NOTES ON DRINFELD'S THEORY OF CLASSIFYING PAIRS

Contents

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The following is supposed to evolve into notes on zastava spaces.

A. Classifying Pairs

1. The setting: Loop Grassmannians and Hilbert schemes of points

Let $\Bbbk$ be the ground ring. By C we will denote a smooth curve over $\Bbbk$ and by X an arbitrary smooth scheme over $\Bbbk$.

1.0.1. *Curve C.* For a smooth point c of a curve C denote $\mathcal{O} = \mathcal{O}(\widehat{c})$ and $\mathcal{K} = \mathcal{O}(\widetilde{c})$ for the formal neighborhood $\widehat{c}$ of c and $\widetilde{c} = \widehat{c} - c$. Then the fiber of the loop Grassmannian at c is

$$\mathcal{G}(G)_c \ \cong\ G_{\mathcal{K}}/G_{\mathcal{O}}.$$

1.0.2. *Group G.* We assume that G is semisimple, split and simply connected algebraic group over $\Bbbk$. We will fix two opposite Borel subgroups $B^{\pm}$ with unipotent radicals $N^{\pm}$ and a common Cartan $T = B^+ \cap B^-$.

For any Borel $B \subseteq G$ the group $H \overset{\text{def}}{=} B/[B,B]$ is canonically independent of the choice of B, For a Cartan T, a choice of a Borel $B \supseteq T$ gives an identification $\iota_B : T \xrightarrow{\cong} H$ as $T \subseteq B \twoheadrightarrow H$. The Lie algebras are $\mathfrak{g}, \mathfrak{b}^{\pm}, \mathfrak{n}^{\pm}, \mathfrak{t}, \mathfrak{h}$ as usual.

Let $\alpha_i \in X^*(H),\ ii$, be the simple roots.

1.1. **Hilbert schemes of colored points $\mathcal{H}_{C\times I}$.** The Hilbert scheme of points of X is graded by the length of the subscheme $\mathcal{H}_X = \sqcup_{n\in\mathbb{N}}\ \mathcal{H}_X^n$. For a smooth curve C, the "Hilbert powers" $\mathcal{H}_C^n = C^{[n]}$ coincide with the symmetric powers $C^{(n)}$.

For a set I the Hilbert scheme of the multiple $X\times I$ decomposes as $\mathcal{H}_{X\times I} \cong\ (\mathcal{H}_X)^I = (\sqcup_{n\in\mathbb{N}}\ \mathcal{H}_X^n) =\ \sqcup_{\alpha\in\mathbb{N}[I]}\ \mathcal{H}_X^\alpha$, where for $\alpha = \sum_{i\in I}\ \alpha_i i \in \mathbb{N}[I]$ we denote by $\mathcal{H}_X^\alpha = X^{[\alpha]} =\ X^\alpha$ the product $\prod_{i\in I}\ X^{[\alpha_i]}$. So, $D \in \mathcal{H}_{X\times I}$ is a system $(D_i)_{i\in I}$ of $D_i \in \mathcal{H}_X$, we also denote it by $D = \sum_{i\in I}\ D_i i$.

In particular we denote $\mathbb{A}^{[\alpha]} \overset{\text{def}}{=}\ (\mathbb{A}^1)^{[\alpha]}$.

1.1.1. *The loop Grassmannians $\mathcal{G}(G) \to \mathcal{R}_C$.* Let $\mathcal{R}_C$ be the Ran space of the curve C, i.e., the moduli of finite subsets E of C. The loop Grassmannians $\mathcal{G}_{\mathcal{R}_C}(G) \to \mathcal{R}_C$ is the moduli of triples $(\mathcal{T}, \tau, E)$ where, $\mathcal{T}$ is a right G-torsor over C, $E \in \mathcal{R}_C$ is a finite subset and τ is a section of $\mathcal{T}$ off E, i.e., defined over $C - E$.

Lemma. For a curve C the fiber of the loop Grassmannian at $E \in \mathcal{R}_C$ is given by maps of pairs

$$\mathcal{G}(G)_E \ =\ Map[(C, C-E), (\mathbb{B}(G), \mathrm{pt})].$$

Proof. For $X' \subseteq X$, and a subgroup $G' \to G$, a map $F \in Map[(X,X'), (\mathbb{B}(G), \mathbb{B}(G'))]$ is a pair of a G-torsor $\mathcal{T}$ over X (i.e., a map $f : X \to \mathbb{B}(G)$) and a G'-torsor $\mathcal{T}'$ over X' (i.e., a map $f' : X' \to \mathbb{B}(G')$), together with a compatibility which is a G'-map $\mathcal{T}' \to \mathcal{T}|_{X'}$, i.e., a reduction on X' of $\mathcal{T}$ to the G'-torsor $\mathcal{T}'$.

So, $Map[(X,X'),(\mathbb{B}(G),\mathbb{B}(G'))]$ is the moduli of a G-torsor on X with a G'-reduction on X'. The lemma is the case $G' = 1$. $\square$

1.1.2. *Global loop Grassmannian* $\widetilde{\mathcal{G}}(G)$. Here we will restate lemma 1.1.1 so that it describes the whole space rather than just a single fiber.

One defines the global loop Grasmannian $\widetilde{\mathcal{G}}(G)$ by passing from triples $(\mathcal{T},E,\tau) \in \mathcal{G}(G)$ to pairs $(\mathcal{T},\tau)$, i.e., by omitting a choice E of an estimate on the singularity of the rational section $\tau^{(1)}$, i.e.,

$$\widetilde{\mathcal{G}}(G) \stackrel{\text{def}}{=} \lim_{\to\ E\in\mathcal{R}_C} \mathcal{G}(G)_E.$$

Then the lemma 1.1.1 says that

$$\widetilde{\mathcal{G}}(G) \ = \ Map[(C,\eta_C),(\mathbb{B}(G),\mathrm{pt})].$$

1.1.3. *Some base changes of* $\mathcal{G}(G) \to \mathcal{R}_C$. For any $Y \to \mathcal{R}_C$ we denote by $\mathcal{G}_Y(G) \to Y$ the corresponding base change of $\mathcal{G}_{\mathcal{R}_C}(G)$.

Example. The support map $\mathrm{supp} : \mathcal{H}_C \to \mathcal{R}_C$ gives the pull-back $\mathcal{G}_{\mathcal{H}_C}(G) \to \mathcal{H}_C$ to $\mathcal{H}_C$ with the fiber at $D \in \mathcal{H}_C$ the moduli of all $(\mathcal{T},\tau)$ such that the rational section τ of $\mathcal{T}$ is defined off $\mathrm{supp}(D)$.

Example. For a smooth curve C, $\mathcal{H}_C$ is a monoid for the schematic union operation $+$ which is given by tensoring the ideals $\mathcal{I}_{D'+D''} \stackrel{\text{def}}{=} \mathcal{I}_{D'}\otimes_{\mathcal{O}_C}\mathcal{I}_{D''}$ of $D',D'' \in \mathcal{H}_C$. So, for a set I the schematic union $\mathcal{H}_{C\times I} \xrightarrow{+} \mathcal{H}_C$ gives a base change $\mathcal{G}_{\mathcal{H}_{C\times I}}(G) \to \mathcal{H}_{C\times I}$. The fiber at $D = (D_i)_{i\in I} \in \mathcal{H}_{C\times I}$ is the moduli of all $(\mathcal{T},\tau)$ such that the rational section τ of $\mathcal{T}$ is defined off the support $\mathrm{supp}_I(D) \stackrel{\text{def}}{=} +_{i\in I}\ D_i$.

1.2. **Maps into a quotient stack.**

1.2.1. *Adjunction.* The following adjunction relates moduli of torsors and the corresponding classifying spaces. For a left G-torsor $\mathcal{P}$ over X, the *$\mathcal{P}$-twist of a G-space* Y over X is the space over X

$$\mathsf{Y}^{\mathcal{P}} \stackrel{\text{def}}{=} {}^{\mathcal{P}^{-1}}\mathsf{Y} \ \stackrel{\text{def}}{=} \ \mathcal{P}^{-1}\times_{X,G}\mathsf{Y} \ = \ G\backslash(\mathcal{P}\times_X\mathsf{Y}).$$

[1] Here, “global” refers to dependence on C only, since we have eliminated the local part of the data $E \in \mathcal{R}_C$.

Remark. One can write all formulas here without $\mathcal{P}^{-1}$ using the diagonal quotients $\mathsf{Y}^{\mathcal{P}} = G\backslash(\mathcal{P}\times_X\mathsf{Y})$. However, $\mathcal{P}^{-1}$ is useful for the "tensor product" notation $\mathsf{Y}^{\mathcal{P}} = \mathcal{P}^{-1}\times_G\mathsf{Y}$ which may be more intuitive. The two are related by $G{\cdot}(p,y)$ corresponding to the G-orbit $\overline{(p^{-1},y)} \in \mathcal{P}^{-1}\times_G\mathsf{Y}$ for $p\in\mathcal{P}$ and $y\in\mathsf{Y}$.

Lemma. For a left G-torsor $\mathcal{P}$ over X and any G-space Y over X one has

$$Map_{X,G}(\mathcal{P},\mathsf{Y}) \cong \Gamma(X,\mathsf{Y}^{\mathcal{P}}).$$

Proof. Denote $\mathcal{P}\xrightarrow{\pi}X$. The correspondence of G-maps $\alpha:\mathcal{P}\to\mathsf{Y}$ over X and sections $\phi\in\Gamma(X,\mathsf{Y}^{\mathcal{P}}) = \Gamma(X, G\backslash[\mathcal{P}\times\mathsf{Y}])$ is written in terms of $x\in X$ and $p\in\mathcal{P}$ lying above x, by

$$\phi(x) \overset{\text{def}}{=} G{\cdot}(p,\alpha(p)) \in G\backslash[\mathcal{P}\times_G\mathsf{Y}].$$

Conversely, $\alpha(p)$ is the unique $y\in\mathsf{Y}$ such that (p,y) lies in the G-orbit $\phi(x)\in G\backslash(\mathcal{P}\times\mathsf{Y})$. $\square$

1.2.2. *Maps into a quotient stack.*

Corollary. For a space X and a G-space Y, the following are the same

- (0) A map $f:X\to G\backslash Y$.
- (1) A G-torsor $\mathcal{P}$ over X and a G-map $\mathcal{P}\xrightarrow{\alpha}Y$.
- (2) A G-torsor $\mathcal{P}$ over X and a section ϕ of $Y^{\mathcal{P}} \overset{\text{def}}{=} {}^{\mathcal{P}^{-1}}Y \overset{\text{def}}{=} \mathcal{P}^{-1}\times_G Y$.

