathfrak{n}$) for the form $Tr(xy)$ on gl_n. This implies that $\mathfrak{g} \perp \mathfrak{g}'$.

1b. In the more interesting direction, assume that $Tr_V(\mathfrak{g}\cdot\mathfrak{g}') = 0$. From this we will prove that that any element x of $\mathfrak{g}'$ is nilpotent as an operator on V.

This implies that the Lie algebra $\mathfrak{g}'$ is nilpotent (proposition 6.4.3.c). Then $\mathfrak{g}$ will be solvable as an extension of an abelian Lie algebra $\mathfrak{g}^{\mathrm{ab}} = \mathfrak{g}/\mathfrak{g}'$ by a nilpotent Lie algebra $\mathfrak{g}'$!

The nilpotency of $x \in \mathfrak{g}'$ will follow from the above linear algebra lemma 6.6.1 in the case when $A\subseteq B\subseteq gl(V)$ is $\mathfrak{g}'\subseteq\mathfrak{g}$. Then $\mathcal{N}$ consists of all $x \in gl(V)$ such that $[x, \mathfrak{g}]\subseteq\mathfrak{g}'$.

[10] However, the 1st version is more general in another sense. When we want to show that $\mathfrak{g}$ is solvable), it allows us to use any (faithful) representation V that we happen to understand.

For instance $\mathcal{N}\subseteq\mathfrak{g}$. The lemma says that the operators in $\mathcal{N}\cap\mathcal{N}^{\perp}$ are nilpotent. So, it suffices to show that $\mathfrak{g}'\subseteq\mathcal{N}$ (true because $\mathfrak{g}\subseteq\mathcal{N}$), and $\mathfrak{g}'\perp\mathcal{N}$.

For the last claim let $x,y\in\mathfrak{g}$ so that $[x,y]\in\mathfrak{g}'$ and $n\in\mathcal{N}$ then the invariance of κ_V gives

$$\kappa_V([x,y],n) = -\kappa_V(y,[x,n]).$$

Here, $n\in\mathcal{N}$ and $x\in\mathfrak{g}$ imply that $[x,n]=-[n,x]$ lies in $\mathfrak{g}'$. So the assumption that $\mathfrak{g}'\perp\mathfrak{g}$ implies that the expression is zero. So, $\mathfrak{g}'\perp\mathcal{N}$.

(2) will follow by applying **(1)** to the algebra $ad(\mathfrak{g})\subseteq gl(\mathfrak{g})$.

The kernel of the adjoint representation $ad:\ \mathfrak{g}\to\mathrm{End}(\mathfrak{g})$ is $Z(\mathfrak{g})$. So, the quotient $\mathfrak{g}/Z(\mathfrak{g})$ of $\mathfrak{g}$ is exactly the image $ad(\mathfrak{g})$ of the map of Lie algebras $ad:\ \mathfrak{g}\to\mathrm{End}(\mathfrak{g})$.

Now, we see that $\mathfrak{g}$ is solvable iff $ad(\mathfrak{g})$ is solvable (if $ad(\mathfrak{g})$ is solvable then $\mathfrak{g}$ is an extension of solvable $\mathfrak{g}$ by abelian $Z(\mathfrak{g})$).

Part **(1)** of the theorem now say that the subalgebra $ad(\mathfrak{g})$ of $gl(\mathfrak{g})$ is solvable iff for the form $\kappa_{\mathfrak{g}}^{ad(\mathfrak{g})}$ on $ad(\mathfrak{g})$ we have

$$\kappa_{\mathfrak{g}}^{ad(\mathfrak{g})}\big(ad(\mathfrak{g}),\ ad(\mathfrak{g})'\big) = 0.$$

However, the left hand side is the same as $\kappa_{\mathfrak{g}}^{\mathfrak{g}}(\mathfrak{g},\ \mathfrak{g}')$ since $ad(\mathfrak{g})'=ad(\mathfrak{g}')$ (because $ad(x),ad(y)=ad([x,y])$) and for $x,y\in\mathfrak{g}$

$$\kappa_{\mathfrak{g}}^{\mathfrak{g}}(x,y) = Tr_{\mathfrak{g}}\big(ad(x)ad(y)\big) = \kappa_{\mathfrak{g}}^{ad(\mathfrak{g})}(ad(a),ad(y)).$$

$\square$

7. Semisimple Lie algebras

7.1. Killing criterion for semisimplicity.

Theorem. A Lie algebra is semisimple iff its Killing form is nondegenerate.

Proof. **A.** Let $\mathfrak{g}$ be semisimple, then we want to show that $rad(\kappa_{\mathfrak{g}}) = 0$. We know that $R = rad(\mathfrak{g}) = \mathfrak{g}^{\perp}$ is an ideal. So, to show that it is zero it suffices to prove that R is solvable (there are no solvable ideals in a semisimple $\mathfrak{g}$). By Cartan's criterion (version (2)) this is equivalent to $\kappa_R(R, [R, R]) = 0$.

Now recall that since R is an ideal its Killing form κ_R is the restriction of the Killing form on $\mathfrak{g}$ (see proposition 3.3.5.b). So, $\kappa_R(R, [R, R]) = \kappa_{\mathfrak{g}}(R, [R, R])$ is zero since $R = rad(\kappa_{\mathfrak{g}})$.

B. Now suppose that $rad(\kappa_{\mathfrak{g}}) = 0$. To prove that $\mathfrak{g}$ is semisimple it suffices to prove that any abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$ is zero. Since $\kappa_{\mathfrak{g}}$ is nondegenerate it suffices to prove that $\mathfrak{a} \perp \mathfrak{g}$. So, for $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$ we need $tr_{\mathfrak{g}}\big(ad(y)ad(x)\big) = 0$.

We will actually prove that $(ad(y)ad(x))^2 = 0$. The point is that $ad(y)ad(x)$ maps $\mathfrak{g}$ to $\mathfrak{a}$ and then also $ad(y)ad(x)$ maps $\mathfrak{a}$ to $[\mathfrak{a}, \mathfrak{a}] = 0$. □

7.1.1. *Action of the radical on irreducible representations.*

Theorem. If (V, π) is an irreducible representation of $\mathfrak{g}$ then any element of $Rad(\mathfrak{g})$ acts on V by a scalar.

Proof. Since $Rad(\mathfrak{g})$ is solvable, it has an invariant line L. Here it acts by some linear functional $\lambda \in Rad(\mathfrak{g})^*$. So, $V_{\lambda}^{Rad(\mathfrak{g})}$ is not zero.

Actually, $V_{\lambda}^{Rad(\mathfrak{g})}$ is a $\mathfrak{g}$-submodule. "This is proved as in theoorf of Lie's theorem".

Now we assume that V is irreducible, hence $V_{\lambda}^{Rad(\mathfrak{g})} = V$ and $\mathfrak{g}$ acts on V by λ. □

Corollary. $[\mathfrak{g}, Rad(\mathfrak{g})]$ acts by 0 on irreducible representations.

Proof. $\pi[\mathfrak{g}, Rad(\mathfrak{g})] = [\pi\mathfrak{g}, \pi\ Rad(\mathfrak{g})] \subseteq [\pi\mathfrak{g}, \Bbbk] = 0$. □

7.1.2. *Levi's theorem.* Here we jsut mention the following theorem.

Theorem. Any Lie algebra $\mathfrak{g}$ can be written as a sum of subspaces which are subalgebras (but $\mathfrak{s}$ need not be an ideal)

$$\mathfrak{g} \cong Rad(\mathfrak{g}) \oplus \mathfrak{s},$$

where $\mathfrak{s}$ is semisimple.

Proof. This is a cohomological statement, the vanishing of the second cohomology $H^2(\mathfrak{s}, \Bbbk)$ for semisimple Lie algebras. The proof is similar to the proof of semisimplicity of representations of semisimple Lie algebras. (This also has cohomological interpretation.) □

Corollary. Any reductive Lie algebra $\mathfrak{g}$ decomposes as a sum of ideals $[\mathfrak{g},\mathfrak{g}]\oplus Z(\mathfrak{g})$ and $[\mathfrak{g},\mathfrak{g}]$ is semisimple.

7.1.3.

Theorem. For a Lie subalgebra $\mathfrak{g}\subseteq sl(V)$, if the form κ_V on $\mathfrak{g}$ is non-degenerate then $\mathfrak{g}$ is reductive.

(So, $\mathfrak{g}$ is semisimple iff the form κ_V on $\mathfrak{g}$ is non-degenerate and $Z(\mathfrak{g})=0$.)

Proof. We will assume that $rad(\kappa_V)=0$ and we will prove that $Rad(\mathfrak{g})$ is central, i.e., $[\mathfrak{g},Rad(\mathfrak{g})]=0$.

The point is that for any $x\in[\mathfrak{g},Rad(\mathfrak{g})]$ we know that it acts by zero in any irreducible representation W. This implies that $\kappa_W(x,-)=0$ for irreducible W.

However, this further implies that $\kappa_U(x,-)=0$ for any representation U. The point is that for a SES $0\mathbb{U}'\to U\to U''\to 0$, we have $\kappa_U(x,y)=\kappa_{U'}(x,y)+\kappa_{U''}(x,y)$.

