Algebra 412, Homework 6

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Due Wed April 4^{th} in class

As always, answers need to be justified.

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Correspondence of ideals in R/I and ideals that contain I

This is the subject of the 1st 3 problems. Notice that the problem 6.3.c is an example for this subject.

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6.1. Let I be an ideal in a ring R and denote by $q: R \to R/I$ the canonical quotient map

$$q(r) = r + I, \quad r \in R.$$

We will find a bijection of the set $\mathcal{J}$ of ideals in R that contain I and the set $\mathcal{K}$ of all ideals in R/I.

(a) Let J be an ideal in R which contains I, i.e., $J \supseteq I$. Prove that the quotient group J/I is an ideal in the quotient ring R/I. [1]

(b) For any subset K of R/I, we define its "pull back to R" to be the subset $\widetilde{K}$ of R consisting of all $r \in R$ which are sent to K by the map q:

$$\widetilde{K} \stackrel{\text{def}}{=} \{r \in R;\ q(R) \in K\} = \{r \in R;\ r + I \in K\} \quad \subseteq \quad R.$$

Prove that if $K \subseteq R/I$ is an ideal in R/I then its pull back $\widetilde{K}$ is an ideal in R and it contains I.

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6.2. Notice that observations (a) and (b) in the preceding problem define two procedures of passing between $\mathcal{J}$ and $\mathcal{K}$, i.e., two functions

(1) $\mathcal{A}: \mathcal{J} \to \mathcal{K}$ by $\mathcal{A}(J) = J/I$, and
(2) $\mathcal{B}: \mathcal{K} \to \mathcal{J}$ by $\mathcal{B}(K) = \widetilde{K}$.

Prove that these two functions are inverse to each other.

[1] Here $J/I \subseteq R/I$ consists of all cosets in R/I with representative in J:

$$J/I \stackrel{\text{def}}{=} \{j + I;\ j \in J\}.$$

6.3. Prove that

(a) Under the bijection in the preceding problem the ideal $J = I$ corresponds to the zero ideal $\{0_{R/I}\}$ in R/I.

(b) [Correspondence is compatible with inclusions.] Under the same bijection, if $J_1 \in \mathcal{J}$ corresponds to $K_1 \in \mathcal{K}$ and $J_2 \in \mathcal{J}$ corresponds to $K_2 \in \mathcal{K}$, then

$$J_1 \subseteq J_2 \quad \text{iff} \quad K_1 \subseteq K_2.$$

(c) Find all ideals J in $\mathbb{Z}$ that contain the ideal $36\mathbb{Z}$. Find all ideals K in $\mathbb{Z}/36\mathbb{Z}$.

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Maximal and prime ideals

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6.4. Let A be a commutative ring. Prove that if I is a maximal ideal in A and a is an element of A which is not in I, then there exist some $i \in I$ and some $x \in A$ such that

$$1 = i + ax.$$

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6.5. Let A be a commutative ring, Prove that

(a) A proper ideal $I \subseteq A$ is maximal iff A/I is a field.

(b) A proper ideal $I \subseteq A$ is prime iff A/I is an integral domain.

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6.6. Explain why the parts (a) and (b) of the preceding problem imply that:

($\star$) In a commutative ring any maximal ideal is a prime ideal.

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Fraction fields

6.7. Prove that $\mathbb{Q}[i] \stackrel{\text{def}}{=} \mathbb{Q} + \mathbb{Q}i$ is a field of fractions of the ring $\mathbb{Z}[i] \stackrel{\text{def}}{=} \mathbb{Z} + \mathbb{Z}i$.