Proof. (0)$\Leftrightarrow$(1) is standard. Then (1) and (2) are related by the above adjunction. Explicitly, the data $X\xleftarrow{\pi}\mathcal{P}\xrightarrow{\alpha}Y$ from (1) can be viewed as a G-map over X

$$\mathcal{P}\xrightarrow{(\alpha,\pi)}Y\times X.$$

Then by the above adjunction it is the same as a section $\phi\in\Gamma(X,(Y\times X)^{\mathcal{P}}) = \Gamma(X,Y^{\mathcal{P}})$ by (for any $x\in X$ and $p\in\mathcal{P}$ lying above x)

$$\phi(x) = G{\cdot}(p,\alpha(p)) \in G\backslash[\mathcal{P}\times Y].$$

So, we get data from (2). Conversely, $\alpha(p)$ is the unique $y\in Y$ such that (p,y) lies in the G-orbit $\phi(x)\in G\backslash(\mathcal{P}\times Y)$. $\square$

1.3. **The conventions.** We will make a choice of positive roots Δ^+ in 1.3.2 and of the embedding $X_*(T)\hookrightarrow\mathcal{G}(G)$, $\lambda\mapsto L_\lambda$.

1.3.1. *The standard notation.* For a Cartan T in a Borel $B = TN$ we will choose the positive part Δ_T^+ of the root system $\Delta_T = \Delta_T(\mathfrak{g})$. This gives the positive coroots $\check{\Delta}_T^+ \stackrel{\text{def}}{=} (\Delta_T^+)^\vee$, the simple (co)roots $\alpha_i^T, \check{\alpha}_i^T,\ i \in I$, the positive cones $Q_T^+ \subseteq Q_T = \mathbb{Z}[\Delta_T]$ and $\check{Q}_T^+ \subseteq \check{Q}_T = \mathbb{Z}[\check{\Delta}_T]$, as well as the notions of dominant weights $X^*(T)^+ = \oplus_{i\in I}\ \mathbb{N}\omega_T^i$ and coweights $X_*(T)^+ = \oplus_{i\in I}\ \mathbb{N}\check{\omega}_T^i$. We write $\alpha \leq_B \beta$ in $X_*(T)$ if $\beta - \alpha \in \check{Q}^+$ and similarly in $X^*(T)$.

The abstract Cartan $H \stackrel{\text{def}}{=} B/[B,B]$ is canonically independent of the choice of a Borel B. Now, via the isomorphism $\iota_B : T \xrightarrow{\cong} H$ (by $T \subseteq B \twoheadrightarrow H$) a choice of Δ_T^+ gives compatible notions for the abstract Cartan H: $\Delta^+ \ni \alpha_i, \check{\Delta}^+ \ni \check{\alpha}_i, Q^+ \subseteq Q, \check{Q}^+ \subseteq \check{Q}, X^*(H) \ni \omega^i, X_*(H)^+ \ni \check{\omega}^i$ and $\leq$.

1.3.2. *The choice of positive roots.* Our choice $\Delta^+ \stackrel{\text{def}}{=} \Delta_T(\mathfrak{n})$ is traditional in representation theory (then the highest weight vectors are fixed by N).(2) The reason is that the semigroup closure $\overline{H}$ defined as the closure of $H = B/N$ in $(G/N)^{\text{aff}}$ will be parameterized (for simply connected G) by positive coroots (see lemma 1.4.a).

1.3.3. *The choice of the embedding* $X_*(T) \hookrightarrow \mathcal{G}(T)$, $\lambda \mapsto L_\lambda$. Such choice gives a parameterization of $N_\mathcal{K}$-orbits by $X_*(T) \ni \lambda \mapsto S_\lambda = N_\mathcal{K} \cdot L_\lambda$. We will use the choice $\lambda \mapsto L_\lambda \stackrel{\text{def}}{=} z^{-\lambda} \cdot G_\mathcal{O} \in \mathcal{G}(G)$. As we will see in 2.2.3, in this case the $N_\mathcal{K}$-orbits have closure relation $\overline{S_\lambda} \supseteq S_\mu$ iff $\lambda \geq_B \mu$.(3)

1.4. **The semigroup closure $\overline{H}$.** We define $\overline{H}$ as the *closure* of $H \cong B/N \subseteq G/N$ in the affinization $(G/N)^{\text{aff}}$.

Lemma. (a) $\overline{H}$ is a semigroup closure of H.

(b) $\mathcal{O}(\overline{H}) \subseteq \mathcal{O}(H)$ is the subspace spanned by the dominant characters of H. In particular, when G is simply connected, the standard description of H extends to

$$\overline{G_m}^I \xrightarrow[\cong]{\prod_{i\in I}\ \check{\alpha}_i} \overline{H}_{sc} \xrightarrow[\cong]{\prod_{i\in I}\ \omega_i} \overline{G_m}^I.$$

(c??) For any H orbit $\mathcal{O}$ in G/N, the closure in $(G/N)^{\text{aff}}$ is described by $\overline{\mathcal{O}} \cong\ \mathcal{O} \underset{H}{\times} \overline{H}$.

Proof. We know how to choose a consistent system frames e_λ in V_λ^N for all λ in the set $X^*(H)^+ = \oplus\ \mathbb{Z}\omega_i$ of

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(2) The opposite choice $\Delta_T^+ \stackrel{\text{def}}{=} \Delta_T(\mathfrak{g}/\mathfrak{b})$ agrees with algebraic geometry in the sense that say, the regular dominant weights correspond to very ample line bundles on G/B.

(3) The closure relation is the same for what may be the simplest choices where $L_\lambda \stackrel{\text{def}}{=} z^\lambda \cdot G_\mathcal{O}$ if Δ_T^+ is chosen in the geometric way as $\Delta_T(\mathfrak{g}/\mathfrak{b})$.

(b') Recall the Tannakian description of $G/N\subseteq(G/N)^{\mathrm{aff}}$. Let $X^+ = X^*(H)^+ \stackrel{\text{def}}{=} X^*(H)\cap \oplus\ \mathbb{Z}\omega_i$ be all dominant characters of H. We can choose a consistent sytem of realizations of standard representations V_λ of G, of frames e_λ in V_λ^N for $\lambda \in X^+$ and of surjections $V_\lambda\otimes V_\mu \xrightarrow{\zeta_{\lambda,\mu}} V_{\lambda+\mu}$ for $\lambda,\mu \in X^+$ (so, ζ is associative and sends $e_\lambda\otimes e_\mu$ to $e_{\lambda+\mu}$).[4]

Then the map $\iota : G/N \to V \stackrel{\text{def}}{=} \prod_{\lambda\in X^+} V_\lambda$ by $G/N \ni gN\mapsto (ge_\lambda)_{\lambda\in X^+}$ identifies G/N with all $v = (v_\lambda)_{\lambda\in X^+} \in V$ that satisfy the corresponding Tannakian equations $\zeta_{\lambda,\mu}(v_\lambda\otimes v_\mu) = v_{\lambda+\mu}$ and $v_\lambda \neq 0$. If we choose a finite system $\mathcal{X}$ of generators of the cone X^+ we can use $\iota_{\mathcal{X}} : G/N\hookrightarrow V_{\mathcal{X}} = \oplus_{\lambda\in\mathcal{X}}\ V_\lambda$. and the corresponding embedding equations are called Pluecker equations.

Now, $(G/N)^{\mathrm{aff}}$ is the closure of G/N in V, i.e., precisely all $v = (v_\lambda)_{\kappa\in\mathcal{X}} \in V$ that satisfy the Tannakian equations $\zeta_{\lambda,\mu}(v_\lambda\otimes v_\mu) = v_{\lambda+\mu}$.

(b") The image $\iota(H)\subseteq\iota(G/N)$ consists of all Tannakian systems v with $0 \neq v_\lambda \in V_\lambda^N$ for $\lambda \in X^+$. By definition $\overline{H}$ is the closure of $\iota(H)$ in $\iota((G/N)^{\mathrm{aff}})$, i.e., all Tannakian systems v with $v_\lambda \in V_\lambda^N$. The functions on $\prod_{i\in I}\ V_\lambda^N$ are the polynomials in variables $\widetilde{\lambda},\ \lambda \in X^+$. The Tannakian equations for $\overline{H}$ are then $\widetilde{\lambda+\mu} = \widetilde{\lambda}\cdot\widetilde{\mu}$. Therefore, the functions on $\overline{H}$ are indeed the span of X^+ in $X^*(H)$.

In particular, when G is simply connected then $\mathcal{O}(\overline{H}) =\ \oplus_{\lambda\ \oplus_{i\in I}\ \mathbb{N}\omega_i}\ \Bbbk\lambda$ are the polynomials in $\omega_i,\ i \in I$.

(c) It follows from (b) by left translations.?? $\square$

Remark. $\overline{H}$ usually does not act on $\mathfrak{n}$ nor $\mathfrak{g}/\mathfrak{b}$ since the weights need not be dominant.

[4] $\square^5\blacksquare$ It suffices to make the choice for simply connected G?

2. **Finitely supported maps**

The loop Grassmannian $\mathcal{G}(G)$ of G is the moduli of finitely supported maps into $\mathbb{B}(G)$. We find that the interesting local moduli of G-torsors (factorization subspaces or local subspaces of loop Grassmannians $\mathcal{G}(G)$) have classifying spaces in the sense of finitely supported maps into pointed stacks $\mathcal{Y}$ that lie above $\mathbb{B}(G)$ (i.e. $\mathcal{Y}$ is of the form $G\backslash Y$).