In particular $\kappa_V(x,-)$ for the above representation V. Since κ_V is nondegenerate this implies that $x=0$. □

Question. If $\mathfrak{g}\subseteq gl(V)$ is semisimple, then we want to show that $rad(\kappa_V)=0$.

"Proof." We know that $R=rad(\kappa_V)=\mathfrak{g}^{\perp}$ is an ideal in $\mathfrak{g}$. So, to show that it is zero it suffices to prove that R is solvable (there are no solvable ideals in a semisimple $\mathfrak{g}$). By Cartan's criterion (version (1)) this is equivalent to $\kappa_V(R,[R,R])=0$.

Now recall that since R is an ideal its Killing form κ_R is the restriction of the Killing form on $\mathfrak{g}$ (see proposition 3.3.5.b). So, $\kappa_R(R,[R,R])=\kappa_{\mathfrak{g}}(R,[R,R])$ is zero since $R=rad(\kappa_{\mathfrak{g}})$. □

7.2. **Semisimple Lie algebras are sums of simple Lie algebras.** Recall that $\mathfrak{g}$ is *simple* if it has no proper ideals and $\mathfrak{g}$ is not abelian. (Equivalently, $\mathfrak{g}$ has no proper ideals and $0\neq\mathfrak{g}\ncong\Bbbk$.)

Lemma. Let I,J be ideals in a Lie algebra $\mathfrak{g}$.

(a) If $I\cap J$ then$I$ and J commute.

(b) If $\mathfrak{g}=I\oplus J$ as a vector space then $\mathfrak{g}=I\oplus J$ as a Lie algebra, i.e., $[,J]=0$.

ideal in

Proof. (a) $[I,J]\subseteq I\cap J$. (b) follows. □

The point is an ideal I in $\mathfrak{g}$ is always of the form

Proposition. Semisimple Lie algebras are the same as sums of simple Lie algebras.

Proof. **A.** To see that simple $\mathfrak{g}$ is semisimple we need $Rad(\mathfrak{g}) = 0$. Since $\mathfrak{g}$ is simple the ideal $Rad(\mathfrak{g})$ has to be 0 or all of $\mathfrak{g}$. We can see that $Rad(\mathfrak{g}) = \mathfrak{g}$ is impossible (so $Rad(\mathfrak{g}) = 0$ and $\mathfrak{g}$ is semisimple).

If $Rad(\mathfrak{g}) = \mathfrak{g}$ then $\mathfrak{g}$ would be solvable. Since $\mathfrak{g} \neq 0$ solvability would imply that $\mathfrak{g}$ has an abelian ideal $\mathfrak{a} \neq 0$. Since $\mathfrak{g}$ has no proper ideals this implies that $\mathfrak{a} = \mathfrak{g}$, so $\mathfrak{g}$ would be abelian.

B. A sum $\mathfrak{g} \stackrel{\text{def}}{=} \oplus_1^N \mathfrak{g}_i$ is semisimple iff all $\mathfrak{g}_i$ are semisimple.

This is simple in terms of the Killing criterion since $\kappa_{\oplus\ \mathfrak{g}_i} = \ \oplus_i \kappa_{\mathfrak{g}_i}$.

C. It remains to prove that any semisimple $\mathfrak{g}$ is a sum of simple Lie algebras.

First, if $\mathfrak{g}$ itself is not simple then it has a proper ideal $0 \neq I \neq \mathfrak{g}$. Then $I^\perp$ is also an ideal, hence so is $I \cap I^\perp$.

C1. The ideal $\mathfrak{s} = I \cap I^\perp$ is zero. It suffices to show that it is solvable and this will follow from Cartan's criterion (version (1)).

Fist, since $\mathfrak{g}$ is semisimple $Z(\mathfrak{g}) = 0$ as it is an abelian ideal. Therefore, $\mathfrak{g} \stackrel{\cong}{\to} ad(\mathfrak{g}) \subseteq gl(\mathfrak{g})$. Now $\kappa_{\mathfrak{g}}(\mathfrak{s}, \mathfrak{s}) \subseteq \kappa_{\mathfrak{g}}(I, I^\perp) = 0$. So, in particular $\mathfrak{s} \perp \mathfrak{s}'$ for $\kappa_{\mathfrak{g}}$ and therefore $\mathfrak{s}$ is solvable.

C2. Now $\mathfrak{g}$ contains the sum of vector spaces $I \oplus I^\perp$. Actually, this is all of $\mathfrak{g}$ since (because $\kappa_{\mathfrak{g}}$ is nondegenerate) $\dim(I \oplus I^\perp) = \ \dim(I) + (\dim(\mathfrak{g}) - \dim(I)) = \dim(\mathfrak{g})$.

Therefore, $\mathfrak{g} = I \oplus I^\perp$ as a sum of vector spaces, but then this is also a sum of Lie algebras by the lemma $\square$

7.2.1. *The adjoint representation of a semisimple* $\mathfrak{g}$. Recall that an ideal in $\mathfrak{g}$ is the same as a $\mathfrak{g}$-subgmodule of $\mathfrak{g}$. So, a

Corollary. Let $\mathfrak{g}$ be a semisimple Lie algebra with a decomposition $\mathfrak{g} = \oplus_1^N \mathfrak{g}_i$ into simple ideals.

(a) All ideals of $\mathfrak{g}$ are classified by subsets S of $\{1, ..., n\}$. here S gives $\mathfrak{g}_S = \oplus_{i \in S} \mathfrak{g}_i$.

(b) The *simple ideals* (in the sense of an ideal which is simple as a Lie algebra) in $\mathfrak{g}$ are exactly the ideals $\mathfrak{g}_i$. (In particular the decomposition into simple Lie subalgebras is unique up to a permutation of indices.)

(c) If $\mathfrak{g}$ is semisimple then $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is all of $\mathfrak{g}$.

(d) Each $\mathfrak{g}_i$ is an irreducible module for $\mathfrak{g}$ and modules $\mathfrak{g}_i$ are mutually non-isomorphic. As a $\mathfrak{g}$-module we have $(\mathfrak{g}_i)^* \cong \mathfrak{g}_i$.

Proof. (a) Let x is an element of an ideal $\mathfrak{a}$ in $\mathfrak{g}$ and $x = \sum x_i$ with $x_i \hookrightarrow \mathfrak{g}_i$. We seethat if $x_i \neq 0$ Then $\mathfrak{a} \supseteq [x, \mathfrak{g}_i] = [x_i, \mathfrak{g}_i]$ and if $x_i \neq 0$ then $[x_i, \mathfrak{g}_i] = \mathfrak{g}_i$ since $\mathfrak{g}_i$ is an irreducible $\mathfrak{g}_i$-module.

(b) follows. For (c) it suffices to prove the case when $\mathfrak{g}$ simple. Then $[\mathfrak{g},\mathfrak{g}]$ is an ideal in $\mathfrak{g}$ which is not zero (then $\mathfrak{g}$ would be abelian), hence $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$. $\square$

(d) For $i\neq j$ modules $\mathfrak{g}_i,\ \mathfrak{g}_j$ are different since $\mathfrak{g}_i$ acts by zero on $\mathfrak{g}_j$ but not on $\mathfrak{g}_i$. Since $\mathfrak{g}_i$ is esmisimple, its Killing form $\kappa_{\mathfrak{g}_i}$ is nondegenerate, hence $\kappa_{\mathfrak{g}_i}: fg_i\to\mathfrak{g})i^*$ is an isomorphism.

7.2.2. *Invariant bilinear forms.*

Lemma. For a semsimple Lie algebra $\mathfrak{g}$, the invariant bilinear forms on $\mathfrak{g}$ are exactly all linear combinations $\sum\ c_i\kappa_{\mathfrak{g}_i}$ of Killing forms on irreducible summands $\mathfrak{g}_i$. (In particular they are all symmetric.)

Proof. The bilinear invariant forms κ on $\mathfrak{g}$ are the same as $\mathfrak{g}$-morphisms $\kappa:\mathfrak{g}\to\mathfrak{g}^*$. However, by part (d) of the lemma 7.2.1, $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}_i,\mathfrak{g}_j^*)\cong\ \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}_i,\mathfrak{g}_j)$ is zero for $i\neq j$ and for $i=j$ it contains the Killing form $\kappa_{\mathfrak{g}_i}$. Then $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}_i,\mathfrak{g}_i)=\Bbbk\kappa_{\mathfrak{g}_i}$ by Schurr's lemma. $\square$

7.3. **All derivations of a semisimple Lie algebra are inner.**

7.3.1. *Lie algebra $Der(\mathfrak{g})$.* A derivation of a Lie algebra $\mathfrak{g}$ is a linear map $D:\mathfrak{g}\to\mathfrak{g}$ such that it satisfies the Leibniz rule for the bracket operation, i.e., $D[x,y]=\ [Dx,y]+[x,Dy]$. All derivations of $\mathfrak{g}$ form a vector space $Der(\mathfrak{g})$.

