Algebra 412, Exam 1.

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Do any 4 of the following problems. Indicate on the cover which 4 problems you want graded.

As always, answers need to be clearly justified.

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1.1. Let A be a commutative ring. We say that an ideal I is a *prime ideal* if whenever a product uv is in I then at least one factor is in I.

(a) Prove that $\{0\}$ is a prime ideal in A iff A is an integral domain.

(b) Find all prime ideals in $\mathbb{Z}$.

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1.2. Let R be a ring.

(a) Consider a family of subrings S_k of R indexed by elements k of some set K. Show that the intersection $\cap_{k\in K} S_k$ is again a subring.

(b) For any subset X of the ring R consider the family of all subrings S in R that contain X. We now know (by (a)), that the intersection of this family

$$\langle X\rangle \stackrel{\text{def}}{=} \cap_{S\subseteq R \text{ is a subring that contains } X} S$$

is a subring of R. Show that $\langle X\rangle$ is the smallest subring of R that contains the subset X. (We call $\langle X\rangle$ the *subring generated by the subset* X.)

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1.3. Prove that $\mathbb{Q}[i] \stackrel{\text{def}}{=} \mathbb{Q}+\mathbb{Q}i$ is a subfield of the field $\mathbb{C}$ of complex numbers.

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1.4. Prove that the ring $\mathbb{Q}[X]/(X^2+1)$ is isomorphic to $\mathbb{Q}[i] \stackrel{\text{def}}{=} \mathbb{Q}+\mathbb{Q}i$. (Use the evaluation function $\phi : \mathbb{Q}[X] \to \mathbb{C}$ given by $\phi(P) = P(i)$ for any polynomial $P \in \mathbb{Q}[X]$.)

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1.5. Let I be an ideal in a ring R and denote by $q:\ R \to R/I$ the canonical quotient map

$$q(r) = \ r+I, \quad r \in R.$$

We will find a bijection of the set $\mathcal{J}$ of ideals in R that contain I and the set $\mathcal{K}$ of all ideals in R/I.

(a) Let J be an ideal in R which contains I, i.e., $J \supseteq I$. Prove that the quotient group J/I is an ideal in the quotient ring R/I. [1]

(b) For any subset K of R/I, we define its "pull back to R" to be the subset $\widetilde{K}$ of R consisting of all $r \in R$ which are sent to K by the map q :

$$\widetilde{K} \stackrel{\text{def}}{=} \{r \in R;\ q(R) \in K\} \ = \ \{r \in R;\ r+I \in K\} \quad \subseteq \quad R.$$

Prove that if $K \subseteq R/I$ is an ideal in R/I then its pull back $\widetilde{K}$ is an ideal in R and it contains I.

(c) Notice that observations (a) and (b) define two procedures of passing between $\mathcal{J}$ and $\mathcal{K}$, i.e., two functions

(1) $\mathcal{A}:\ \mathcal{J} \to \mathcal{K}$ by $\mathcal{A}(J) = \ J/I$, and
(2) $\mathcal{B}:\ \mathcal{K} \to \mathcal{J}$ by $\mathcal{B}(K) = \ \widetilde{K}$.

Prove that these two functions are inverse to each other.

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1.6. Let A be a commutative ring. We say that A is a *principal ideal domain* if every ideal in A is principal, i.e., of the form $(a) \stackrel{\text{def}}{=} aA$ for some $a \in A$.

Prove that if F is a field then the ring of polynomials $F[X]$ is a principal ideal domain. (Use the division of polynomials.)

[1]Here $J/I \subseteq R/I$ consists of all cosets in R/I with representative in J :

$$J/I \ \stackrel{\text{def}}{=} \ \{j+I;\ j \in J\}.$$