We consider the moduli $\mathcal{M}_{\mathcal{Y}}(C) = Map_{gt}[C, (\mathcal{Y}, \mathrm{pt})]$ of "*generically trivialized*" maps from a curve C into a given pointed stack $(\mathcal{Y}.\mathrm{pt})$. When $(\mathcal{Y}, \mathrm{pt})$ has a presentation (Y, a) with $\mathcal{Y} = G\backslash Y$ we define the loop Grassmannian of G with the condition Y to be $\mathcal{G}(G, Y) \stackrel{\text{def}}{=} \mathcal{M}_{G\backslash Y}$ (2.2). This notation contains redundancy but it has the relation to the usual loop Grassmannian $\mathcal{G}(G)$ which is just $\mathcal{G}(G, \mathrm{pt})$ and when Y is separated then $\mathcal{G}(G, Y)$ is a subfunctor of $\mathcal{G}(G)$.[(6)]

In 2.1 we notice that in general the moduli $\mathcal{M}_{\mathcal{Y}}$ has a structure of a (colored) factorization space over curves (under Drinfeld's conditions on the pointed stack $(\mathcal{Y}, \mathrm{pt})$).

In the reminder of this section we describe the classifying spaces of standard subspaces of the loop Grassmannian $\mathcal{G}(G)$. These are the closures of $S_\lambda = N_{\mathcal{K}}{\cdot}L_\lambda$ (3.3) etc. The origin of the present point of view is Drinfeld's description of the zastava space in terms of classifying spaces (theorem **??**).

2.0.1. *Moduli $\mathcal{G}(G, Y)$ and orbits in $\mathcal{G}(G)$.* We will reproduce in the form $\mathcal{G}(G, Y)$ certain moduli of G-torsors (with extra structures) that are local spaces. For this, the space Y (or the classifying space $\mathcal{Y} = G\backslash Y$) will be produced from G. Say, Y could be a semigroup closure of G or the affinization $(G/A)^{\mathrm{aff}}$ of a homogeneous space.

Typically, the connected components of our moduli $\mathcal{G}(G, Y)$ will be certain orbits in $\mathcal{G}(G)$, their closures and intersections of such. In particular, the closure relations on orbits will be more transparent from the description via classifying spaces.

Example. For a subgroup $A{\subseteq}G$ the orbits in $\mathcal{G}(G)$ of the subgroup $\ddot{A} = N_G(A)_{\mathcal{O}}{\cdot}A_{\mathcal{K}}$ of $G_{\mathcal{K}}$ are related to homogeneous space $Y^o = G/A$, the closures of orbits are then related to some partial compactification Y of G/A. So, in some sense "*probing a space $G\backslash Y^o$ with curves leads to its partial completion Y*".

For instance for a parabolic $P = U \ltimes L$ with the unipotent radical U and a Levi factor L we have $\ddot{U} = U_{\mathcal{K}} \ltimes L_{\mathcal{O}}$. The extreme cases of this are the disc group $G_{\mathcal{O}}$ (here $P = G$ and Y is the Vinberg semigroup of G) and $T_{\mathcal{O}}N_{\mathcal{K}}$ for a Borel $B = NT$ (the "semi-infinite orbits", here $P = B$ and $Y = (G/N)^{\mathrm{aff}}$).

[6] In this way we restate the usual theory of moduli of torsors in the more flexible and general terms of maps into (classifying spaces). In terms of physics this is the slogan that "All quantum field theories are Σ-models."

Example. $\square$[7]$\blacksquare$ ANOTHER example? The connected component of $A_{\mathcal{K}}$ for a subgroup $A{\subseteq}G$.

2.1. **Moduli of finitely supported maps.** We are interested in various moduli of G-torsors over a curve C that are local spaces over C. As observed by Beilinson and Drinfeld, the relevant spaces Y are usually of the form $\mathcal{M}_{\mathcal{Y}}(C)$, the moduli of finitely supported, i.e., *generically trivialized* maps into some pointed stack $(\mathcal{Y},\mathrm{pt})$ built from G. (We usually omit pt from notation.)

2.1.1. *Versions of the moduli.* A pointed stack $(\mathcal{Y},\mathrm{pt})$ will give a functor (an algebro-geometric "sigma model") that associates to each source space X the moduli of *generically trivialized* maps from X to $\mathcal{Y}$. We first define the *global version*, for the generic point η_X of X this is the moduli

$$\mathcal{M}_{gl}^{X}(\mathcal{Y}) \overset{\text{def}}{=} Map[(X,\eta_X),(\mathcal{Y},\mathrm{pt})].$$

We will now assume that X is a curve and we denote it C. Then "generically trivialized" is the same as *finitely supported*. We will consider two versions depending on how one organizes these finite supports into an algebro-geometric objects. The *Ran space* $\mathcal{R}_X$ of X is the moduli of finite subsets of X.

• (1) *Factorization space version* $\mathcal{M}(\mathcal{Y})$. To a curve C it associates the space $\mathcal{M}^C(\mathcal{Y})\to\mathcal{R}_C$ over the Ran space of C. The fiber at $E\in\mathcal{R}_C$ is

$$\mathcal{M}^C(\mathcal{Y})_E = Map[(C-E),(\mathcal{Y},\mathrm{pt}).$$

• (2) *The "filtered" or "local space" version* $f\mathcal{M}(\mathcal{Y})$ *of* $\mathcal{M}(\mathcal{Y})$. It is only defined under the *Drinfeld condition* that the pt $\to Y$ is an open inclusion. Then we can define its boundary $\partial(\mathrm{pt}) \overset{\text{def}}{=} \overline{\mathrm{pt}}-\mathrm{pt}$ which is closed in $\overline{\mathrm{pt}}$. So, one has the *singularity* map

$$\mathcal{M}_{gl}^{C}(\mathcal{Y}) \xrightarrow{\pi} \mathcal{H}_C, \quad \pi(f) \overset{\text{def}}{=} f^{-1}\partial(\mathrm{pt}).$$

(Since f is generically in pt, $\pi(f)$ is a proper closed subscheme of C, hence it is a finite subscheme.)

Now, $f\mathcal{M}^C(\mathcal{Y})$ is the space $\mathcal{M}_{gl}^{C}(\mathcal{Y})$ considered with the structure map π, i.e., considered as a family of spaces indexed by finite subschemes $D\in\mathcal{H}_C$:

$$f\mathcal{M}^C(\mathcal{Y})_D \overset{\text{def}}{=} \pi^{-1}D = \{f:(C,\eta_C)\to(\mathcal{Y},\mathrm{pt});\ f^{-1}[\partial(\mathrm{pt})]=D\}.$$

2.1.2. *Properties of the moduli* $\mathcal{M}^C(\mathcal{Y})$.

[7] $\boxed{!}$

Lemma. (a) For a pointed space $(\mathcal{Y}, \mathrm{pt})$ the space $\mathcal{M}^C(\mathcal{Y}) \to \mathcal{R}_{\mathcal{C}}$ is indeed a factorization space over C.

(b) If the point pt of $\mathcal{Y}$ is open in Y then $f\mathcal{M}^C(\mathcal{Y})$ is a local space over C whose associated factorization space is $\mathcal{M}^C(\mathcal{Y})$.

Moreover, if the boundary $\partial(\mathrm{pt}) = \mathcal{Y} - \mathrm{pt}$ is a divisor with irreducible components $\mathcal{Y}_i,\ i \in I$, then $f\mathcal{M}^C(\mathcal{Y})$ can be defined with a structure $f\mathcal{M}^C(\mathcal{Y}) \to \mathcal{H}_{C\times I}$ of an I-colored local space over C.

Proof. (a) To simplify the notation denote $\mathcal{M}_E = \mathcal{M}^C(\mathcal{Y})_E$. The locality structure is a consistent system of isomorphisms for I-disjoint[8] $E_i \in \mathcal{H}_{C\times I}$,

$$\iota :\ \mathcal{M}_{E_1} \times \mathcal{M}_{E_2} \xrightarrow{\cong} \mathcal{M}_{E_1 \sqcup E_2}.$$

To (f_1, f_2) in the LHS it associates f in the RHS so that on $C - E_1$ one has $f = f_2$ and on $C - E_2$ one has $f = f_1$ (on the intersection $C - (E_1 \sqcup E_2)$ both are equal to y).

(b) The locality structure, i.e., the gluing for $f\mathcal{M}_{\mathcal{Y}}(C)_D$ is the same as in (a), one just needs to observe that $\pi(f) = \pi(f_1) \sqcup \pi(f_2)$.

When the boundary divisor $\partial(\mathrm{pt})$ has irreducible components $\mathcal{D}_i$ we can refine π to a collection of $\pi_i(f) = f^{-1}(\mathcal{D}_i) \in \mathcal{H}_C$ so that now $\pi : \mathcal{M}^C_{gl}(\mathcal{Y}) \to (\mathcal{H}_C)^I = \mathcal{H}_{C\times I}$.

For a local space $Z \to \mathcal{H}_C$ with a "growth" structure $Z'_D \to Z_D$ for $D' \subseteq D$, there is an associated factorization space $Z^{fac} \to \mathcal{R}_C$ with the fibers

$$Z^{fac}_E \overset{\mathrm{def}}{=} \lim_{\to\ \mathrm{supp}(D)\subseteq E} Z_D, \quad E \in \mathcal{R}_C.$$

In our case this is

$$([\mathcal{M}^C(\mathcal{Y})]^{fac})_E \ = \ \lim_{\to\ \mathrm{supp}(D)\subseteq E} f\mathcal{M}^C(\mathcal{Y})_D = \ Map[(C, C-E), (\mathcal{Y}, \mathrm{pt})] \ = \ \mathcal{M}^C(\mathcal{Y})_E.$$

$\square$

Example. (a) The standard example is $\mathcal{Y} = \mathbb{B}(G)$ (we will see that then $\mathcal{M}_{\mathcal{Y}} = \mathcal{G}(G)$). Here, the map $\mathrm{pt} \to \mathcal{Y}$ is not an open inclusion, so the factorization space $\mathcal{M}_{\mathbb{B}G}(C) = \mathcal{G}(G) \to \mathcal{H}_C$ does not have a filtered version.

(b) The local version appears for $\mathcal{Y} = G\backslash(G/N)^{\mathrm{aff}}$ (3.3), $Y = N\backslash\mathcal{B}$ (**??**) and $Y = G\backslash\overline{G}$ (3.5).

(c) [Ra] For any X, the moduli $\mathcal{M}^X_{gl}(G_m\backslash\mathbb{A}^1)$ (with the point $1 \to G_m\backslash\mathbb{A}^1$ is the moduli of effective Cartier divisors in X.

