Algebra 412, Homework 5 (also the 1st Sample Exam)

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Due at the time of the exam.

As always, answers need to be justified.

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An ideal I in a ring R is said to be *proper* if I is not the whole ring: $I \neq A$. An ideal I in a ring R is said to be *maximal* if it is proper and the only ideals J between I and R are the obvious ones: I and R. In other words if $I \subseteq J \subseteq R$ implies $J = I$ or $J = R$.

5.1. (a) Let A be a commutative ring, Prove that $\{0\}$ is a maximal ideal in A iff A is a field.

(b) Find all maximal ideals in $\mathbb{Z}$.

5.2. Let S be a polynomial over a field $\mathbb{k}$ with $\deg(S) > 0$. Consider the quotient ring $A = \mathbb{k}[x]/(S)$ where, we denote by (A) the corresponding principal ideal $(S) = S \cdot \mathbb{k}[x]$.

(1) (a) Let $P_1, P_2 \in \mathbb{k}[x]$ be two polynomials and denote by R_1, R_2 the remainders of dividing P_1, P_2 by S. Show that in $\mathbb{k}[x]/(S)$ the two cosets of P_1 and P_2 are the same if and only if the remainders are the same:
$$P_1 + (S) \ = \ P_2 + (S) \quad \Leftrightarrow \quad R_1 = R_2.$$
(b) Let $\mathcal{R} \subseteq \mathbb{k}[X]$ be the subset consisting of all polynomials P such that $\deg(P) < \deg(S)$. Show that the function $\phi : \mathcal{R} \to \mathbb{k}[x]/(S)$ is a bijection.

5.3. Over the field $F = \mathbb{Z}_2$ consider the polynomial $P = X^2 + X + 1$ and let $\mathcal{A} = F[X]/(P)$.

(1) Show that P has no roots in F. [A "root of P" means the same as "solution of $P = 0$" and "zero of P"]
(2) Show that the following function is a bijection:P
$$f : \ F^2 \to \mathcal{A}, \ f(a, b) = a + bX + (P).$$
(3) Show that $\mathcal{A}$ has no zero divisors.
(4) Show that $\mathcal{A}$ is a finite field with 4 elements.

5.4. Let R be a ring.

(a) Consider a family of ideals I_k in R indexed by elements k of some set K. Show that the intersection $\cap_{k\in K}\ I_k$ is again an ideal.

(b) For any subset S of the ring R consider the family of all ideals I in R that contain S. We now know that the intersection of this family

$$\widetilde{S} \stackrel{\text{def}}{=} \cap_{I\supseteq S}\ I$$

is an ideal. Show that this is the smallest ideal that contains the subset S. (We call $\widetilde{S}$ the *ideal generated by* S.)

(c) For any two ideals I, J in R, the subset

$$I+J \stackrel{\text{def}}{=} \{x+y;\ x\in I \quad \text{and} \quad y\in J\} \subseteq R\ ;$$

is an ideal in R.

5.5. Consider the subring $\mathbb{Z}$ of the ring $\mathbb{C}$ of complex numbers.

(a) Show that $A=\mathbb{Z}+\mathbb{Z}i\sqrt{5}$ is a subring of $\mathbb{C}$.

(b) Show that the function $f:\ A\to A$ defined by $f(a+i\sqrt{5}b) = a-i\sqrt{5}b$ is an automorphism of the ring A.

(c) Show that the function $\nu:\ A\to\mathbb{Z}$ defined by $\nu(\alpha) = \ \alpha{\cdot}f(\alpha)$ has the following properties:

- (i) $\nu(\alpha{\cdot}\beta)\ =\ \nu(\alpha){\cdot}\nu(\beta)$ for any $\alpha,\beta\in A$;
- $\nu(\alpha)\geq 0$ for any $\alpha\in A$;
- $\nu(1)=1$.