Algebra 412, Homework 7

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Due Wednesday April 11^{th} in class

As always, answers need to be clearly justified.

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Irreducible elements, factorizations and prime elements

This is about how the ideas of *prime integers* and *factorization into prime integers* extend to general integral domain rings.

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An element a of an integral domain A is said to be *irreducible* if

- $a \neq 0$ and a is not invertible, and (ii) a is not a product of two non-invertible elements, i.e., $a = bc$ implies that at least one of factors is invertible.

An element a of a commutative ring A is said to be a *prime* if

- $a \neq 0$ and a is not invertible, and (ii) if p divides a product ab the p divides one of factors a or b.

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7.1. Recall the subring $A = \mathbb{Z}+\mathbb{Z}i\sqrt{5}$ of $\mathbb{C}$ and the automorphism $f: A \to A$ of the ring A defined by $f(a+i\sqrt{5}b) = a - i\sqrt{5}b$ for $a, b \in \mathbb{Z}$. We also know that the *norm* function $\nu: A \to \mathbb{Z}$ defined by $\nu(\alpha) = \alpha{\cdot}f(\alpha)$ satisfies the following properties

- (i) $\nu(\alpha{\cdot}\beta) = \nu(\alpha){\cdot}\nu(\beta)$ for any $\alpha, \beta \in A$;
- (ii) $\nu(\alpha) \geq 0$ for any $\alpha \in A$;
- (iii) $\nu(1) = 1$.

(a) Show that $\alpha \in A$ is invertible iff $\nu(\alpha) = 1$. List all invertible elements of A.

(b) Prove that 3 is an irreducible element of A.

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7.2. Consider the ring $A = \mathbb{Z}[\sqrt{5}i]$ from the preceding problem.

(a) Prove that A is a *factorization domain* in the sense that any element of A which is not 0 nor invertible factors into a product of irreducible elements.

(b) Prove that $\alpha = 2 + i\sqrt{5}$ and $\beta = 2 - i\sqrt{5}$ are irreducible elements of A.

(c) Explain why A is not a *unique factorization domain*. In other words, show that there is an element of A that has two non-equivalent factorizations into a product of irreducible elements. [Hint: Use the irreducible elements of A that you already know.]

Here two factorizations $a = p_1 \cdots p_n = q_1 \cdots q_m$ into irreducible factors p_i and q_j are said to be *equivalent* if $m = n$ and (after possibly reordering q_j's), p_i, q_i only differ by invertible elements in the sense that $p_i = u_i q_i$ for $i = 1, ..., n$ with u_i invertible.

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7.3. (a) Let A be an integral domain. Prove that any prime element in A is irreducible.

(b) Give an example of an integral domain A an an element a in A which is irreducible but it is not prime.

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7.4. Show that the following ideals are maximal:

(1) $(X^2 - 11)$ in $\mathbb{Q}[X]$.
(2) $(X^2 + 1)$ in $\mathbb{R}[X]$.

Non-principal ideals

7.5. Consider the ring $\mathbb{R}[X, Y]$ of polynomials in two variables X, Y over the field $\mathbb{R}$ of real numbers.[(1)] Let I be the subset of $\mathbb{R}[X, Y]$ consisting of all polynomials P such that $P(0, 0) = 0$.

(a) Show that I is an ideal in $\mathbb{R}[X, Y]$.

(b) Show that I is not a principal ideal.

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[1] A polynomial A in X and Y is a finite linear combination of monomials $X^i Y^j$, i.e.,

$$A \;=\; a_{0,0} + a_{1,0}X + a_{0,1}Y + a_{2,0}X^2 + a_{1,1}XY + a_{0,1}Y^2 + \cdots.$$

We can write it simply as a sum with two indices $A = \sum_{i,j \geq 0} a_{i,j} X^i Y^j$ and we should keep in mind that all but finitely many coefficients $a_{i,j} \in \mathbb{R}$ are zero, so that the sum is actually finite. Also, in order to remember that A is a polynomial of variables X and Y we sometimes denote it as $A(X, Y)$.