Proof. For $f : X \to \mathbb{A}^1/G_m$, the pull back $f^{-1}0 = \ X \times_{\mathbb{A}^1/G_m} 0 \subseteq X$ is a Cartier divisor in X. $\square$

[8] Define I-disjoint.

2.2. **Subfunctor $\mathcal{G}(G,Y){\subseteq}\mathcal{G}(G)$ given by "condition Y".** Consider a G-space Y with a point y[9] It gives a pointed stack $(G\backslash Y,\mathrm{pt})$ where pt is the composition $y\in Y\to G\backslash Y$. For any X We consider the corresponding moduli of fintely supported maps

$$\mathcal{G}(G,Y)\stackrel{\text{def}}{=}\mathcal{M}(G\backslash Y)$$

(we omit the base point y from the notation). To a curve C it associates the factorization space $\mathcal{G}^C(G,Y)=\mathcal{M}^C(\mathcal{Y})$ over C with the fiber at $E\in\mathcal{R}_C$

$$\mathcal{G}(G,Y)_E \;=\; \mathcal{G}^C(G,Y)_E \;\stackrel{\text{def}}{=}\; Map[(C,C-R),(G\backslash Y,\mathrm{pt})].$$

Here we study the functor $\mathcal{G}(G,-)$ on the category $Sp_\bullet(G)$ of G-spaces with a point (Y,y) By definition, $\mathcal{G}(Y,\mathrm{pt})$ is the loop Grassmannian $\mathcal{G}(G)$. We will see that when Y is separated then $\mathcal{G}(G,Y)$ is a subfunctor of $\mathcal{G}(G)$ (lemma 2.2.2.b). Actually, the interesting subfunctors of $\mathcal{G}(G)$ are usually of this form.

2.2.1. *Loop Grassmannians $\mathcal{G}(G,Y)$ with the "condition Y".* If Y is a scheme near y then the stabilizer of y is a subgroup A of G and the orbit $Y^o\stackrel{\text{def}}{=}G{\cdot}y{\subseteq}Y$ is a well defined subscheme of Y isomorphic to G/A.

The Drinfeld setting is the case when the orbit Y^o is free, i.e., $A=1$, i.e., $\mathrm{pt}\to G\backslash Y$ is an open embedding. Then lemma 2.1.1.b provides a refinement of the factorization space functor $\mathcal{G}(G,Y)$ has to a local space functor $f\mathcal{G}(G,Y)\stackrel{\text{def}}{=}f\mathcal{M}(G\backslash Y)\to\mathcal{H}_C$ with the fiber at $D\in\mathcal{H}_C$

$$f\mathcal{G}(G,Y)_D \;=\; \{f:(C,\eta_C)\to(G\backslash Y,\mathrm{pt});\;\; f^{-1}[\partial(\mathrm{pt})]=D\}.$$

We start with some formal properties.

Lemma. (a) [Redundancy in the notation $\mathcal{G}(G,Y)$.] For a normal subgroup $K{\subseteq}G$ one has $\mathcal{G}(G,Y)\cong\mathcal{G}(K\backslash G,K\backslash Y)$.

(b) [Fibered products.] We consider a system of groups $G_i\to G_0$ and a compatible system $Y_i\xrightarrow{a_i}Y_0$ of G_k-spaces Y_k.

(b1) The general formula for two factors is

$$\begin{aligned}\mathcal{G}(G_1,Y_1)\times_{\mathcal{G}(G_0,Y_0)}\mathcal{G}(G_2,Y_2) \;&\cong\; \mathcal{G}[G_0,(G_0{\times}_{G_1}Y_1)\times_{Y_0}(G_0{\times}_{G_2}Y_2)]\\ &\cong\; \mathcal{G}[G_1,Y_1{\times}_{Y_0}(G_0{\times}_{G_2}Y_2)] \;\cong\; \mathcal{G}[G_2,(G_0{\times}_{G_1}Y_1){\times}_{Y_0}Y_2].\end{aligned}$$

(b2) If $G_0=G_i/K_i$ is a quotient of G_i for $1\le i<n$, then

$$\prod_1^n\,[\mathcal{G}(G_i,Y_i)\to\mathcal{G}(G_0,Y_0)] \;\cong\; \mathcal{G}[\prod_1^n\,G_i/G_0,\;\prod_1^n\,Y_1/Y_0].$$

(b3) [Products.] $\mathcal{G}(G_1,Y_1){\times}\mathcal{G}(G_2,Y_2)\xrightarrow{\cong}\mathcal{G}(G_1{\times}G_2,Y_1{\times}Y_2)$.

[9] y is a point of the underlying space of the G-space Y, i.e., $y\in Y$ need not be a G-map.

Proof. (a) is obvious since $(K\backslash G)\backslash(K\backslash Y) \ \cong\ G\backslash Y$.

(b) holds because $Map(X,-)$ preserves fibered products. For instance when in (b1) one calculates the first component of objects in the moduli $\mathcal{G}(G_1,Y_1)\times_{\mathcal{G}(G_0,Y_0)}\mathcal{G}(G_2,Y_2)$ one gets

$$Map(C,G_1\backslash Y_1)\times_{Map(C,G_0\backslash Y_0)}Map(C,G_2\backslash Y_2) \ =\ Map[C,\ G_1\backslash Y_1 \times_{G_0\backslash Y_0} G_2\backslash Y_2].$$

By 12.1 the target is

$$\begin{aligned} G_1\backslash Y_1 \times_{G_0\backslash Y_0} G_2\backslash Y_2 \ &=\ G_0\backslash[(G_0\times_{G_1}Y_1) \times_{Y_0} (G_0\times_{G_2}Y_2) \\ =\ G_1\backslash[Y_1\times_{Y_0} (G_0\times_{G_2}Y_2) \ &\cong\ G_2\backslash[(G_0\times_{G_1}Y_1)\times_{Y_0} Y_2. \end{aligned}$$

The claim (b2) follows by induction from its case $n=2$. For $n=2$ we use (b1) and $G_0=G_1/K_1$ to identify

$$\begin{aligned} \mathcal{G}(G_1,Y_1)\times_{\mathcal{G}(G_0,Y_0)}\mathcal{G}(G_2,Y_2) \ &=\ \mathcal{G}[G_2,(G_0\times_{G_1}Y_1) \times_{Y_0} Y_2] \\ &\cong\ \mathcal{G}[G_2,K_1\backslash Y_1 \times_{Y_0} Y_2]. \end{aligned}$$

Since K_1 acts trivially on Y_0 this is

$$\cong\ \mathcal{G}[G_1,(K_1\times 1)\backslash\big(Y_1\times_{Y_0} Y_2\big)]$$

and since $(K_1\times 1)\backslash(G_1\times_{G_0}G_2) \ =\ K_1\backslash G_1\times_{G_0}G_2=G_2$, by (a) we get

$$\cong\ \mathcal{G}[G_1\times_{G_0}G_2,Y_1\times_{Y_0}Y_2].$$

Finally, (b3) is a special case of (b2) when $G_0=1$ and $Y_0=\mathrm{pt}$. $\square$

2.2.2. $\mathcal{G}(G,Y)$ *when* Y *is a separated scheme.* Under this assumption one has the following lemma.

Lemma. (a) In terms of torsors, the fiber $\mathcal{G}(G,Y)_D$ is the moduli of all $(\mathcal{T},\tau)\in\mathcal{G}(G)_D$ such that the A-reduction $A{\cdot}\tau\in\Gamma(U,A\backslash\mathcal{T})$ of $\mathcal{T}$ over $C-D$, extends to a section of $Y^{\mathcal{T}}$ over C,[(10)] in the sense of the embedding (that is given by the choice of $y\in Y^o$):

$$A\backslash\mathcal{T} \ =\ G\backslash(\mathcal{T}\times G/A) \ =\ (G/A)^{\mathcal{T}} \ \cong\ (Y^o)^{\mathcal{T}}\subseteq Y^{\mathcal{T}}.$$

(b) $\mathcal{G}(G,Y)$ is a subfunctor of $\mathcal{G}(G)$ and it carries the induced structure of a factorization space.

(c) If Y is affine then $\mathcal{G}(G,Y)$ is a closed subfunctor of $\mathcal{G}(G)$.[(11)] If Y is quasiaffine and Y^{aff} is separated then the subfunctor $\mathcal{G}(G,Y)\subseteq\mathcal{G}(G,Y^{\mathrm{aff}})$ is open,

[10] The inverse map from the above submoduli of $\mathcal{G}(G)$ to $\mathcal{G}(G,Y)=Map_{gt}(C,G\backslash Y)$ sends $(\mathcal{T},\tau)$ to a triple $(\mathcal{T},\phi,\tau)$ consisting of a map $(\mathcal{T},\phi):C\to G\backslash Y$ and the trivialization τ of the map on U. Here, $\phi\in\Gamma(C,Y^{\mathcal{T}})$ is the unique extension of $\overline{\tau}\in\Gamma(C-D,A\backslash\mathcal{T})$. In the opposite direction we have the projection $(\mathcal{T},\phi,\tau)\mapsto(\mathcal{T},\tau)$.

[11] This is not true for arbitrary Y, For a counterexample let $Y=G/A$ so that $\mathcal{G}(G,G/A)\cong\mathcal{G}(A)$ according to the corollary 2.3.2. When A is a proper parabolic $P\neq G$, then $\mathcal{G}(P)$ is not closed in $\mathcal{G}(G)$.

As one can see in the proof of (a), the problem arises because $Map(d^*,\mathsf{Y})$ is not local in Y for nonaffine Y. So, when $Y=G/A$ is not affine then there is a difficulty in defining $Map(d^*,\mathsf{Y})$ as a geometric object.

(d) If Y is a scheme near a then the map $\mathcal{G}(G,\overline{G\cdot a})\to\mathcal{G}(G,Y)$ is an isomorphism.

