Algebra 412, Homework 8
The same as the Second Sample Exam

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Due Thursday April 17^{th} at the exam

As always, answers need to be clearly justified.

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The exam covers

(1) Extensions of fields that one gets from irreducible polynomials. [See section 20 in the book and in the notes this is chapter 7. *Construction of Field Extensions from polynomials.*]
(2) Prime ideals, maximal ideals, irreducible elements of a commutative ring, prime elements of a commutative ring. [This is section 21 in the book.]

Notice: The book covers more material than what we have covered.

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8.1. *Maximal and prime ideals in polynomials $\mathbb{F}[X]$ over a field F.* We say that a polynomial $P \in F[X]$ is *monic* if its leading coefficient is 1. We say that a polynomial $P \in F[X]$ is *irreducible* if $\deg(P) > 0$ and P can not be written as a product $P = UV$ with $U, V \in F[X]$ and the degrees $\deg(U)$, $\deg(V)$ both strictly lesser than $\deg(P)$. Prove that

(a) In $F[X]$ any ideal I is either $\{0\}$ or it is of the form $(P) = PF[X]$ for a unique monic polynomial P.

(b) For a monic polynomial P, the following is equivalent:

(1) Ideal (P) in $F[X]$ is prime.
(2) Ideal (P) in $F[X]$ is maximal.
(3) Polynomial P is irreducible,

8.2. Prove that

(a) Maximal ideals I in $F[X]$ are exactly the ideals of the form $I = (P)$ with P an irreducible monic polynomial.

(b) An ideal I in $F[X]$ is prime if and only if $I = \{0\}$ or I is of the form $I = (P)$ with P an irreducible monic polynomial.

(c) For a monic polynomial P, the ring $A = F[X]/(P)$ is a field iff P is irreducible.

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8.3. For an element a of a commutative ring A prove that a is a prime in A iff the principal ideal (a) is a prime ideal in A.

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8.4. Show that for a field $\mathbb{k}$,

- (a) a quadratic polynomial $P \in \mathbb{k}[X]$ is irreducible iff P has no zeros in F.
- a cubic polynomial $Q \in \mathbb{k}[X]$ is irreducible iff Q has no zeros in F.

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Problems from the book:

8.5. Problem 20.1.

8.6. Problem 20.2. Notice that the book uses a shorthand "domain" for "integral domain".

8.7. Problem 20.3.

8.8. Problem 20.6. Let F be a field and let $P \in F[X]$ be an irreducible polynomial. Then we know that $F[X]/(P)$ is a field and there is a natural injective map of rings

$$\phi : F \to F[X]/(P) \text{ by } \phi(a) = \overline{a} \stackrel{\text{def}}{=} a + (P)$$

for $a \in F$.

Let K be the field obtained from $F[X]/(P)$ by replacing for each $a \in F$ the element $\phi(a) = \overline{a}$ in $F[X]/(P)$ by a. So, K is an extension of F and K is naturally isomorphic to $F[X]/(P)$.

(a) If the degree of P is n, prove that any element $\alpha \in K$ has a unique representation of the form

$$\alpha = a_0 + a_1\overline{X} + \cdots + a_{n-1}\overline{X}^{n-1}$$

with $a_0, ..., a_{n-1}$ in F.

(b) Show that if F is a finite field with q elements then K has q^n elements.

8.9. Problem 20.7.