Algebra 412, Homework 1

The main purpose of these homeworks is to increase your understanding of the subject. The solutions should be written in sufficient detail so that (i) you clarify to yourself what is happening, what is true and why, and (ii) a reasonable reader[1] can follow the reasoning.

In particular, if you are certain that writing down the solution of a certain problem is a waste of time for you, you can just state so clearly. You will receive the full credit but I reserve the right to talk with you about this problem.

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Due Wednesday January 31, in class.

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1.1. If $(R, +_R, \cdot_R)$ and $(S, +_S, \cdot_S)$ are rings prove that $R{\times}S$ with operations

$$(r,s)+(r',s') \stackrel{\text{def}}{=} (r +_R r', s +_S s') \quad \text{and} \quad (r,s)\cdot(r',s') \stackrel{\text{def}}{=} (r\cdot_R r', s\cdot_S s')$$

is also a ring.

1.2. If $(R, +, \cdot)$ is a ring prove that $M_n(R)$, the set of $n{\times}n$ matrices with values in R, is also a ring where operations on matrices are defined as usual:

$$(A+B)_{ij} \;=\; A_{ij}+B_{ij} \quad \text{and} \quad (A\cdot B)_{ij} \;=\; \sum_{k=1}^{n} A_{ik}\cdot B_{kj}.$$

1.3. Show that

$$\mathbb{Z}[i,\sqrt{2}] \;=\; \{a+bi+c\sqrt{2}+di\sqrt{2};\; a,b,c,d\in\mathbb{Z}\}$$

is a subring of the ring $\mathbb{C}$ of complex numbers.

[1] For instance me, or you a year from now.

1.4. Show that the following 2x2 matrices with complex entries

$$1 \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad I \stackrel{\text{def}}{=} \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \quad J \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad K \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right);$$

satisfy

(1) $1^2 = 1 \quad \text{and} \quad I^2 = J^2 = K^2 = -1,$
(2) $1{\cdot}X = X = X{\cdot}1 \;\; \text{for} \quad X = 1, I, J, K,$
(3) $I{\cdot}J = K, \quad J{\cdot}K = I, \quad K{\cdot}I = J$ and $K{\cdot}I = -K, \quad K{\cdot}J = -I, \quad I{\cdot}K = -J.$
(4) Show that the subset of $M_2(\mathbb{C})$ defined by

$$\mathbb{H} \stackrel{\text{def}}{=} \mathbb{R}1 + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K \;=\; \{a1 + bI + cJ + dK;\; a, b, c, d \in \mathbb{R}\}$$

is a subring.