Proof. (a) **(1)** A map $(C,C-D)\xrightarrow{F}(G\backslash Y,\mathrm{pt})$ is a map $C\xrightarrow{f}G\backslash Y$ together with the commutativity constraint for the square

$$\begin{array}{ccc} C & \xrightarrow{f} & G\backslash Y \\ \uparrow{\scriptstyle\subseteq} & & {\scriptstyle i}\uparrow \\ U & \xrightarrow{q} & \mathrm{pt}, \end{array}$$

where the map i is the composition $[\mathrm{pt}=a\in Y\to G\backslash Y]$ and f is a pair of a G-torsor $\mathcal{T}$ on X and $\phi\in\Gamma(X,Y^{\mathcal{T}})$.

We will see that for $f:C\to G\backslash Y$ a completion to the above diagram, i.e., a factorization of $f|_U$ through the point pt, is the same as a U-trivialization $\tau\in\Gamma(U,\mathcal{T})$ of $\mathcal{T}$ whose image in $\Gamma(U,A\backslash\mathcal{T})$ extends to a section of $Y^{\mathcal{T}}$ on C.

(**2**) The map $\mathrm{pt}\xrightarrow{i}G\backslash Y$ is defined as $(\mathrm{pt}\xrightarrow{a}Y\overset{i'}{\twoheadrightarrow}G\backslash Y)$. Here, i' is represented by the trivialized G-torsor $G\times Y$ over Y and the trivial Y-section of the (trivial twist) $Y^{G\times Y}=G\times Y$, i.e., the map id_Y.

So, $iq=i'aq$ is represented by the trivial G-torsor $(aq)^*(G\times Y)=G\times U$ over U and the U-section of the trivial twist $Y^{G\times U}=Y\times U$, i.e., a map $a':U\to Y$, which is the constant map with value $a\in Y$.

(**3**) Now, the commutativity constraint is an isomorphism of G-torsors $\tau:\ G\times U\to\mathcal{T}|_U$, i.e., a section $\tau\in\Gamma(U,\mathcal{T})$, such that the corresponding trivialization of the $\mathcal{T}$-twist $\widetilde{\tau}:\ Y\times U\to Y^{\mathcal{T}|_U}$ takes the constant section a' of $Y\times U$ to the section $\phi|_U$ of $Y^{\mathcal{T}|_U}$.

The twist is $Y^{\mathcal{T}|_U}=G\backslash(\mathcal{T}\times Y)$ and $\widetilde{\tau}(a')=G\cdot(\tau,a')$. So, the data for a map F are a G-torsor $\mathcal{T}$ over Y, a section $\tau\in\Gamma(U,\mathcal{T})$ and a section $\phi\in\Gamma(C,Y^{\mathcal{T}})$ such that $\widetilde{\tau}^{-1}[\phi(u)]=G\cdot(\tau(u),a),\ u\in U$.

(**4**) *ϕ is an extension of $\overline{\tau}$.* Since $\widetilde{\tau}^{-1}\phi$ is representable with a pair $(\tau,a)\in\Gamma(U,\mathcal{T}\times Y^o)$ we know that $\phi(U)$ lies in $(Y^o)^{\mathcal{T}}$. Moreover, as we identify $(Y^o)^{\mathcal{T}}$ with $G\backslash(\mathcal{T}\times G/A)\cong A\backslash\mathcal{T}$ we see that $\phi|_U\in\Gamma[U,(Y^o)^{\mathcal{T}}]$ identifies with the image $\overline{\tau}\in\Gamma[U,A\backslash\mathcal{T}]$ of $\tau\in\Gamma(U,G^{\mathcal{T}})$. So, ϕ is an extension of $\overline{\tau}$ to a C-section of $Y^{\mathcal{T}}$.

(**5**) Since U is dense, such extension is unique provided that Y is separated. Therefore, the data reduce to a G-torsor $\mathcal{T}$ over C and a section $\tau\in\Gamma(U,\mathcal{T})$ such that the image $\overline{\tau}\in\Gamma(U,A\backslash\mathcal{T})=\Gamma(U,(Y^o)\mathcal{T})$ extends to a C-section of $Y^{\mathcal{T}}$. [12]

(b-c) We now know that $\mathcal{G}(G,Y)$ is the submoduli of $\mathcal{G}(G)$, given by the property of the existence of an extension of the image $\overline{\tau}\in\Gamma(U,A\backslash\mathcal{T})\ \cong\ \Gamma(U,(G/A)^{\mathcal{T}})$ of the section τ

[12] If we only know that Y is a scheme near a then the steps (1-4) of this argument are still valid. This only tells us that there is a map $\mathcal{G}(G,Y)\to\mathcal{G}(G)$ by $(\mathcal{T},\phi,\tau)\mapsto(\mathcal{T},\tau)$ such that $\phi|_U$ is $A\cdot\tau$.

to an X-section of $Y^{\mathcal{T}}$. The factorization claim is known (lemma 2.1.1). So, we only need to show that $\mathcal{G}(G,Y)$ is closed.

The "extension" condition on maps is closed when the target is affine; for any *affine* scheme Y, the functor $Map(\eta_C,\mathsf{Y})$ has a canonical structure of an indscheme such that $Map(C,\mathsf{Y})$ is a closed subscheme.

This claim follows from its local version. It says that $Map(d^*,\mathsf{Y})$ is canonically an indscheme such that $Map(d,\mathsf{Y})$ is a closed subscheme.

The first example of this is when $\mathsf{Y}=\mathbb{A}^1$, here $Map(d^*,\mathbb{A}^1)=\Bbbk((z))$ and the functions that extend to the formal disc d are $\mathcal{O}=\Bbbk[[z]]$. The general case follows by embedding Y into $\mathbb{A}^n$.

(d) Moduli $\mathcal{G}(G,Y)$ consists of all $(\mathcal{T},\tau)\in\mathcal{G}(G)$ and $\phi\in\Gamma(d,Y^{\mathcal{T}})$ such that ϕ extends the section $A{\cdot}\tau\in\Gamma(C-D,A\backslash\mathcal{T})=\Gamma(C-D,(Y^o)^{\mathcal{T}})$. Since $\phi|_{C-D}$ has values in $(Y^o)^{\mathcal{T}}$, all values of ϕ are in $\overline{(Y^o)^{\mathcal{T}}}=(\overline{Y^o})^{\mathcal{T}}$. $\square$

Remarks. (0) In terms of the global Grassmannians one states the inclusion in (b) as

$$\widetilde{\mathcal{G}}(G)\supseteq\widetilde{\mathcal{G}}(G,Y)\ \stackrel{\text{def}}{=}\ Map[(C,\eta_C),(G\backslash Y,\mathrm{pt})].$$

(1) The inclusion map $\mathcal{G}(G,Y)\ \to\mathcal{G}(G)$ is realized on the level of moduli of maps $Map[(C,\eta_C),(G\backslash Y,\mathrm{pt})]\to Map[(C,\eta_C),(G\backslash\mathrm{pt},\mathrm{pt})]$ by the G-map $Y\to\mathrm{pt}$.

(2) *Algebraic structure on* $\mathcal{G}(G,Y)$. When Y is a quasiaffine scheme the the part(c) of the lemma provides sucg structure, Here, we will only define the algebraic structure in some additional special cases.

Corollary. (a) At a point $c\in C$, $\mathcal{G}(G,Y)_c\subseteq\mathcal{G}(G,c)$ is the quotient $\mathcal{G}\widetilde{(G,Y)}_c/G_{\mathcal{O}}\subseteq G_{\mathcal{K}}/G_{\mathcal{O}}$ where

$$\mathcal{G}\widetilde{(G,Y)}_c\ \stackrel{\text{def}}{=}\ \{g\in G_{\mathcal{K}};\ ga:d^*\to Y^o\text{ extends to }d\to Y\}/G_{\mathcal{O}}.$$

(b) The functor $\mathcal{G}(G,-)$ in (Y,a) preserves fibered products. Moreover, on separated schemes Y the functor $\mathcal{G}(G,-)$ takes all morphisms into inclusions. and the fibered products are taken to intersections in $\mathcal{G}(G)$:

$$\mathcal{G}(G,Y_1{\times}_Y Y_2)\ \cong\mathcal{G}(G,Y_1){\times}_{\mathcal{G}(G,Y)}\ \mathcal{G}(G,Y_2)\ =\ \mathcal{G}(G,Y_1)\cap_{\mathcal{G}(G)}\ \mathcal{G}(G,Y_2).$$

Proof. (a) The fiber $\mathcal{G}(G,Y)_c$ consists of all G-torsors $\mathcal{T}$ over $d=\hat{a}$, with a section τ over $d^*=\widetilde{a}$ such that $A\tau\in\Gamma(d^*,A\backslash\mathcal{T})$ exends to a section of $Y^{\mathcal{T}}$ over d.

Since we are working locally near c the torsor $\mathcal{T}$ is trivial and we can assume that $\mathcal{T}=G{\times}d$. Then $\tau:d^*\to G$ is an element g of $G_{\mathcal{K}}$. The condition on g is that $A\tau\in\Gamma(d^*,Y)$, i.e., $ga\in Map(d^*,Y^o)$ extends to a d-section of $Y^{\mathcal{T}}=Y{\times}d$.

(b) The first claim is a special case of the lemma 2.2.1.b2, when $n = 2$ and all G_i equal G, so that $G_1 \times_{G_0} G_2 = G$. The rest follows since $\mathcal{G}(G, Y)$ is a submoduli of $\mathcal{G}(G)$ for separated schemes Y (lemma 2.2.2.b). □

2.2.3. *Examples.* Because of the lemma 2.2.1.b we will usually consider (Y, a) such that the G-orbit $G\cdot a \subseteq Y$ is dense, hence open. Then the stack $G\backslash Y$ will have an open dense part $G\backslash Y^o \cong \mathbb{B}(A)$.

(0) When Y is a semigroup closure $\overline{G}$ then $\mathcal{G}(G, \overline{G})$ can be thought of as the "loop Grassmannian of $\overline{G}$". This gives extension of Langlands duality to reductive semigroups on the level of the geometric Satake mechanism.

(1) The other class of Y's are the affinizations of homogeneous spaces G/V which are quasiaffine. Examples come from parabolic subgroups $P = U \ltimes L$, then V is a normal subgroup of P such that $P' \supseteq V \supseteq U$. The interesting cases $V = U$ and $V = P'$.