Lemma. $Der(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$,

Proof. For $D',D''\in Der(\mathfrak{g})$ and $x,y\in\mathfrak{g}$ we have $D'D''\ [x,y]=\ D'([D''x,y]+[x,D''y])=[D'D''x,y]+[D''x,D'y]+[D'x,D''y]+[x,D'D''y]$. Therefore,

$$\begin{aligned}[D',D'']([x,y])\ &=\ ([D'D''x,y]+[D''x,D'y]+[D'x,D''y]+[x,D'D''y])\\ &=\ [[D',D'']x,y]\ +\ [x,[D',D'']y].\quad\square\end{aligned}$$

7.3.2. *Inner derivations.* For any $x\in\mathfrak{g}$ the operator $ad(x)$ on $\mathfrak{g}$ is a derivation of $\mathfrak{g}$ because of the Leibniz rule. So, the image $ad(\mathfrak{g})$ of $ad:\mathfrak{g}\to gl(\mathfrak{g})$ lies in $Der(\mathfrak{g})$ and this Lie subalgebra is called the *inner derivations.*

Lemma. $ad(\mathfrak{g})\subseteq Der(\mathfrak{g})$ is an ideal.

Proof. For $D\in Der(\mathfrak{g})$ and $x\in\mathfrak{g}$ we have $[D,ad(x)]\ =\ ad(Dx)$:

$$[D,ad(x)]y\ =\ D[x,y]-[x,Dy]=\ [Dx,y]$$

since D is a derivation. $\square$

Theorem. In a semisimple Lie algebra $\mathfrak{g}$ all derivations are inner:

$$Der(\mathfrak{g}) \;=\; ad(\mathfrak{g}).$$

Proof. For a subspace U of a vector space V which carries a bilinear symmetric form κ, we may denote by $U^{\perp}_{V,\kappa}$ the subspace of V consisting of vectors orthogonal to U for κ, Here, we will use $ad(\mathfrak{g})^{\perp}$ in the sense of $ad(\mathfrak{g})^{\perp}_{Der(\mathfrak{g}),\kappa_{Der(\mathfrak{g})}}$.

Now, recall that $\mathfrak{g}\xrightarrow{\cong}ad(\mathfrak{g})$ and that $ad(\mathfrak{g})$ is an ideal in $Der(\mathfrak{g})$. This gives another ideal $ad(\mathfrak{g})^{\perp}$ in $Der(\mathfrak{g})$.

Now, the restriction of $\kappa_{Der(\mathfrak{g})}$ to $ad(\mathfrak{g})$ is $\kappa_{ad(\mathfrak{g})}$ because $ad(\mathfrak{g})$ is an ideal in $Der(\mathfrak{g})$. So, the restriction of $\kappa_{Der(\mathfrak{g})}$ to $ad(\mathfrak{g})$ is nondegenerate. This implies a decomposition of a vector space

$$Der(\mathfrak{g}) = \; ad(\mathfrak{g}) \oplus ad(\mathfrak{g})^{\perp}.$$

So, it suffices to show that $ad(\mathfrak{g})^{\perp}$ is zero.

For $D \in \; ad(\mathfrak{g})^{\perp}$ and $x \in \mathfrak{g}$ we need $Dx = 0$, It suffices that $ad(Dx) = 0$. However,

$$ad(Dx) \;=\; [D, ad(x)]$$

and $D,\; ad(x)$ commute since $ad(\mathfrak{g})^{\perp}$ and $ad(\mathfrak{g})$ are ideals in $Der(\mathfrak{g})$ and their intersection is zero. □

7.4. **Jordan decomposition in semisimple Lie algebras.**

Theorem. If $\mathfrak{g}$ is semisimple Lie algebra then any element has a unique Jordan decomposition.

Proof. For semisimple $\mathfrak{g}$ we have

$$\mathfrak{g} \;\underset{\cong}{\xrightarrow{ad}}\; ad(\mathfrak{g}) \;=\; Der(\mathfrak{g}) \subseteq \mathrm{End}(\mathfrak{g}).$$

Now, for $x \in \mathfrak{g}$, the operator $ad(x) \in \mathrm{End}(\mathfrak{g})$ has a Jordan decomposition $S + N$.

By lemma 6.5.4 we know that S, N lie in $Der(\mathfrak{g})$. Since $Der(\mathfrak{g}) = ad(\mathfrak{g})$ we have $S = ad(s),\; N = ad(n)$ for some $s, n \in \mathfrak{g}$.

Now, $ad(x) = ad(s) = ad(n) = ad(s + n)$ implies that $x = s + n$ (since $Z(fg) = 0$). Moreover, $[s, n] = 0$ since $ad[s, n] = [ad(s), ad(n)] = [S, N] = 0$. Finally operator $ad(s) = S$ is semisimple and $ad(n) = N$ is nilpotent.

The uniqueness of the Jordan decomposition in $\mathfrak{g}$ is a general property of all Lie algebras with $Z(\mathfrak{g}) = 0$ (lemma 6.5.2). □

7.5. **Semisimplicity theorem for semisimple Lie algebras.**

7.5.1. *Invariant forms κ_V for semisimple $\mathfrak{g}$.* Let (V, π) be a finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$. Then $\mathrm{Ker}(\pi)$ is an ideal in $\mathfrak{g}$ so it is a sum of some simple summands of $\mathfrak{g}$ (see corollary 7.2.1). Hence the sum of the remaining simple ideals $\mathfrak{g}_V \subseteq \mathfrak{g}$ is canonically isomorphic to the quotient $\pi(\mathfrak{g})$.

Lemma. (a) The form $\kappa = \kappa_V^{\mathfrak{g}_V}$ on $\mathfrak{g}_V$ is nondegenerate on $\pi(\mathfrak{g})$.

(b) The Casimir element $C_V \in U(\mathfrak{g}_V)$ corresponding to the form κ on $\mathfrak{g}_V$ lies in in $Z[U\mathfrak{g}]$.

(c) C_V acts on V as $\dim(\pi(\mathfrak{g}))$.

Proof. (a) The radical of κ is (by Cartan criterion!) a solvable ideal in $\mathfrak{g}_V$. Since $\mathfrak{g}_V$ is semisimple this makes it zero.

(b) We know that $C^\kappa \in U(\mathfrak{g}_V)$ commutes with $\mathfrak{g}_V$. It commutes with $\mathrm{Ker}(\pi)$ because $\mathfrak{g}_V$ commutes with $\mathrm{Ker}(\pi)$ (sums of disjoint families of simple ideals in $\mathfrak{g}$). Finally, if we write C as a sum $\sum x_p x^p$ for κ-dual bases of $\mathfrak{g}_V$ then $Tr_V(C) = \sum Tr_V(x_p x^p) = \sum \kappa(x_p, x^p) = \dim(\mathfrak{g}_V$. □

Theorem. For a semisimple Lie algebra $\mathfrak{g}$ any finite dimensional representation is semisimple.

Proof. We need to split all extensions $0 \to E' \to E \to E'' \to 0$. By general arguments it suffices to split such extensions when $E'' = \Bbbk$ and E' is irreducible.

Let $C = C_E$ be the Casimir associated to the representation E in the preceding lemma 7.5.1. It acts on irreducible E' by a scalar c (Schurr lemma) and it clearly acts on $\Bbbk$ by 0.

If $c \neq 0$ then the 0-eigenspace $\mathrm{Ker}(C)$ of C is a $\mathfrak{g}$-invariant complement of E' in E.

If $c = 0$ then $Tr_E(C) = c\dim(E')$ is zero, but by lemma 7.5.1.c this implies that $\mathfrak{g}$ acts on E by 0, hence E would split! □

Remark. This proof is much more abstract then the one we used for sl_n. That one used the knowledge of all irreducible finite dimensional representations of $\mathfrak{g}$!

7.5.2. *Preservation of Jordan decomposition.*

Remark. (a) The image of a semisimple Lie algebra $\mathfrak{g}$ in any representation U lies in $sl(U)$.

(b) For a linear operator x on W, with a Jordan decomposition $s+n$ we have $s, n \in sl(V)$ iff $x \in sl(V)$.

Proof. (a) For any semisimple Lie subalgebra $\mathfrak{s}$ of $gl(U)$ we have $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \subseteq [gl(V), gl(V)] = sl(V)$.

(b) We have $Tr(x) = Tr(s)$. □

Theorem. (a) Any representation (V, π) of a semisimple Lie algebra then preserves Jordan decomposition, i.e., if $x = s + n$ is a Jordan decomposition in V then $\pi x = \pi s + \pi n$ is a Jordan decomposition in linear operators on V.

(b) If $\mathfrak{g}_i$ are semisimple then any Lie algebra map $\mathfrak{g}_1 \xrightarrow{\alpha} \mathfrak{g}_2$ preserves Jordan decomposition.

Proof. (b) follows from (a) by using the adjoint representation of $\mathfrak{g}_2$ (and remembering that it is faithful).

In (a) it suffices to consider the case when the representation V is faithful since any representation V of $\mathfrak{g}$ reduces to a faithful representation of a semisimple Lie subalgebra $\mathfrak{g}_V$ of $\mathfrak{g}$ (see 7.5.1).

