Algebra 412, Homework 2

Due Wednesday Feb 7, in class.

♡

Reading assignment. Read sections 16 and 17 in the book and the Chapter 1 in notes.

♡

2.1. A subset I of a ring R is said to be an *ideal* if

(1) it is a subgroup of $(R,+)$ (i.e., it is nonempty and if $x,y \in I$ then $x-y \in I$), and
(2) it is closed under multiplication with elements of R, i.e.,

$$x \in I \textit{ and } a \in R \quad \Rightarrow \quad ax, xa \in \; I.$$

(a) Prove that the set of cosets $R/I = \{r+I;\ r \in R\}$ is then a ring for the operations

$$(a+I) +_{R/I} (b+I) \ \stackrel{\text{def}}{=}\ (a+b)+I \quad \text{and} \quad (a+I)\cdot_{R/I}(b+I) \ \stackrel{\text{def}}{=}\ ab+I.$$

(b) When $R=\mathbb{Z}$ show that for any $n \in \mathbb{Z}$, $I = n\mathbb{Z}$ is an ideal in $\mathbb{Z}$.

(c) Let n be a positive integer and denote $\mathbb{Z}_n = \{0,1,...,n-1\}$. Show that the function

$$\iota : \mathbb{Z}_n \ \to \mathbb{Z}/n\mathbb{Z}, \quad io(x) = x + n\mathbb{Z},$$

is an ismorphism of rings.

(d) Write addition and multiplication tables for $\mathbb{Z}_2$. □

Quaternions. Recall that $\mathbb{H} \stackrel{\text{def}}{=} span_{\mathbb{R}}\{1, I, J, K\} \ = \ \{a1 + bI + cJ + dK;\ a,b,c,d \in \mathbb{R}\}$ is a subring of $M_2(\mathbb{C})$. □

2.2. (a) The *centralizer* of a subset A of a ring R is the set $Z_R(A)$ consisting of all $r \in R$ which commute with all elements of A :

$$(\forall a \in A) \ \ ar \ = \ ra.$$

Show that for any subset $A \subseteq R$, its centralizer $Z_R(A)$ is a subring.

(b) The center of a ring R is the subset $Z(R)$ consisting of all $a \in R$ which commute with all elements of R. Show that the center $Z(R)$ is a subring.

(c) For the ring $\mathbb{H}$ of quaternions determine the center $Z(\mathbb{H})$ and the centralizer $Z_I(\mathbb{H})$ of the element I of $\mathbb{H}$.

2.3. (*Vector spaces.*) (a) Write the definition of "V is a vector space over $\mathbb{R}$".

(b) Write the definition of a basis of a vector space V over $\mathbb{R}$.

(c) Show that the multiplication of elements of $\mathbb{H}$ by elements of $\mathbb{R}$

$$r\cdot(a1 + bI + cJ + dK) \stackrel{\text{def}}{=} (ra)1 + (rb)I + (rc)J + (rd)K \quad \text{for} \quad a, b, c, d, r \in \mathbb{R};$$

makes $\mathbb{H}$ into a vector space.

(d) Show that $1, I, J, K$ is a basis of the vector space $\mathbb{H}$ over $\mathbb{R}$.

[Hint: if you do not remember (a) and (b) from Linear Algebra, look for the definition in the book.]

2.4. Denote by $\mathbb{H} \ni x \mapsto \overline{x} \in \mathbb{H}$ the function that sends $x = a1 + bI + cJ + dK$ to $\overline{x} = a1 - bI - cJ - dK$. This operation is called *conjugation of quaternions.*

(a) Show that conjugation is an *antihomomorphism of rings*, i.e., that

$$\overline{x+y} \;=\; \overline{x} + \overline{y} \quad \text{and} \quad \overline{x\cdot y} \;=\; \overline{y}\cdot\overline{x} \quad \text{for} \quad x, y \in \mathbb{H}.$$

(b) Show that conjugation is an involution, i.e., that $\overline{\overline{x}} \;=\; x$.

(c) Define the *norm* of $x = a1 + bI + cJ + dK \in \mathbb{H}$ to be

$$|x| \stackrel{\text{def}}{=} \sqrt{a^2 + b^2 + c^2 + d^2} \;\in \mathbb{R}_{\geq 0}.$$

Show that

(1) $x\cdot\overline{x} = \overline{x}\cdot x = |x|^2\cdot 1$.
(2) $|x\cdot y| \;=\; |x|\cdot |y|, \quad x, y \in \mathbb{H}$.
(3) If $x \in \mathbb{H}$ is not zero, then the quaternion $\frac{1}{|x|^2}\cdot\overline{x}$ is the inverse of x. (So, $\mathbb{H}^* = \mathbb{H} - \{0\}$.)

[Hint: It is preferable to prove (3) using (1) then by computing with the formula $x = a1 + bI + cJ + dK$, since you will otherwise need to duplicate the work you did for (1).]