Corollary. The moduli of line bundles with a nonvanishing section is $\mathcal{G}(G_m, \mathbb{A}^1)$ (with the base point $a = 1 \in \mathbb{A}^1$). This is preceisely the punctual Hilbert scheme:

$$\widetilde{\mathcal{G}}(G_m, \mathbb{A}^1) = \mathcal{H}_C.$$

Proof. Here, $\widetilde{\mathcal{G}}(G_m, \mathbb{A}^1) = Map[(C, \eta_C), (G_m\backslash\mathbb{A}^1, \mathrm{pt})]$ is the moduli of G_m-torsors $\mathcal{L}$ on C with a section ϕ of $(\mathbb{A}^1)^{\mathcal{L}}$ which is generically in $G_m)^{\mathcal{L}}$. These are precisely the pairs of a line bundle L $(= (\mathbb{A}^1)^{\mathcal{L}})$ with a generically nonvanishing section ϕ.

The pair (L, ϕ) gives an effective divisor $\phi^{-1}0 \in \mathcal{H}_C$. Conversely, for any $D \in \mathcal{H}_C$ we get a pair of a line bundle $\mathcal{O}_C(D)$ with a section 1. □

2.2.4. *When is $\mathcal{G}(G, Y)$ (quasi)projective.* The following conjecture is I believe actually a result of Drinfeld.

Conjecture. The connected components of $\mathcal{G}(G, Y)$ are quasiprojective iff Y^o is a G-torsor, i.e., $\mathrm{pt} \to \mathcal{Y} = G\backslash Y$ is open, i.e., $\dim(\mathcal{Y}) = 0$. Then they are projective iff Y is also affine.

Example. [Zhijie] The connected components of $\mathcal{G}(N, \overline{\mathcal{B}}) = \mathcal{G}(N \times H, (G/N)^{\mathrm{aff}}) = \sqcup_\lambda S_0 \cap \overline{S_\lambda^-}$ are <u>not</u> projective.

Question. Possibly the claim is only correct for a reductive group G? Say, if we rewrite the formula by induction to replace N by G then

$$\mathcal{G}(N, \overline{\mathcal{B}}) = \mathcal{G}(G \times H, G \times_N (G/N)^{\mathrm{aff}})$$

is (I guess) included into

$$\mathcal{G}(G \times H, [G \times_N G/N]^{\mathrm{aff}}) \cong \mathcal{G}(G \times H, [G/N \times G/N]^{\mathrm{aff}}) = \mathcal{G}(G \times H, (G/N)^{\mathrm{aff}} \times (G/N)^{\mathrm{aff}}) = \sqcup \overline{S_0} \cap \overline{S^-\lambda}$$

which does have projective connected components. □

Remark. The assumption that Y^o is a G-torsor, i.e., that $A = 1$, is exactly the Drinfeld setting, i.e., the case when the factorization space $\mathcal{G}(G, Y)$ has a local space filtration.

2.2.5. *A case when $\mathcal{G}(G, Y^o)\subseteq\mathcal{G}(G, Y)$ is dense.* □[13]■

Lemma. If ∂Y^o is in codimension 2 then the subfunctor $\mathcal{G}(G, Y^o)\subseteq\mathcal{G}(G, Y)$ is dense.

In particular we have a surjection $\pi_1(A) \ = \ \pi_0[\mathcal{G}(G, Y^o)] \twoheadrightarrow \pi_0[\mathcal{G}(G, Y)]$.

Proof. $\mathcal{G}(G, Y)_c$ consists of all $(\mathcal{T}, \tau) \in \mathcal{G}(G)_c$ such that the section $A\tau \in \Gamma(d^*, (Y^o)^{\mathcal{T}})$ extends to a section (call it ϕ) in $\Gamma(d, Y^{\mathcal{T}})$. Actually, we can assume that $\mathcal{T}$ is trivialized on d, and then we can think of ϕ as a map $d \to Y$ which is generically in Y^o.

So, the claim reduces to

When $Z = \partial Y^o\subseteq Y$ is in codimension 2 then in $Map(d, Y)$
the maps that meet Z are in codimension one.

□[14]■ □[15]■ Actually, if $\dim(\Sigma) = d$ and $Z\subseteq Y$ is in codimension c, then in $Map(\Sigma, Y)$ the maps that meet Z are in codimension $c - d$.[16]

The reason is that for $s \in \Sigma$, $Map(\Sigma, Y)_s = \{f : \ f(s) \in Z\}$ is in codimension c, hence $\cup_{s\in\Sigma} \ Map(\Sigma, Y)_s$ is in codimension $c - d$. □

Remark. Claim fails in codimension one, say for $\mathcal{G}(G_m, \mathbb{A}^1) = \mathcal{H}_d$. Here, $\phi \in \mathcal{H}_d$ is a monic polynomial $z^d + \sum_1^d \ s_i z^{d-i}$ with s_i nilpotent. Now, $\phi(0)$ lies in $Y^o = G_m$ iff $d = 0$. However, the connected components of $\mathcal{H}_d$ are given by the degree d of ϕ. □

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[13] $\boxed{!}$ The following does not make sense until algebaic structure on $\mathcal{G}(G, Y)$ is defined? Is there a notion of a dense subfunctor of a functor?

[14] $\boxed{!}$ Still needs a proof

[15] $\boxed{!}$ This reformulation only works when the moduli is connected!!!
The claim could be that (?)

(1) The closure of the subfunctor $\mathcal{G}(G, Y^o)\subseteq\mathcal{G}(G, Y)$ is open and closed in $\mathcal{G}(G, Y^o)\subseteq\mathcal{G}(G, Y)$, i.e., a union of connected components.
This would agree with the strange expectation that the connected components of $\mathcal{G}(G, Y)$ are locally closed in $\mathcal{G}(G)$.
(2) Another way is to require that *in each connected component of $Map(d, Y)$ the maps that meet Z are in codimension one.*

The original calim is obviously false when $\mathcal{G}(G, Y^o) = \mathcal{G}(A)$ is connected, i.e., $\pi_1(A) = 0$, but $\mathcal{G}(G, Y)$ is not connected.

[16] This explains why, for $\Sigma = d$, we needed codimension 2.

Question. For any Y we have inclusions

$$\mathcal{G}(G,Y) \subseteq \mathcal{G}(G,Y^{\text{aff}}) \overset{\textit{closed}}{\subseteq} \mathcal{G}(G).$$

When is the first inclusion always dense?

We know this when Y is quasiaffine and $\mathcal{G}(G,Y)$ is connected $\square$[17]$\blacksquare$. However, denseness also holds when $Y = Y^o$ is a partial flag variety G/P since then the map $\mathcal{G}(G,G/P) \to \mathcal{G}(G,(G/P)^{\text{aff}}) = \mathcal{G}(G,\text{pt})$ is $\mathcal{G}(P) \hookrightarrow = \mathcal{G}(G)$ which is known to be dense.

Question. Suppose that $(Y_1,a_1) \to (Y_2,a_2)$ is proper and that the generic fiber $A_2/A-1$ is connected. Is $\mathcal{G}(G,Y_1) \subseteq \mathcal{G}(G,Y_2))$ dense?[(18)]

2.3. **Restriction, induction and symmetries of $\mathcal{G}(G,Y)$.** Here we study the functoriality of $\mathcal{G}(G,Y)$ in the group G.

A. Restriction of the condition (Y,y) to a subgroup. We consider a subgroup $K \subseteq G$.

2.3.1. *Loop Grassmannians embedd.*

Lemma. (a) For any subgroup $K \subseteq G$, its loop Grassmannian $\mathcal{G}(K)$ embeds into $\mathcal{G}(G)$ as a subfunctor by the induction functor that takes $(\mathsf{S},\sigma) \in \mathcal{G}(K)$ to $Ind_K^G(S,\sigma) \overset{\text{def}}{=} (G \times_K \mathsf{S},\sigma)$.

(b) The image of $\mathcal{G}(K) \hookrightarrow \mathcal{G}(G)$ is the submoduli $\mathcal{G}(G;K) \subseteq \mathcal{G}(G)$ consisting of all $(\mathcal{T},\tau) \in \mathcal{G}(G)$ that satisfy the equivalent conditions that

- the image of the section $\tau \in \Gamma(C-D,\mathcal{T})$ in $\Gamma(C-D,K\backslash\mathcal{T})$ extends to a C-section of $K\backslash\mathcal{T}$;
- the closure $\mathcal{T}_{K,\tau}$ of $K\cdot\tau$ in $\mathcal{T}$ is a K-subtorsor (i.e., a reduction of $\mathcal{T}$ from G to K).

(c) The inverse map $\mathcal{G}(G,K) \to \mathcal{G}(K)$ sends $(\mathcal{T},\tau)$ to the pair $(\mathcal{T}_{K,\tau},\tau)$.

Proof. In (c) one observes that $\mathcal{T}_{K,\tau}$ has a meromorphic section τ. $\square$

Corollary. When Y is a single G-orbit G/A then $\mathcal{G}(G,G/A) \cong \mathcal{G}(A)$ and $\mathcal{G}(G,G/A)$ is exactly the above $\mathcal{G}(G;A)$. In particular, $\mathcal{G}(G,\text{pt}) = \mathcal{G}(G)$ and $\mathcal{G}(G,G) = \mathcal{G}(1) = \text{pt}$.

Proof. The main claim is clear from the lemma. Then $\mathcal{G}(G,\text{pt}) = \mathcal{G}(G,G/G) = \mathcal{G}(G)$ and $\mathcal{G}(G,G) = \mathcal{G}(G,G/1) = \mathcal{G}(1) = \text{pt}$. $\square$

[17] $\boxed{!}$?

[18] The question requires a natural algebraic structure on $\mathcal{G}(G,Y_2)$.

2.3.2. *Restriction to subgroups and the T-fixed points in $\mathcal{G}(G,Y)$.* For a subgroup $K\subseteq G$ denote the K-orbit through $y\in Y$ and its closure in Y by

$$Res_K^G(Y^o) \ = \ Y_K^o \ \stackrel{\text{def}}{=} \ K{\cdot}a \ \cong \ K/(K\cap A) \quad \text{and} \quad Res_C^G Y \ = \ Y_K \ \stackrel{\text{def}}{=} \overline{Y_K^o}.$$

If Y is affine then so is Y_K.

