Algebra 412

♡

Homework 3

♡

Due Wednseday Feb 14, in class.

♡

Read: section 18 in the book.

♡

3.1. (a) [Product of rings is related to its factors.] For rings S_1 and S_2 consider the maps

$$S_1 \xrightarrow{i_1} S_1{\times}S_2 \xleftarrow{i_2} S_2, \quad i_1(u) \overset{\text{def}}{=} (u, 0_{S_2}) \quad \text{and} \quad i_2(v) \overset{\text{def}}{=} (0_{S_1}, v)$$

and

$$S_1 \xleftarrow{p_1} S_1{\times}S_2 \xrightarrow{p_2} S_2, \quad p_1(u,v) \overset{\text{def}}{=} u \quad \text{and} \quad p_2(u,v) \overset{\text{def}}{=} v.$$

Show that

(1) p_1, p_2 are morphisms of rings.
(2) i_1, i_2 are "morphisms of rings without unity", i.e., they preserve addition and mutilpication but they do not preserve units.

(b) [Map into a product is the same as a pair of maps into factors.] Show that for any ring R the map

$$\iota : \ \mathrm{Hom}(R, S_1{\times}S_2) \ \to \mathrm{Hom}(R, S_1){\times}\mathrm{Hom}(R, S_2), \quad \iota(f) \overset{\text{def}}{=} \ (p_1{\circ}f, p_2{\circ}f);$$

is well defined and it is a bijection.

3.2. Let m, n be positive integers. Show that

(a) There is precisely one homorphism of rings

$$\phi : \ \mathbb{Z}_{mn} \to \mathbb{Z}_m{\times}\mathbb{Z}_n.$$

(b) If m, n are relatively prime then ϕ is an isomorphism.

(c) If m, n are not relatively prime then the rings $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m{\times}\mathbb{Z}_n$ are not isomorphic.

[Hint: A homomorphism of rings is in particular a homomorphism of additive groups. The part when you only consider the morphisms of abelian groups should be familiar from 411.]

3.3. Show that in the following pairs, the two rings are not isomorphic

(1) $\mathbb{R}$ and $\mathbb{C}$.
(2) $\mathbb{Z}$ and $\mathbb{R}$.
(3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_8$.

3.4. Show that for rings R and S

(1) If $I \subseteq R$ and $J \subseteq S$ are ideals show that $I \times J = \{(x, y);\ x \in I \quad \text{and} \quad y \in J\}$ is an ideal in $R \times S$.
(2) Any ideal K in $R \times S$ is equal to $I \times J$ for some ideals $I \subseteq R$ and $J \subseteq S$.