So, we are now in a case where $\mathfrak{g}$ is a semisimple Lie subalgebra of some $gl(V)$.

(S1) Let $x \in \mathfrak{g}$ and let $x = S + N$ be its Jordan decomposition as an operator on V. The we know that $ad_{gl(V)}(x) = ad_{gl(V)}(S) + ad_{gl(V)}(N)$ is Jordan decomposition in operators on $gl(V)$. $S, N \in gl(V)$, of the The reason is that Since $ad_{gl(V)}(x)$ preserves $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, we get that $ad_{gl(V)}(x)$, $ad_{gl(V)}(x)$ preserve $\mathfrak{g}$, so S, N lie in the normalizer subalgebra $N \stackrel{\text{def}}{=} N_{gl(V)}(\mathfrak{g})$ of $gl(V)$.

(S2) We will actually prove that S, N lie in a smaller subalgebra $N' \subseteq N$. First for a $\mathfrak{g}$-submodule W of V we define

$$L_W \stackrel{\text{def}}{=} \{y \in gl(V);\ y \text{ preserves } W \text{ and } Tr_W(y) = 0\}.$$

Then we choose N' as the intersection of the normalizer N with all subalgebra L_W corresponding to submodules $W \subseteq V$.

Certainly, operator x preserves any submodule $W \subseteq V$, hence so do S and N. Moreover, the image of the semisimple Lie algebra $\mathfrak{g}$ in $gl(W)$ is a semisimple subalgebra of $gl(W)$ so it lies in $sl(W)$. Therefore, $x|_W$ is in $sl(V)$ and then $x|_V = S|_V + N|_V$ implies that $0 = Tr_W(x)$ equals $Tr_w(S)$. So, restrictions of S, N to W are in L_W. Therefore, S, N lie in N'.

(S3) We will now prove that $N' = \mathfrak{g}$ hence we will get that $S, N \in \mathfrak{g}$. This will imply that they form a Jordan decomposition of x in the Lie algebra $\mathfrak{g}$.

For this we will use the semisimplicity theorem. It guarantees that for the inclusion of $\mathfrak{g}$-modules $\mathfrak{g} \subseteq N'$ there is a complementary $\mathfrak{g}$-submodule X.

Here, X is a trivial $\mathfrak{g}$-module as $[X, \mathfrak{g}] \subseteq X$ and $[X, \mathfrak{g}] \subseteq [N, \mathfrak{g}] \subseteq \mathfrak{g}$ (since Lie algebra $\mathfrak{g}$ is an ideal in its normalizer $N = N_{gl(V)}(\mathfrak{g})$). so it is an ideal in $N' \subseteq N$.

... $\square$

8. Structure of semisimple subalgebras

We follow the point of view for which the structure of any semisimple Lie algebra over a closed field $\Bbbk$ is completely parallel to that of sl_n.

While for sl_n facts can be checked by computation with matrices, for semisimple Lie algebras in general we will "at each step" use the the Killing form $\kappa_{\mathfrak{g}} = \kappa$ since its nondegeneracy is an equivalent definition of semisimple Lie algebras. (This definition is "more concrete" and "positive" than the absence of solvable ideals.)

8.1. Cartan subalgebras.

8.1.1. *Toral subalgebras.* A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is said to be *toral* if all its elements are $ad_{\mathfrak{g}}$-semisimple. A Lie algebra $\mathfrak{h}$ is said to be *toral* if it is toral as a subalgebra of itself, i.e., if for all its elements are $ad_{\mathfrak{h}}$-semisimple. Clearly if $\mathfrak{h}$ is a toral subalgebra of some $\mathfrak{g}$ then it is a toral Lie algebra.

Lemma. Toral algebras are abelian.

Proof. For $x \in \mathfrak{h}$, ad-semisimplicity says that $\mathfrak{g}$ is the sum of α-eigenspaces $\mathfrak{g}^x_\alpha$. So we need to see that for $\alpha \neq 0$ the α-eigenspace $\mathfrak{g}^x_\alpha$ of any $x \in \mathfrak{g}$ is 0.

If $y \neq 0$ is an eigenvector $[x, y] = \alpha y$ then $ad(y)x = -\alpha y$, hence $ad(y^2)x = 0$. The semisim[licity of $ad(y)$ then gurantees that $ad(y)x = 0$ hence $\alpha = 0$.

8.1.2. *Maximal toral subalgebras.* Let $\mathfrak{h}$ be a *Cartan subalgebra* of a Lie algebra $\mathfrak{g}$, i.e., a maximal toral subalgebra of $\mathfrak{g}$. By the lemma, $ad(\mathfrak{h})$ is a commutative family of semisimple operators, hence $\mathfrak{g} = \oplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}^{\mathfrak{h}}_\alpha$. This defines the *weights* $\mathcal{W}(\mathfrak{g})$ of $\mathfrak{h}$ in $\mathfrak{g}$ and *roots* $\Delta = \Delta^{\mathfrak{h}}(\mathfrak{g}) \stackrel{\text{def}}{=} \mathcal{W}(\mathfrak{g}) - \{0\}$. So,

$$\mathfrak{g} = \ Z_{\mathfrak{g}}(\mathfrak{h}) \oplus \oplus_{\alpha\in\Delta} \mathfrak{g}^{\mathfrak{h}}_\alpha.$$

Lemma. (a) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

(b) For $\alpha \in \Delta$, any $x \in \mathfrak{g}_\alpha$ is nilpotent.

(c) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$.

(d) For any α, the restriction of κ to a pairing of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ is nondegeenrate.

Remark. The following is a key observation. It says that a semisimple Lie algebra has a "large" toral subalgebra $\mathfrak{h}$. The meaning of "large' here is that anything that commutes with $\mathfrak{h}$ has to be in $\mathfrak{h}$ itself!

We will then see in 8.2 that such large toral subalgebra plays the same role in $\mathfrak{g}$ as the diagonal matrices in sl_n.

Proposition. $Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Proof. (S1) For any $x \in \mathcal{C} \stackrel{\text{def}}{=} Z_{\mathfrak{g}}(\mathfrak{h})$ its semisimple and nilpotent parts s, n lie in $\mathcal{C}$.

The point is that when x commutes with $\mathfrak{h}$ then so do s and n.

(S2) All semisimple elements of $\mathcal{C}$ lie in $\mathfrak{h}$. The reason is that if $s \in \mathcal{C}$ is ad-semisimple then $\mathfrak{h} + \Bbbk s$ is toral!.

(S3) Denote by $\mathfrak{h}^{\perp} \stackrel{\text{def}}{=} \mathfrak{h}^{\perp}_{\mathcal{C},\kappa_{\mathfrak{g}}}$ the part of $\mathcal{C}$ that is orthogonal to $\mathfrak{h}$ for $\kappa_{\mathfrak{g}}$. It contains all nilpotent elements of $\mathcal{C}$, i.e., all $y \in \mathcal{C}$ with $ad_{\mathfrak{g}}$ nilpotent.

The point is that for $h \in \mathfrak{h}$ the operator $ad_{\mathfrak{g}}(h)ad_{\mathfrak{g}}(y)$ is nilpotent (hence has trace zero), since $ad_{\mathfrak{g}}(y)$ is nilpotent and $ad_{\mathfrak{g}}(h), ad_{\mathfrak{g}}(y)$ commute.

(S4) The restrictions of the Killing form κ to $\mathcal{C}$ and $\mathfrak{h}$ are non-degenerate.

The first statement is the part (d) of the preceding lemma. The second statement says that $\mathfrak{h} \cap \mathfrak{h}^{\perp} = 0$. It will follow from the first once we show that $\mathfrak{h} \cap \mathfrak{h}^{\perp}$ is orthogonal to $\mathcal{C}$,

However, for any $x \in \mathcal{C}$ with the Jordan decomposition $x = s + n$ in $\mathfrak{g}$ we have $s, n \in \mathcal{C}$ by (S1). Now $n \perp \mathfrak{h}$ by (S3) and $s \in \mathfrak{h}$ by (S2). hence $s \perp \mathfrak{h}^{\perp}$. Therefore, $x \perp \mathfrak{h} \cap \mathfrak{h}^{\perp}$.

(S5) Lie algebra $\mathcal{C}$ decomposes as a sum of ideals $\mathcal{C} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. Also, $\mathfrak{h}^{\perp}$ is the set of all $ad_{\mathfrak{g}}$ nilpotent elements of $\mathcal{C}$.

First, $\mathfrak{h}$ is an ideal in $\mathcal{C}$ (the brackets of $\mathcal{C}$ with $\mathfrak{h}$ are zero), hence $\mathfrak{h}^{\perp}$ is also an ideal in $\mathcal{C}$ (because the restriction of $\kappa_{\mathfrak{g}}$ to $\mathcal{C}$ is an invariant form). The decomposition as a vector space $\mathcal{C} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ holds because κ is non-degenerate on $\mathcal{C}$ and $\mathfrak{h}$.