Example. When K is a torus T in G the T-orbit Y_T^o is canonically identified with the torus $\mathsf{T}=T/(T\cap A)$ and its closure Y_T in Y is then a toric variety for the torus T.

Proposition. For a subgroup $K\subseteq G$ the intersection $\mathcal{G}(G,Y)\cap\mathcal{G}(K)$ of subfunctors of $\mathcal{G}(G)$ is $\mathcal{G}(K,Y_K)$.

Proof. We know that the fiber at $D\in\mathcal{H}_C$ of

$$\mathcal{G}(G,Y)\cap_{\mathcal{G}(G)}\mathcal{G}(K) = \ \mathcal{G}(G,Y)\cap_{\mathcal{G}(G,\text{pt})}\mathcal{G}(G,G/K) = \ \mathcal{G}(G,Y{\times}_{\text{pt}}G/K)$$

is

$$Map[(C,C-D),\big(G\backslash(Y{\times}G/K),\text{pt})] \cong \ Map[(C,C-D),\big(sC\backslash Y),\text{pt}])$$

and this is $\mathcal{G}(K,Y)_D$ which is the same as $\mathcal{G}(K,Y_K)_D$.

Example. The inclusion $\mathcal{G}(G')\subseteq\mathcal{G}(G)$ need not be closed. For instance this fails when G' is a proper parabolic P in a reductive G.

2.3.3. *Fixed points in $\mathcal{G}(G)$.*

Lemma. (a) For the centralizer $\mathcal{Z}\stackrel{\text{def}}{=}Z_G(K)$, we have $\mathcal{G}(G)^K\supseteq\mathcal{G}(\mathcal{Z})$ and

$$\mathcal{G}(G,Y)^K \supseteq \mathcal{G}(\mathcal{Z},Y_{\mathcal{Z}}).$$

(b) When K is reductive the inclusions in (a) are equalities. For instance, when K is a Cartan $T\subseteq G$, then $\mathcal{Z}=T$ and $Y_{\mathcal{Z}}=Y_T$ is a toric variety for the torus $T/(A\cap T)$. So,

$$\mathcal{G}(G)^T\supseteq\mathcal{G}(T) \quad \text{and} \quad \mathcal{G}(G,Y)^T \supseteq \mathcal{G}(T,Y_T).$$

(c) [Conj.] If $s\in G$ is semisimple then $\mathcal{G}(G)^s \ = \ \mathcal{G}[Z_G(s)$.

Proof. (a) Since K acts trivially on $\mathcal{Z}$ we have $\mathcal{G}(G)^K\supseteq\mathcal{G}(\mathcal{Z})$. Then

$$\mathcal{G}(G,Y)^K \supseteq \mathcal{G}(G,Y)\cap\mathcal{G}(\mathcal{Z}) \ \cong \ \mathcal{G}(\mathcal{Z},Y_{\mathcal{Z}}).$$

In (b) we just need to see that when K is reductive then $\mathcal{G}(G)^K=\mathcal{G}[Z_G(K)]$. This follows from (c) since the semisimple elements are dense in a reductive K, hence

$$\mathcal{G}(G)^K = \cap_{s\in K}\mathcal{G}(G)^s = \cap_{s\in K}\mathcal{G}[Z_G(s)] = \mathcal{G}[\cap_{s\in K}Z_G(s)] = \mathcal{G}(Z_G(K)).$$

(c) $\square$

B. Orbits of $\ddot{A}$ in $\mathcal{G}(G,Y)$

2.3.4. *Orbits of* $\ddot{A}\subseteq N_G(A)_{\mathcal{O}}\cdot A_{\mathcal{K}}$ *in* $\mathcal{G}(G,Y)\subseteq\mathcal{G}(G)$. The automorphism group of the G-space G/A is $N_G(A)/A$ acting by right multiplication. In particular, the normalizer $N_G(A)$ acts on $Y^o=G/A$ by conjugation. Let $\dot{A}\subseteq N_G(A)$ consist of all elements $g\in N_G(A)$ such that the actions of g,g^{-1} on $Y^o=G/A$ by right multiplication extends to an action on Y. When Y is separated the extensions are unique, $\dot{A}$ is a group and its conjugation action on Y^o extends to an action on Y. In $G_{\mathcal{K}}$ we will consider the subgroup $\ddot{A}\overset{\text{def}}{=}\dot{A}_{\mathcal{O}}A_{\mathcal{K}}\cong\dot{A}_{\mathcal{O}}\rtimes_{A_{\mathcal{O}}}A_{\mathcal{K}}$.

Example. (0) When Y is obtained from Y^o by some canonical construction then the symmetries extend automatically hence $\dot{A}=N_G(A)$. For instance, when G/A is quasiaffine we can take $Y=(G/A)^{\text{aff}}$.

(1) When Y is not a canonical construct from Y^o then $\dot{A}$ can be A small part of $N_G(A)$. An example is $\mathcal{G}(G{\times}H,(G/N^+)^{\text{aff}}{\times}(G/N^-)^{\text{aff}}$ with $a=(N^+,N^-))$. Here, $(G{\times}H)a$ is a torsor for $\mathcal{V}=(G{\times}H)/Z(G)$, so $A=\Delta_{Z(G)}$. Hence, $\dot{A}$ is the part of $N_{G\times H}(Z(G))=G{\times}H$ whose right action on $\mathcal{V}\subseteq(G/N^+)^{\text{aff}}{\times}(G/N^-)^{\text{aff}}$ extends to an action on $(G/N^+)^{\text{aff}}{\times}(G/N^-)^{\text{aff}}$. This is small and indeed $\mathcal{G}(G{\times}H,(G/N^+)^{\text{aff}}{\times}(G/N^-)^{\text{aff}}$ has little symmetry. □

Remark. For a torus T in $\dot{A}$ the closure $Y_T\overset{\text{def}}{=}\overline{T\cdot a}$ in Y is a toric variety for the torus $\mathsf{T}=T/(T\cap A)$. So, Y_T is a semigroup closure $\overline{\mathsf{T}}$ of T iff Y_T is affine.

Lemma. (a) $\ddot{A}$ acts on $\mathcal{G}(G,Y)\subseteq\mathcal{G}(G)$.

(b) Suppose that a Cartan T of G lies in $\dot{A}\subseteq N_G(A)$ and that Y_T is affine so that it is a semigroup closure $\overline{\mathsf{T}}$ of the torus $\mathsf{T}=T/(T\cap A)$. Then for $\mu\in X_*(T)$, the point $L_\mu\in\mathcal{G}(G)^T$ lies in $\mathcal{G}(G,Y)$ iff the morphism of groups $G_m\xrightarrow{-\lambda}T\twoheadrightarrow\mathsf{T}$ extends to a morphism of semigroups $(\mathbb{A}^1,\cdot)\to\overline{\mathsf{T}}$.

(c) If each $\ddot{A}$-orbit in $\mathcal{G}(G)$ defined over the ground field contains a point in $X_*(T)$ then $\mathcal{G}(G)$ has a stratification by orbits $\ddot{A}\cdot L_\lambda$ and this restricts to a stratification of $\mathcal{G}(G,Y)$ over all $L_\lambda\in\mathcal{G}(G,Y)$.[19]

Proof. (a) $G_{\mathcal{K}}$ acts on $\mathcal{G}(G)$ by changing the d^*-trivializations $\tau\in\Gamma(d^*,\mathcal{T})$ by $\tau\mapsto g\cdot\tau$. The submoduli $\mathcal{G}(G,Y)_c$ is given by the condition that $A\tau\in\Gamma(d^*,A\backslash\mathcal{T})=\Gamma(d^*,(Y^o)^{\mathcal{T}})$ extends to a d-section of $Y^{\mathcal{T}}$. Clearly, $g\in A_{\mathcal{K}}$ does not change the coset $A\cdot\tau=A\cdot g\tau$, so it preserves $\mathcal{G}(G,Y)\subseteq\mathcal{G}(G)$.

On the other hand, the action of G on $\mathcal{T}$ induces an action of $N_G(A)$ on $A\backslash\mathcal{T}\cong G\backslash(\mathcal{T}{\times}G/A)=Y^{o\mathcal{T}}$ so that $g\in N_G(A)$ acts in the first realization by $g\cdot At=A\cdot gt$ for $t\in\mathcal{T}$. The second realization is related by $At\leftrightarrow\overline{(t,A)}$. Therefore, $g\cdot\overline{(t,xA)}$ corresponds to $g\cdot A(x^{-1}t)=A(gx^{-1}t)$, hence

$$g\cdot\overline{(t,xA)}=\overline{(gx^{-1}t,A))}=\overline{(t,xg^{-1}A))}=\overline{(gt,gxAg^{-1}))}=\overline{\big(gt,\ {}^g(xA)\big)}.$$

[19] This has versions for $\ddot{A}_{red}$-orbits in $\mathcal{G}(G,Y)_{red}\subseteq\mathcal{G}(G)_{red}$.

This uses the conjugation action so it extends to an action of $\dot{A}$ on $Y^{\mathcal{T}}$. Therefore, the action of $\dot{A}_{\mathcal{O}}$ on sections $\tau \in \Gamma(d^*, \mathcal{T})$ preserves the property that $A\tau$ extends to a d-section of $Y^{\mathcal{T}}$.

(b) We know that $\mathcal{G}(G,Y)^T = \mathcal{G}(T, Y_T)$ where $Y_T = \overline{T\cdot a}\subseteq Y$ is a toric variety for $\mathsf{T} = T/T\cap A$.

For $\lambda \in X_*(T)$, the point L_λ in $\mathcal{G}(T)_c$ is the trivial T-torsor $\mathcal{T} = T\times d$ over d with the section $\tau = z^{-\lambda} : d^* \to T$ over d^* which is the composition of a local parameter $z : d^* \to G_m$ with $-\lambda \in \mathrm{Hom}(G_m, T)$.

