Algebra 411.2, Homework 4

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All answers should be justified.

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Due Wednesday February 28, in class.

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4.1. Prove that

- (a) If A is a commutative ring then for any $a \in A$ the subset
$$(a) \ = \ aA \ \stackrel{\text{def}}{=} \ \{ax;\ x \in A\}$$
is an ideal in A.
(One says that (a) is the *ideal generated by* a. Such ideals, i.e., ideals which are generated by one element, are called *principal ideals.*)
- (b) The ring $A = \mathbb{R}[X]/(X^2+1)$ is isomorphic to $\mathbb{C}$.
[*Hint*: One starts with a map $\phi:\ \mathbb{R}[X] \to \mathbb{C}$, $\phi(P) = P(i)$; and shows that (i) it is a homomorphism, (ii) $Im(\phi) = \mathbb{C}$ and $\mathrm{Ker}(\phi) = \ (X^2+1)$.]

4.2. Recall that $\mathbb{Q}[\sqrt{2}]$ is a subring of $\mathbb{R}$ which consists of all sums $a+b\sqrt{2}$ with $a,b \in \mathbb{Q}$. Prove that

- (a) $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{R}$.
- (b) Ring $A = \mathbb{Q}[X]/(X^2-2)$ is isomorphic to $\mathbb{Q}[\sqrt{2}]$.
[Hint. This is similar to the preceding problem. One starts with the evaluation map $\phi:\ \mathbb{Q}[X] \to \mathbb{R}$, $\phi(P) = P(\sqrt{2})$ and shows that ...]

4.3. Over the ring $F = \mathbb{Z}_3$ consider the polynomial $P = X^2 + 1$,

(1) Show that P has no roots in F.
(2) In $F[X]$ consider the principal ideal $I = (X^2 + 1)$. How many elements does the ring $A = F[X]/(X^2 + 1)$ posses? Show that the function

$$f: F^2 \to A, \; f(a, b) = (a + bX) + I$$

is a bijection.
(3) Show that A is an integral domain, i.e., if $a + bX + I \neq 0_A$ and $\alpha + \beta X + I \neq 0_A$, then $(a + bX + I) \cdot (\alpha + \beta X + I) \neq 0_A$.[1]
(4) Show that A is a finite field with 9 elements.

[*Hint*: This problem uses division of polynomials!]

4.4. Let I and J are ideals in a ring R such that $I \subseteq J$. Prove that

- (a) The function

$$q: R/I \to R/J, \; q(r + I) \stackrel{\text{def}}{=} r + J$$

is well defined,
- (b) q a homomorphism of rings,
- (c) q is surjective.
- (d) The kernel of q is $J/I \stackrel{\text{def}}{=} \{r + I \in R/I; \; r \in J\}$.

[1] One possible strategy is the following: First discuss the case when $b = 0$ (easy!). Once you check the claim in this case you will know that it is also true in the case $d = 0$. It remains to consider the case when $b \neq 0$ and $d \neq 0$, here you can factor out b and d and reduce to the case when $b = d = 1$. Finally, the case when $b = 1 = d$, will use part (1).

4.5. Let $R \supseteq J \supseteq I$ be a ring with two ideals as in the preceding problem.

- (a) [“*Cancellation isomorphism.*”] J/I is an ideal in R/I and the quotient

$$(R/I)\big/(J/I)$$

is isomorphic to R/J.

- (b) Prove that for any integers a, b

$$(\mathbb{Z}/ab\mathbb{Z})\big/(a\mathbb{Z}/ab\mathbb{Z}) \;\cong\; \mathbb{Z}/a\mathbb{Z}.$$

- (c) Prove that the ring $\mathbb{Z}_{28}$ has an ideal I with 7 elements, such that $\mathbb{Z}_{28}/I$ is isomorphic to $\mathbb{Z}_4$.

4.6. Prove that that the following equations have no solutions in $\mathbb{Z}$.

- (a) $X^4 + X^3 + X^2 + X + 1 = 0$.
- (b) $X^3 + 10X^2 + 6X + 1 = 0$.