If $y \in \mathfrak{h}^{\perp}$ then its Jordan parts s_y, n_y in $\mathfrak{g}$ lie in $\mathcal{C}$ by (S1), hence $s_y \in \mathfrak{h}$ by (S2). But this will imply that $s_y = 0$ (hence $y = n_y$ is ad-nilpotent). The reason is that $\kappa(\mathfrak{h}, n_y) = 0$ by (S3) and therefore since $\kappa(\mathfrak{h}, s_y) = \kappa(\mathfrak{h}, y) = 0$. So, $s_y \in \mathfrak{h} \cap \mathfrak{h}^{\perp} = 0$.

(S6) Lie algebra $\mathcal{C}$ is nilpotent.

Since $\mathcal{C} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ and $\mathfrak{h} \subseteq Z(\mathcal{C})$, it suffices that $\mathfrak{u}$ be nilpotent. This is true by Engel's theorem since for any $y \in \mathcal{C}$ the operator $ad_{\mathcal{C}}(x)$ is nilpotent (it is a restriction of $ad_{\mathfrak{g}}(y)$).

(S7) The center of $\mathfrak{u}$ is zero.

This will follow from $Z(\mathfrak{h}^{\perp}) \perp \mathcal{C}$. First notice that $Z(\mathfrak{h}^{\perp}) \subseteq \mathbb{Z}(\mathcal{C})$ since $\mathcal{C} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$.

Now, for $z \in \mathbb{Z}(\mathfrak{u})$ and any $x \in \mathcal{C}$ the operator $ad_{\mathfrak{g}}(z)ad_{\mathfrak{g}}(x)$ is nilpotent (the factors commute ad $ad_{\mathfrak{g}}(z)$ is known to be nilpotent by (S5)). So, $\kappa(z, x) = Tr_{\mathfrak{g}}[ad_{\mathfrak{g}}(z)ad_{\mathfrak{g}}(x) = 0$.

(S8) $\mathfrak{h}^{\perp} = 0$. The reason is that a non-zero nilpotent algebra always has has a non-zero center (the last non-zero term in the lower central series lies in the center!). $\square$

8.2. **Roots.**

8.2.1. *Elements $\kappa^{-1}\alpha$ of $\mathfrak{h}$ ($\alpha \in \Delta$).* In the following, κ could be any non-degenerate symmetric bilinear invariant form on $\mathfrak{g}$. The important results will be independt from the choiceof such κ. For convinience we will use the Killing form $\kappa = \kappa_{\mathfrak{g}}$.

Since $\kappa|_{\mathfrak{h}}$ is non-degenerate, the corresponding linear operator $ka_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}^*$ is invertible. We will use its inverse $\kappa^{-1} : \mathfrak{h}^* \xrightarrow{\cong} \mathfrak{h}$.

Lemma. (a) Δ spans $\mathfrak{h}^*$.

(b) $-\Delta = \Delta$.

(c) For $\alpha \in \Delta$ and $x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\alpha}$ one has

$$[x, y] \ = \ \kappa(x, y) \cdot \chi^{-1}\alpha.$$

In particular,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \ = \ \Bbbk \cdot \chi^{-1}\alpha.$$

(d) For $\alpha \in \Delta$ we have $\langle \alpha, \chi^{-1}\alpha \rangle \neq 0$. (By definition $\langle \alpha, \chi^{-1}\beta \rangle \ = \ \kappa(\chi^{-1}\alpha, \chi^{-1}\beta \rangle$.)

Proof. (b) follows from non-degeneracy of κ and the lemma 8.1.2.c. $\square$

8.2.2. *Vector $\check{\alpha} \in \mathfrak{h}$ corresponding to a root α.* We will use $ka_{\mathfrak{h}} : \mathfrak{h} \xrightarrow{\cong} \mathfrak{h}^*$ to transfer the bilinear form κ from $\mathfrak{h}$ to $\mathfrak{h}^*$. This gives a bilinear form on $\mathfrak{h}^*$ denoted

$$(\lambda, \mu) \ \stackrel{\text{def}}{=} \ \kappa(\kappa^{-1}\lambda, \kappa^{-1}\mu), \qquad \lambda, \mu \in \mathfrak{h}^*.$$

Now we can define for each root α an element $\check{\alpha} \in \mathfrak{h}$

$$\check{\alpha} \ \stackrel{\text{def}}{=} \ \frac{2}{(\alpha, \beta)}\ \kappa^{-1}\alpha \ = \ \frac{2}{\kappa(\kappa^{-1}\alpha, \kappa^{-1}\beta)}\ \kappa^{-1}\alpha.$$

Lemma. (a) The construction $\alpha \mapsto \check{\alpha}$ is *canonical*, i.e., it is independent of the choice of the invariant bilinear form κ.

(b) $\langle \alpha, \check{\alpha} \rangle = 2$.

Proof. First notice that if we replace κ by a multiple $c\kappa$ (for invertible c), then the form $(\lambda, \mu) \ = \ \kappa(\kappa^{-1}\alpha, \kappa^{-1}\beta)$ gets multiplied by c^{-1} and the same holds for $\kappa^{-1}\alpha$. So, the multiple c cancels.

In general, we use the decomposition $\mathfrak{g} = \oplus_1^N \mathfrak{g}_i$ of $\mathfrak{g}$ into a sum of simple ideals. Then any invariant bilinear form κ on $\mathfrak{g}$ is a linear combination $\sum c_i \kappa_{\mathfrak{g}_i}$. Clearly κ is nondegenerate iff all c_i are invertible. Since $\alpha \in \Delta$ lives in a single $\mathfrak{g}_i$ we can use the initial obhservation inthis proof. $\square$

Remark. From (b) we see that the operator s_α on $\mathfrak{h}^*$ defined by

$$s_\alpha(\beta) \stackrel{\text{def}}{=} \beta - \langle \beta, \check{\alpha} \rangle \alpha$$

is a reflection.

8.2.3. *sl_2-subalgebras $\mathfrak{s}_\alpha$.*

Lemma. (a) For $0 \neq x \in \mathfrak{g}_\alpha$ there exists a $y \in \mathfrak{g}_{-\alpha}$ such that $(x, y) = 2/(\alpha, \alpha)$. Then $\mathfrak{s}_\alpha \stackrel{\text{def}}{=} \Bbbk \cdot x \oplus \Bbbk \cdot y \oplus \Bbbk \cdot [x, y]$ is a Lie subalgebra with a basis $x, y, [x, y]$ isomorphic to the Lie algebra sl_2 with the basis e, f, h.

(b) For such x, y the bracket $[x, y]$ is $\check{\alpha}$ (so, it does not depend on the choice of x, y or κ).

(c) $(-\alpha)\check{} \; = \; -\check{\alpha}$.

Proposition. (a) For $\alpha \in \Delta$, $\dim(\mathfrak{g}_\alpha) = 1$. (So, the subalgebra $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha . \mathfrak{g}_{-\alpha}]$ really only depends on α.)

(b) If $\alpha, c\alpha \in \Delta$ then $c = \pm 1$.

8.2.4. *Reflections s_α.*

Lemma. For roots α, β

(a) $\langle \alpha, \check{\beta} \rangle = \langle \alpha, \chi^{-1} \beta \rangle$ is an integer.

(b) $s_\alpha(\beta) \stackrel{\text{def}}{=} \beta - \langle \beta, \check{\alpha} \rangle \alpha$ is again a root.

Corollary. If $\alpha, \beta, \alpha + \beta \in \Delta$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

8.2.5. *α-string of roots through β.*

Proposition. For $\alpha, \beta \in \Delta$ we have

$$\langle \beta, \check{\alpha} \rangle \; = \; r - s$$

where

- s is the maximum of all $p \in \mathbb{N}$ such that $\beta + p\alpha \in \Delta$ and
- r is the maximum of all $q \in \mathbb{N}$ such that $\beta - q\alpha \in \Delta$.

(b) For any $p \in [-r, s]$, $\beta + p\alpha$ is a root.

Corollary. $\mathfrak{g}$ is generated by root spaces $\mathfrak{g}_\alpha, \; \alpha \in \Delta$.

8.2.6. *The Killing form on $\mathfrak{h}_\mathbb{Q}$.* Let $\mathfrak{h}_\mathbb{Z} \stackrel{\text{def}}{=} \operatorname{span}_\mathbb{Z} \check{\Delta}$ and similarly for $\mathfrak{h}_Q$ and $\mathfrak{h}_\mathbb{R}$.

Lemma. (a) $\mathfrak{h}_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of $\mathfrak{h}$, i.e., the map $\mathfrak{h}_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathfrak{h}$ is an isomorphism. [Also, we will see later that $\mathfrak{h}_{\mathbb{Z}}$ is a $\mathbb{Z}$-form of $\mathfrak{h}$ (or of $\mathfrak{h}_{\mathbb{Q}}$).]

(b) The Killing form $\kappa_{\mathfrak{g}} : \mathfrak{h} \times \mathfrak{h} \to \Bbbk$ is *integral* in the sense that it restricts to $\kappa_{\mathfrak{g}} : \mathfrak{h}_{\mathbb{Z}} \times \mathfrak{h}_{\mathbb{Z}} \to \mathbb{Z}$. $\mathfrak{h}_{\mathbb{R}}$ is positive definite.

(c) The restriction of the Killing form to $\mathfrak{h}_{\mathbb{R}}$ is positive definite.

