Math 370 , Homework 7

due Thursday April 3rd.

This assignment contains elements of Mathematical Writing and Latex.

The goal is to write a text on the definition of the mathematical notion of *groups*. You can start by copying the text below into your latex template and then you will need to reorganize it into a good mathematical writing. This includes adding whatever explanations are missing and writing in a logical order. The key part is that you greatly improve the text following the criteria below.

A. Target audience. Your intended readership are high school seniors. You should assume that they know what are sets and the simplest set notations (such as $a \in B$ $C = \{x \in \mathcal{C};\ x^2 + 7 = 11\}$). Start with the title

B. Writing. You should be mathematically correct. You should write in complete sentences in understandable English.

C. Presentation of Mathematics.

1. Any mathematical symbol without a fixed meaning has to be introduced.[1] This refers to
   - (i) specifying what kind of object it is. [For instance "a is an element of the set B" introduces a if B has been already introduced, while "f is a function from B to C" introduces f ...]
   - (ii) with which quantifier it appears (the *existential* quantifier means that we are talking of a specific object, the *universal* quantifier means that we are talking of all objects of a certain kind).

   For instance the following is not a good writing: "If $\circ$ is an operation on a set X then an element e of X is said to be a *neutral element* if multiplication with e does not affect elements of X, i.e., $e \circ a = a$."

   For the symbol a that appears in the above equation one has to specify: (i) what is a and (ii) do you mean a particular a or all of them. For (i), here you can say, "a is an element of X". However, for (i) and (ii) you say "for any a in X we have $e \circ a = a$.".

   By the way, this definition is not quite right mathematically.

2. The most important parts may be stated both in a formula and in words.[2]
   [Notice that in the above example we had both "$e \circ a = a$" and also "multiplication with e does not affect elements of X".]

3. Illustrate definitions with examples. Here it is required that after *each* definition there is one or more examples of objects that satisfies the definition (a "positive example") and also if the definition is abstract (i.e., if it is difficult to understand what is its content) an example of an object which does not quite satisfy the definition (a "negative example").
   [For instance, after the definition of groups there should be an example of a monoid which is a group and of a monoid which is not. The examples should come from the last sentence.]

4. Create "newtheorem environments" called Definition, Example, Lemma, Use a separate counter for each of these environments. Each paragraph should be in one of these environments.

[1] The exception are the symbols with a fixed meaning such as "$\in$" or "$\mathbb{R}$".

[2] Both words and formulas have advantages. While we are well accustomed to words the symbols used in formulas are more concise. The process of stating the meaning of a Formula in your own words is how a formula becomes your property – it literally becomes a part of you since this process has carved a pathway into your brain.

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The notion of groups in mathematics

A binary operation on a set X is a function $m : X{\times}X \to X$, i.e., it is a rule m which combines any pair a, b of elements of X into some element of X which we denote $m(a, b)$. One usually denotes $m(a, b)$ as $a{\circ}b$ for some symbol $\circ$.
[To make this clear you will need to insert a paragraph that defines product of sets $A{\times}B$.]

$\circ$ is associative if $(a{\circ}b){\circ}c = a{\circ}(b{\circ}c)$.

If $\circ$ is an operation on a set X then an element e of X is said to be a *neutral element* if multiplication with e does not affect elements of X, i.e., $e{\circ}a = a$.

Lemma. If operation $\circ$ has a neutral element then this neutral element is unique.

Proof. If e, f are two neutral elements then

$$e \stackrel{(1)}{=} e{\circ}f \stackrel{(2)}{=} f.$$

Here, equality (1) holds because f is a neutral element so it does not affect e. Equality (2) holds because e is a neutral element so it does not affect f. Therefore, $e = f$.
[Here, $\stackrel{x}{7}$ is obtained by \stackrel{x}{7}.]

Now that we know that the neutral element is unique we can use the notation e for the neutral element.

A *monoid* is a set X with a binary operation $\circ$ which is both associative and has a neutral element.

For an element a in a monoid $(X, \circ)$ we say that $b \in X$ is an *inverse* of a if $a{\circ}b = e = b{\circ}a$.

Lemma. If an element a of a monoid $(X, \circ)$ has an inverse element then this inverse element is unique.

Proof. If b and c are both inverses of a then .... $b = c$.
[Replace the dots by a detailed proof modeled on the above proof of the preceding lemma. Use $(b{\circ}a){\circ}c = b{\circ}(a{\circ}c)$.]

A *group* is a monoid $(X, \circ)$ such that every element of X has an inverse for the operation $\circ$.

Some examples of X and $\circ$ are: $(\mathbb{R}.+)$ and $(\mathbb{R}, \cdot)$ as well as $n{\times}n$ matrices with addition and multiplication $(M_n(\mathbb{R}), +)$ and $(M_n(\mathbb{R}), \cdot)$.

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The title format should be (here "?" is a place to put section number or homework number):
\title{
\Large{ Math 370.\Huge{?}. Homework ?? }
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The notion of groups in mathematics
}
\author{Your Name}
\maketitle