Proof. (a)

(b) has been observed above. In (c), for $h, h' \in \mathfrak{h}$ we have

$$\kappa_{\mathfrak{g}}(h, h') \;=\; Tr_{\mathfrak{g}}(h, h') = \sum_{\alpha \in \Delta \sqcup 0} \langle \alpha, h \rangle \cdot \langle \alpha, h' \rangle.$$

So, for $h \in \mathfrak{h}_{\mathbb{R}}$ we have $\kappa_{\mathfrak{g}}(h, h) \;=\; \sum_{\alpha \in \Delta \sqcup 0} \langle \alpha, h \rangle^2$ which is ≥ 0 since $\langle \alpha, h \rangle^2 \in \mathbb{R}$. Also, the only way it could be zero is if h is orthogonal to Δ but then $h = 0$. $\square$

9. **Root systems**

9.1. **Root systems.** We consider an Euclidean vector space V with a fixed inner product $(-,-)$.

9.1.1. *Reflections.* A non-zero vector $\alpha \in V$ defines a vector

$$\check{\alpha} \stackrel{\text{def}}{=} \frac{2}{(\alpha,\alpha)}\alpha \ \in V$$

and a linear operator $s_\alpha : V \to V$ by

$$s_\alpha x \ = \ x - (\check{\alpha}, x)\alpha \ = \ x - 2\frac{(\alpha,x)}{(\alpha,\alpha)}\alpha.$$

This is a reflection in the hyperplane H_α of all vectors orthogonal to α. The reflection s_α is *orthogonal*, i.e., it preserves the inner product on V. (Because $V = \mathbb{R}\alpha \oplus H_\alpha$ is an orthogonal decomposition and s_α is ± 1 on summands.)

Remark. One can identify V and V^* via $(-,-)$. Then $\check{\alpha} \in V^*$ is given by $\check{\alpha} = \ 2\frac{(\alpha,-)}{(\alpha,al)}$.

9.1.2. *Root systems.* A *root system* in a real vector space V with an inner product is a finite subset $\Delta \subseteq V - 0$ such that

- For each $\alpha \in \Delta$, reflection s_α preserves Δ.
- For $\alpha, \beta \in \Delta$, $\langle \alpha, \check{\beta} \rangle = \ \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.
- Δ spans V.

We say that a root system is *reduced* if $\alpha \in \Delta$ implies that $2\alpha \notin \Delta$. The non-reduced root systems appear in more complicated representation theories. When we say *root system* we will mean a reduced root system.

We call $\dim(V)$ the rank r of the root system Δ.

The sum of two root systems (V_i, Δ_i) is $(V_1 \oplus V_2, \Delta_1 \sqcup \Delta_2)$. We say that (V, Δ) is *irreducible* if it is not a sum.

Remark. $-\Delta = \Delta$ since $s_\alpha = -\alpha$.

Lemma. (a) Root subsystems are closed under intersections.

(b) Any subset $X \subseteq \Delta$ generates a root subsystem (Δ_X, V_X) where $V_X = span(X)$ and Δ_X is obtained by applying products $s_{\alpha_n} \cdots s_{\alpha_1}$ of reflections in roots $\alpha_i \in \mathbb{X}$ to roots in X.

9.1.3. *The dual root system.*

Lemma. $\check{\Delta} \stackrel{\text{def}}{=} \{\check{\alpha};\ \alpha \in \Delta\}$ is also a root system in V (called the *dual* root system).

9.1.4. *Example: roots of semisimple Lie algebras.*

Theorem. For a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$, the set of roots $\Delta \subseteq \mathfrak{h}^*_{\mathbb{R}}$ is a root system.

Proof. We have already checked all requirements. In ... we constructed an invariant inner product $(-,-)$ on $\mathfrak{h}^*_{\mathbb{R}}$ ... $\square$

9.2. **Bases, positive roots and chambers.** The following three notions will turn out to be equivalent encodings of the same data:

• A *base* of Δ is a subset $\Pi \subseteq \Delta$ such that Π is an $\mathbb{R}$-basis of V and

$$\Delta \subseteq span_{\mathbb{N}}(\Pi) \sqcup -span_{\mathbb{N}}(\Pi).$$

• A *system of positive rots* in a root system Δ is a subset $\Delta^+ \subseteq \Delta$ such that

- $\Delta = \Delta^+ \sqcup -\Delta^+$ and
- Δ^+ is closed under addition within Δ, i.e., If $\alpha, \beta \in \Delta^+$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta^+$.

Then $\Delta^{\pm} = \Delta \cap \pm span_{\mathbb{N}}(\Pi)$ are called the *positive* and *negative* roots. We often write "$\alpha > 0$" for "$\alpha \in \Delta^+$".

• For a root system Δ in V, a *chamber* in V is a connected component of $V - \cup_{\alpha \in \Delta} H_\alpha$ where $H_\alpha = \alpha^\perp$ is the hyperplane orthogonal to vector α.

9.2.1. *Equivalence of notions.*

Lemma. For a root system (Δ, V) the following data are equivalent:

- a base Π;
- a system of positive roots Δ^+;
- a chamber $\mathcal{C}$.

The canonical bijections are given by

(1) a base Π gives Δ^+ as $span_{\mathbb{N}}(\Pi) \cap \Delta$;
(2) A system of positive roots Δ^+ gives a base $\Pi \stackrel{\text{def}}{=} \Delta^+ - (\Delta^+ + \Delta^+)$ which is the set of all elements of Δ^+ that are *minimal for addition.*
(3) The chamber corresponding to Δ^+ or Π can be described as

$$\mathcal{C} = \{v \in V;\ (\alpha, v) > 0 \text{ for } \alpha \in \Delta^+\} = \{v \in V;\ (\alpha, v) > 0 \text{ for } \alpha \in \Pi\}.$$

(4) A chamber $\mathcal{C}$ gives a system of positive roots $\Delta^+ = \{\alpha \in \Delta;\ (\alpha, v) > 0 \text{ for } v \in \mathcal{C}\}$.

Remarks. (0) We can restate a part of the lemma in the following way. We say that $\gamma \in V$ is *regular* if $(\gamma, \alpha) \neq 0$ for $\alpha \in \Delta$ (i.e., if it lies in one of the chambers in V). For any regular γ

$$\Delta_\gamma^+ \stackrel{\text{def}}{=} \{\alpha \in \Delta; (\gamma, \alpha) > 0\}$$

is a system of positive roots.

(1) In particular, any root system has a base.

9.3. **Weyl groups.**

9.3.1. *The* Weyl group W *of the root system* Δ. This is the subgroup of $GL(V)$ generated by the reflections s_α for $\alpha \in \Delta$.

Lemma. (a) W preserves Δ.

(b) W is finite.

Theorem. The Weyl group W of Δ acts simply transitively on each of the following classes of objects:

- (a) systems of positive roots in Δ,
- (b) bases of Δ;
- (c) chambers in V.

Remark. One consequence is that *all* bases of Δ behave the same so it suffices to consider one.

9.4. **Classification of root systems.**

9.4.1. *Bases and Dynkin diagrams.*

Lemma. For $\alpha \neq \beta$ in a base Π we have

(a) $(\alpha, \beta) \leq 0$, i.e., the angle is $\geq \pi/2$;

(b) $\alpha - \beta$ is not a root. □

To base Π we associate its *Cartan matrix* $C : \pi^2 \to \mathbb{Z}$ defined by

$$C_{\alpha\beta} \stackrel{\text{def}}{=} (\alpha, \check{\beta}).$$

We also encode it as the *Dynkin diagram* of Π. It is a graph whose vertices are given by the set Π of simple roots. If $|\lambda| \geq |be|$ we connect α to β with $|(\alpha, \check{\beta})|$ bonds. If $|\lambda| > |be|$ we also put an arrow from α to β over these bonds.

Notice that if α and β are not connected in the Dynkin diagram iff $\alpha \perp \beta$. For the reason the Dynkin diagram of a sum of root systems Δ_i is a disjoint union of Dynkin diagrams of Δ_i's,

Theorem. (a) A root system Δ is completely determined by its Dynkin diagram.

(b) A root system Δ is irreducible iff its Dynkin diagram is connected.

(c) The irreducible root systems fall into 4 infinite series called A_n, B_n, C_n, D_n for $n = 1, 2....$ and 5 more ("exceptional") root systems called E_6, E_7, E_8, F_4, G_2.

9.4.2. *Classical series of root systems.* The following are all infinite series of irreducible root systems.

Let Here $E = \oplus_1^n \mathbb{R}\varepsilon_i$ for orthonormal ε_i. **Type A.** Here, $V = \sum c_i\varepsilon_i$ with $\sum c_i = 0$. The roots are all $\pm\varepsilon_i \pm \varepsilon_j$ where $1 \leq i < j \leq n$. The rank is $n-1$ and root system is called A_{n-1}. We have see that these are roots of the Lie algebra sl_n with respect to the diagonal Cartan $\mathfrak{h}$.

Notice that $\check{\Delta} = \Delta$ as $\check{\alpha} = \alpha$ for each root α.

Type B. Here $V = E$ and Δ consists of all $\pm\varepsilon_i$ and $\pm\varepsilon_i \pm \varepsilon_j$ for $i < j$.

Type C. Here $V = E$ and Δ consists of all $\pm 2\varepsilon_i$ and $\pm\varepsilon_i \pm \varepsilon_j$ for $i < j$.

Type D. ...

9.5. **More on root systems.**

9.5.1. *Rank ≤ 2 root systems.*

Lemma. (a) If rank is 1 the root system is isomorphic to A_1.

(a) If rank is 2 the root system is isomorphic to $A_1 \oplus A_1,\ B_2 = C_2,\ G_2$. $\square$

Corollary. The angles between two roots can be $\pi,\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{4},\ \frac{\pi}{6}, 0$ and also $\frac{2\pi}{3},\ \frac{3\pi}{4},\ \frac{5\pi}{6}$.

9.5.2. *The α-string of roots through β.*

Proposition. For $\alpha, \beta \in \Delta$ not proportional, we have

$$\langle \beta, \check{\alpha} \rangle \ = \ r - s$$

where

- s is the maximum of all $p \in \mathbb{N}$ such that $\beta + p\alpha \in \Delta$ and
- r is the maximum of all $q \in \mathbb{N}$ such that $\beta - q\alpha \in \Delta$.

(b) For any $p \in [-r, s]$ $\beta + p\alpha$ is a root.

Remark. Root strings are of length ≤ 4. Length 4 is found in G_2 only.

10. Classification of semisimple Lie algebras and their finite dimensional representations

10.1. Classification of semisimple Lie algebras.

Theorem. (a) For a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$, the set of roots $\Delta \subseteq \mathfrak{h}^*_{\mathbb{R}}$ is a root system.

(b) The root system of $\mathfrak{g}$ determines $\mathfrak{g}$ up to an isomorphism.

(c) Any root system comes from a semisimple Lie algebra.

Remarks. (i) The root system of a sum $\oplus\ \mathfrak{g}_i$ of semisimple Lie algebras $\mathfrak{g}_i$ is the sum of root systems of the summands $\mathfrak{g}_i$. In particular, the Dynkin diagram of $\oplus\ \mathfrak{g}_i$ is a disjoint union of Dynkin diagrams of $\mathfrak{g}_i$'s.

(ii) A semisimple Lie algebra is simple iff its root system is irreducible, i.e., iff its Dynkin diagram is connected.

Corollary. The semisimple Lie algebras over $\Bbbk = \mathbb{C}$ are classified the same as root systems or Dynkin diagrams.

Remarks. (0) Each root $\alpha \in \Delta$ encodes an sl_2-subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}$. The geometry of the root system gives all information on how the sl_2-subalgebras $\mathfrak{s}_\alpha$ are related and how to reconstruct $\mathfrak{g}$ from these subalgebras.

10.2. Semisimple groups.

10.2.1. *Groups to Lie algebras.* In general, to a Lie group G one can associate its Lie algebra $\mathfrak{g}$ as the tangent space T_eG at unity, then the commutator in $\mathfrak{g}$ is the limit of commutators in G. Group G acts on itself by conjugation, this preserves the neutral element e and this in turn gives an action of G on $\mathfrak{g} = T_eG$ called the *adjoint* action. This action preserves the Lie algebra structure so we get a homomorphism of groups $ad : G \to \mathrm{Aut}(\mathfrak{g})$ into the group $\mathrm{Aut}(\mathfrak{g}) \subseteq GL(\mathfrak{g})$ of automorphisms of the Lie algebra $\mathfrak{g}$.

The Lie algebra records what is happening in the group G near the origin e. This turns out to be sufficient (since G is generated by a neighborhood of e to control the connected component G_0 of G (this means that G_0 is the connected component of G which contains e, it is itself an open subgroup of G).

10.2.2. *Lie algebras to groups.* The reverse direction from $\mathfrak{g}$ to G is in general more complicated. However, for a semisimple Lie algebra $\mathfrak{g}$ it is easy to find a group corresponding to $\mathfrak{g}$.

Theorem. The group $\mathrm{Aut}(\mathfrak{g}) \subseteq GL(\mathfrak{g})$ of automorphisms of the Lie algebra has Lie algebra $\mathfrak{g}$. $\square$

Remarks. (0) The point of the theorem is that the semisimple Lie algebras are "maximally non-commutative", so the Lie algebra $\mathfrak{g}$ itself is recorded in its adjoint action on the vector space $\mathfrak{g}$. For this reason we can expect that a group associated to $\mathfrak{g}$ will be recorded in its action on the vector space $\mathfrak{g}$.

Proposition. (a) The smallest connected Lie group with the Lie algebra $\mathfrak{g}$ is the connected component $\mathrm{Aut}_{LieAlg}(\mathfrak{g})_0$ of $\mathrm{Aut}_{LieAlg}(\mathfrak{g})$. It is called the *adjoint* group associated to G and sometimes it is denoted G_{ad}.

(b) The largest connected group G with the Lie algebra $\mathfrak{g}$ is the *universal cover* of $G_{ad} = \mathrm{Aut}_{LieAlg}(\mathfrak{g})_0$. It is called the *simply connected* group associated to G and sometimes denoted G_{sc}.

(c) The center $Z(G_{sc})$ of G_{sc} is finite (and it coincides with the fundamental group $\pi_1(G_{ad})$ of the adjoint version). All connected groups G with the Lie algebra $\mathfrak{g}$ correspond to all subgroups $\mathcal{Z}$ of $Z(G_{sc})$, a subgroup $\mathcal{Z}$ gives the group $G_{sc}/\mathcal{Z}$.

Remark. Each of connected groups G with Lie algebra $\mathfrak{g}$ has $G_{ad} = \mathrm{Aut}_{LieAlg}(\mathfrak{g})_0$ as a quotient, so G acts on $\mathfrak{g}$ via this quotient map $G \to G_{ad}$.

Example. For $\mathfrak{g} = sl_n$ the simply connected group G_{sc} associated to $\mathfrak{g}$ is SL_n. The adjoint group G_{ad} is $SL_n/Z(SL_n)$, it is isomorphic to $GL_n/Z(GL_n)$ which is called the *projective general linear group* and denoted PGL_n.

10.3. **Classification of finite dimensional representations of a semisimple Lie algebra.** A Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ gives the corresponding root system Δ. Any root $\alpha \in \Delta$ gives a vector $\check{\alpha} \in \mathfrak{h}$.

We define the *integral weights* $P \subseteq \mathfrak{h}^*$ to consist of all $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \check{\Delta} \rangle \subseteq \mathbb{Z}$.

A choice of base Π of Δ gives the *dominant integral weights* cone $P^+ \subseteq P$ consisting of all $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \check{\Pi} \rangle \subseteq \mathbb{N}$.

10.3.1. *Borel subalgebras and Verma modules.* A choice of a system of positive roots $\Delta^+ \subseteq \Delta$ give a Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \quad \text{for} \quad \mathfrak{n} = \oplus_{\alpha \in \Delta^+} \, f g_\alpha.$$

Then any $\alpha \in \mathfrak{h}^*$ defines an $\mathfrak{g}$-module, the Verma module

$$M(\lambda) \stackrel{\text{def}}{=} U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{k}_\lambda.$$

Here $\mathfrak{k}_\lambda$ denotes the 1-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts by λ and $\mathfrak{n}$ by zero.

The base Π of Δ corresponding to Δ^+ defines Let us

Theorem. (a) Any Verma module M_λ has a unique irreducible quotient L_λ.

(b) $L(\lambda)$ is finite dimensional iff $\lambda \in P^+$, i.e., iff λ is the the dominant integral cone.

(c) All irreducible finite dimensional representations are exactly all $L(\lambda)$, $\lambda \in P^+$.

Proof. We have proved this theorem for sl_n. In the general case most of the proof is the same. The difference is in one direction in part (b), where we need to show that for $\lambda \in P^+$ the representation $L(\lambda)$ is finite dimensional.

For sl_n we proved this by an explicit construction of $L(\lambda)$ that starts with the fundamental weights $\lambda = \omega_i$. This method does not extend well to the general case because we do not understand the fundamental representations so well.

It is actually easier to construct irreducible finite dimensional representations of a semisimple Lie algebra $\mathfrak{g}$ using the associated group G (then to do it using the Lie algebra itself). This approach is sketched in 10.4. $\square$

10.4. **The Borel-Weil-Bott construction of irreducible representations.**

10.4.1. *The flag variety $\mathcal{B}$ of a semisimple algebra.* For a semisimple Lie algebra $\mathfrak{g}$ let $G = G_{sc}$ be the simply connected group associated to G. Inside G one finds subgroups B, H, N with Lie algebras $\mathfrak{b}, \mathfrak{h}, \mathfrak{n}$.[11]

The quotient $\mathcal{B} \stackrel{\text{def}}{=} G/B$ is called the *flag variety* of $\mathfrak{g}$.[12]

10.4.2. A character of the Cartan group H is a homomorphism $\chi : H \to G_m = GL_1$. The characters of H form a group denoted $X^*(H)$.

The differential of a character χ at $e \in H$ is a linear map $d_e\chi : \ T_eH = \mathfrak{h} \to T_eG_m = \Bbbk$, so it is a linear functional $d_e\chi \in \ \mathfrak{h}^*$ on $\mathfrak{h}$.

Lemma. Taking the differential gives an isomorphism of the character group $X^*(H)$ and the group of integral weights $P \subseteq \mathfrak{h}^*$. $\square$

From now on we will identify any integral weight $\lambda \in P$ with the corresponding character of H which we will also denote λ.

The canonical map of Lie algebras $\mathfrak{b} \to \mathfrak{h}$ (zero on $\mathfrak{n}$) gives a canonical map of Lie groups $B \to H$ (with kernel N). So for any $\lambda \in P$ we get a 1-dimensional representation $\Bbbk_\lambda$ of B via $B \twoheadrightarrow H \xrightarrow{\lambda} G_m = Gl(\Bbbk)$.

[11] Say, B consists of all $g \in G$ that preserve the subspace $\mathfrak{b}$ of $\mathfrak{g}$ and H consists of all $g \in G$ that fix each element of $\mathfrak{h}$. Then N is the unique subgroup of B complementary to H.

[12] The letter $\mathcal{B}$ stands for "Borel". The reason is that the flag variety $\mathcal{B}$ can be identified with the set of all conjugates ${}^g\mathfrak{b} \subseteq \mathfrak{g}$ of the above Borel subalgebra $\mathfrak{b}$ under elements g in G. All these conjugates are called Borel subalgebras of $\mathfrak{g}$ and the particular one $\mathfrak{b}$ that we started with can be called the "standard" Borel subalgebra.

As in a construction of Verma modules we now induce this to a representation of G. The "induction" is in this case slightly different and it is called coinduction. To a representation $\Bbbk_\lambda$ of the group B we associate G-equivariant line bundle $\mathcal{L}_\lambda$ over the flag variety G/B. This is called the associated bundle

$$\mathcal{L}_\lambda \stackrel{\text{def}}{=} (G\times\mathcal{L}_\lambda)/B \quad \to \quad (G\times\text{pt})/B = \mathcal{B}.$$

Because G acts on the line bundle $\mathcal{L}_\lambda$, it also acts on the space of global section of the line bundle $\mathcal{L}_\lambda$

$$Coind_B^G(\mathcal{L}_\lambda) \stackrel{\text{def}}{=} \Gamma(\mathcal{B}, \mathcal{L}_\lambda).$$

Theorem. (a) [Borel-Weil] When $\lambda \in P^+$ then $\Gamma(\mathcal{B}, \mathcal{L}_\lambda)$ is an irreducible finite dimensional representation of G and therefore also of the Lie algebra $\mathfrak{g}$.

(b) As a representation of $\mathfrak{g}$ the space $\Gamma(\mathcal{B}, \mathcal{L}_\lambda)$ has highest weight λ.

Remarks. (0) This implies that $\Gamma(\mathcal{B}, \mathcal{L}_\lambda)$ is the irreducible representation $L(\lambda)$ which was defined as the unique irreducible quotient of the Verma module $M(\lambda)$.

(1) If λ is not dominant then $\Gamma(\mathcal{B}, \mathcal{L}_\lambda) = 0$.

(2) Bott's contribution is the calculation of all cohomology groups of line bundles $L(\lambda)$.

11. Geometric Representation Theory

11.0.1. *The nilpotent cone $\mathcal{N}$ in $\mathfrak{g}$.* For a semisimple Lie algebra $\mathfrak{g}$, there is a geometric object that encodes the Weyl group, this is the so called *nilpotent cone* $\mathcal{N}$ of $\mathfrak{g}$. It consists of all elements of $\mathfrak{g}$ which are nilpotent (i.e., *ad*-nilpotent.

$\mathcal{N}$ is a singular affine algebraic variety. The group $G = \mathrm{Aut}(\mathfrak{g})$ of automorphisms of the Lie algebra $\mathfrak{g}$ preserves $\mathcal{N}$. It has finitely many orbits in $\mathfrak{g}$, these are called the *nilpotent orbits*. Each of these orbits has a canonical symplectic structure $\omega_{\mathcal{O}}$ which is G-equivariant. One of these orbits $\mathcal{O}_r$ (the regular orbit) is dense, so we can say that $\mathcal{N}$ is "generically symplectic".

As an illustration of the representation theoretic information of the nilpotent cone we state

Theorem. [Springer] (a) To each nilpotent orbit $\mathcal{O}$ one can associate an irreducible representation $\pi_{\mathcal{O}}$ of the Weyl group W.

(b) For sl_n this is a bijection of G-orbits in $\mathcal{N}$ and $Irr(W)$.

Remarks. (0) For sl_n we have $G = GL_n/G_m \stackrel{\text{def}}{=} PGL_n$. $\mathcal{N}$ is the variety of all nilpotent operators on $\Bbbk^n$. Its equation is that $\chi(A) = \det(T - A)$ is T^n.

The nilpotent orbits of G in $\mathcal{N}$ are the same as the GL_n orbits in nilpotent matrices, so they are indexed by partitions λ of n, i.e., decompositions $n = \lambda_1 + \cdots + \lambda_p$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$. The corresponding orbit $\mathcal{O}_\lambda$ consists of nilpotent matrices with Jordan blocks of lengths λ_i.

(1) For general W all irreducible representations π of W are attached to data $(\mathcal{O}, \mathcal{L})$ of a nilpotent orbit $\mathcal{O}$ and an irreducible local system $\mathcal{L}$ on $\mathcal{O}$.

Actually, this does not use all irreducible local system $\mathcal{L}$ on nilpotent orbits. The remaining irreducible local systems encode information of more complicated representation theories (of Hecke algebras and finite groups of Lie type).

(2) The explicit construction of irreducible representations of W from nilpotent orbits uses another geometric object, the symplectic resolution $\widetilde{\mathcal{N}}$ of $\mathcal{N}$.

11.0.2. *Symplectic resolution $\widetilde{\mathcal{N}}$ of $\mathcal{N}$.* There is a canonical "symplectic resolution" $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$. Being a *resolution* means that $\widetilde{\mathcal{N}}$ is smooth, map μ is generically an isomorphism and μ is proper (i.e., its fibers are compact). Being a *symplectic resolution* means that $\widetilde{\mathcal{N}}$ has a symplectic structure ω which generically coincides with the generically symplectic structure on $\mathcal{N}$.

Here, the meaning of "generic" is that μ is is an isomorphism above the regular nilpotent orbit $\mathcal{O}_r \subseteq \mathcal{N}$ and then ω coincides with $\omega_{\mathcal{O}_r}$ above $\mathcal{O}_r$.

Remark. Okounkov pointed out that each symplectic singularity (not only the nilpotent cones of semisimple Lie algebras) gives rise to a representation theory. So, he calls symplectic singularities the *Lie algebras of 21⋆ century.*

The representation theories associated to symplectic singularities give a huge generalization of representation theories associated to semisimple Lie algebras which captures ideas beyond representation theory itself. One of such developments is the Bezrukavnikov-Okounkov program of reformulating representation theory in geometric terms of *quantum cohomology.*

11.0.3. *Geometric Representation Theory.* The above phenomenon of a representation theoretic information being encoded in singularities of algebraic varieties is a general phenomenon "throughout" representation theory.

This observation is one aspect of what is now called *Geometric Representation Theory.* This is a general method (containing a number of deep examples) that one studies representation theoretic questions by encoding them into algebro geometric objects.

Example. One example is the *Geometric Langlands program* which is a modern approach to Number Theory.

Example. Another is that in physics the Quantum Field Theories which can be well understood are the ones with much symmetry ("super symmetry") and all of these turn out to be examples of Geometric Representation Theory.

11.1. **"Higher" Representation Theory.** In this text we were concerned with finite dimensional representations of a semisimple Lie algebra G or its Lie algebra $\mathfrak{g}$.

In some sense this is the 0-dimensional part of representation theory. The basic way more complicated representation theories occur is that one replaces the complex group $G(\mathbb{C})$ with the groups of maps $Map(X, G)$ from some space X to G. (This is "d-dimensional representation theory" for $d = \dim(M)$.) These appear in Number Theory, Geometry and Physics.

Example. X could be the spectrum of a commutative ring A. Then $Map(X, G)$ is essentially the group $G(A)$ of elements of G with values in ring A. (Think of G as a group of matrices: $G \subseteq GL_n$, then $G(A)$ means matrices in G with entries in A.) This case covers groups $G(\mathbb{R})$, $G(\mathbb{F}_q)$, $G(\mathbb{Q}_p)$ and the groups $G\big(\Bbbk((z))\big)$ called *loop groups.*

Example. The case when X is 1-dimensional ("a curve") is paramount for number theory. The relevant rings are the so called rigs of adeles.

Example. In physics the most interesting case is when X is a 4 dimensional real manifold. The groups $Map(X, G)$ are called *gauge groups.* When X is a circle S^1 these are again called *loop groups.*

11.2. **Loop Grassmannians.**

11.3. **Quivers.**