Now, L_λ lies in $\mathcal{G}(T, Y_T)_c$ iff the function $A\cap T\cdot z^{-\lambda} : d^* \to A\cap T\backslash T = \mathsf{T}$ extends to a section of $(Y_T)^{\mathcal{T}}$. Since $\mathcal{T}$ is trivial this means extending to a function $d \to Y_T = \overline{\mathsf{T}}$. This is equivalent to $G_m \xrightarrow{-\lambda} T \to \mathsf{T}$ extending to $\mathbb{A}^1 \to \overline{\mathsf{T}}$. □

2.3.5. *Parabolic $\frac{\infty}{2}$ orbits $S^P_\lambda \overset{def}{=} \ddot{U}\cdot L_\lambda$ for $\ddot{U} = U_{\mathcal{K}} \ltimes L_{\mathcal{O}}$.* Let $P\supseteq B\supseteq T$ be a parabolic, Borel and Cartan subgroups. For the unipotent radical U of P we have $\ddot{U} \overset{\text{def}}{=} U_{\mathcal{K}}\cdot P_{\mathcal{O}} = U_{\mathcal{K}}\ltimes L_{\mathcal{O}}$ for any Levi subgroup $P\supseteq L\supseteq T$. The orbits of $\ddot{U}$ in the loop Grassmannian are parameterized by the orbits in $X_*(T)$ of the the Weyl group $W_T(L)$ of L via $S^P_\lambda \overset{\text{def}}{=} L_{\mathcal{O}}U_{\mathcal{K}}\cdot L_\lambda$.

Remarks. (0) One source of interest in these orbits is that the sets of irreducible components of intersections appear as canonical bases in the representation theory of the dual group $\check{G}$. For instance $Irr[\overline{\mathcal{G}_\lambda} \cap \overline{S_\nu}]$ functions as a natural basis of the ν weight space of the standard finite dimensional representations $W^{\check{G}}_\lambda(\nu)$ and $S^{\check{G}}_\lambda(\nu)$ (over integers).

(1) We will eventually describe these orbits and their closures as moduli. This will give a transparent proof of their closure relations. Here we just recall the notation.

C. Induction of pairs (Y,a)**.** For a map of groups $\phi : G' \to G$ we have the pull-back $\phi^* :\ Sp_\bullet(G) \to Sp_\bullet(G')$ where $\phi^*(Y,a) = (Y,a)$ with the G'-action on Y via ϕ. We also have the direct image

$$\phi_*(Y',a') \overset{\text{def}}{=} G\times_{G'} Y', \overline{(1,a')}).$$

We will also use the affinization operation $Sp_\bullet(G) \to Sp_\bullet(G)$ by $(Y,a)^{\text{aff}} = (Y^{\text{aff}},a)$. While affinization can enlarge or diminish the space Y, any map $Y \to Y'$ gives inclusion of functors $\mathcal{G}(G,Y)\subseteq\mathcal{G}(G,Y'))$. Moreover, when Y is quasiaffine then $\mathcal{G}(G,Y^{\text{aff}})) = \overline{\mathcal{G}(G,Y}$.

2.3.6. *Induction.* For a subgroup $K\overset{\iota}{\hookrightarrow}G$ the direct image is called induction ind^G_K : $Sp_\bullet(K) \to Sp_\bullet(G)$. We can also compose it with affinization for the "affine induction"

$$Aind^G_K(\mathsf{Y}) \overset{\text{def}}{=} (G\times_K \mathsf{Y})^{\text{aff}}.$$

So, lemmas ... say that in

$$\mathcal{G}(G, ind^G_K(\mathsf{Y})) \overset{(1)}{\subseteq} \mathcal{G}(G, AInd^G_K(\mathsf{Y})) \overset{(2)closed}{\subseteq} \mathcal{G}(G)$$

inclusion (2) is closed and inclusion (1) is dense if $Ind_K^G(\mathsf{Y}))$ is quasiaffine.

Lemma. (a) $\mathcal{G}(G, Ind_{\mathrm{sc}}^G(\mathsf{Y})) \;\cong\; \mathcal{G}(K,\mathsf{Y})$.

(b) Ind and Res are adjoint.(?)

xxx

$$\mathcal{G}(L,\mathsf{Y}) \;\to \mathcal{G}(P,\mathsf{Y}) \;=\; \mathcal{G}(G, G\times_P\mathsf{Y}) \;\to \mathcal{G}[G,(G\times_P\mathsf{Y})^{\mathrm{aff}}]$$

Really, this is

$$\mathcal{G}(L,\mathsf{Y},\mathsf{a}) \;\to \mathcal{G}(P,\mathsf{Y},\mathsf{a}) \;=\; \mathcal{G}(G, G\times_P\mathsf{Y}, P\times_P\mathsf{a}) \;\to \mathcal{G}[G,(G\times_P\mathsf{Y})^{\mathrm{aff}}, P\times_P\mathsf{a}]$$

Lemma. (?) For the maximal central torus Z_L in L,

$$\mathcal{G}[G, G\times_P\mathsf{Y}]^Z \;=\; \mathcal{G}(L,\mathsf{Y}) \quad \text{and} \quad \mathcal{G}[G,(G\times_P\mathsf{Y})^{\mathrm{aff}}]^Z \;=\; \mathcal{G}(L,\mathsf{Y}^{\mathrm{aff}}).$$

2.3.7. *How do $A, \dot{A}, \ddot{A}$ induce?* Consider $A_L\subseteq L$. Going from L to P one does not change the spaces $Y^o\subseteq Y$, therefore $P/A_P \cong L/A_L$ says that $A_P = \;(P\twoheadrightarrow L)^{-1}A = U \ltimes A$.

In the opposite direction *by fixed points*, one can consider $\mathcal{G}(L)$ as $\mathcal{G}(P)^{Z_L}$. Then

$$\mathcal{G}(P,Y_P)^{Z_L} = \mathcal{G}(P,Y_P)\cap\mathcal{G}(L) = \mathcal{G}(L,(Y_P)_L).$$

Next, from P to G.

2.3.8. *The parabolic induction.* For a parabolic $P = U\ltimes L$ we have $G\overset{i}{\hookleftarrow}P\overset{q}{\twoheadrightarrow}P/U \overset{\mathrm{def}}{=} \overline{P} \cong L/$ The "parabolic induction" from $P/U\cong L$ to G is i_*q^* and its affine enlargement.

□[20]■ the group $\ddot{U} = U_{\mathcal{K}}\ltimes L_{\mathcal{O}}$ seems to come from the "induction" of $\mathcal{G}(\mathcal{V}_L,\overline{\mathcal{V}}_L)$ to $\mathcal{G}(\mathcal{V}_G,[\mathcal{V}_G\times_{PH}\overline{\mathcal{V}}_L]^{\mathrm{aff}})$.

One starts with $A_{\mathcal{V}_L}=1$ hence $\dot{A}_{\mathcal{V}_L} = N_{\mathcal{V}_L}(A_{\mathcal{V}_L}) = \mathcal{V}_L$ and $\ddot{A}_L = (\mathcal{V}_L)_{\mathcal{O}}$.

[20] $\boxed{!}$

3. Closures of orbits

3.1. Is $S_0 \cap S_{-\alpha}$ dense in $\overline{S_0} \cap \overline{S_{-\alpha}}$?

3.2. Summary.

3.2.1. *$K_{\mathcal{K}}$ orbits in $\mathcal{G}(G)$.*

Lemma. (a) For a subgroup K of G the "trivial" orbit $K_{\mathcal{K}} \cdot L_0$ in $\mathcal{G}(G)$ is $\mathcal{G}(K) = \mathcal{G}(G, G/C)$. If G/K is quasiaffine then its closure is $\overline{\mathcal{G}(K)} = \mathcal{G}[G, (G/K)^{\text{aff}}]$.

3.3. The closure of a semi-infinite orbit S_0 and the G-space $(G/N)^{\text{aff}}$.

The closures of the so called *semi-infinite orbits*

$$S_\lambda \stackrel{\text{def}}{=} N_{\mathcal{K}} \cdot L_\lambda, \ (\lambda \in X_*(T)),$$

will be given a modular description. This modular description will be used to extend the semi-infinite filtration to the relative Grassmannian $\mathcal{G}(G) \to \mathcal{R}_C$. For $\lambda = 0$ the classifying space of the closure will be described in terms of G but for general λ we will use the the Vinberg group $\mathcal{V} = \mathcal{V}(G)$.

3.3.1. *The closure of S_0.*

Let a be a smooth point of a curve C and denote $d \stackrel{\text{def}}{=} \widehat{a} \supseteq d^* \stackrel{\text{def}}{=} \widetilde{a}$.

Proposition. The closure $\overline{S_0}$ is the loop Grassmannian $\mathcal{G}[G, (G/N)^{\text{aff}}]_c$ with the condition $(G/N)^{\text{aff}}$. $\square$[21]$\blacksquare$ So it has equivalent descriptions as

(1) (*Torsors.*) The moduli of all $(\mathcal{T}, \tau)$ in $\mathcal{G}(G)_c$, such that the image in $N\backslash\mathcal{T}$ of the section τ of $\mathcal{T}$ extends from d^* to d as a section of the relative affinization $(N\backslash\mathcal{T})^{\text{aff}}$ of $N\backslash\mathcal{T}$ over C.
(2) (*Classifying spaces.*) The moduli of maps of pairs $Map[(d, d^*), (G\backslash(G/N)^{\text{aff}}, \text{pt})]$.

Proof. We know that $\mathcal{G}(G, (G/N)^{\text{aff}})$ is closed in $\mathcal{G}(G)$ since $(G/N)^{\text{aff}}$ is affine (lemma 2.2.1.b). Also, $S_0 = \mathcal{G}(N) = \mathcal{G}(G, G/N)$ is dense in $\mathcal{G}(G, (G/N)^{\text{aff}})$ by the lemma 2.2.5, so $\mathcal{G}(G, (G/N)^{\text{aff}}) = \overline{S_0}$.

The moduli $\mathcal{G}(G, (G/N)^{\text{aff}})_c$ is defined as in (2). The lemma 2.2.1.b provides a reformulation as the moduli of all $(\mathcal{T}, \tau) \in \mathcal{G}(G)_c$ such that the N-reduction $N\tau \in \Gamma(U, N\backslash\mathcal{T})$ of $\mathcal{T}$ over U, extends to a section of $[(G/N)^{\text{aff}}]^{\mathcal{T}}$ over C.[(22)] This is the same as the formulation in (1) since $[(G/N)^{\text{aff}}]^{\mathcal{T}} = [(G/N)^{\mathcal{T}}]^{\text{aff}} = (N\backslash\mathcal{T})^{\text{aff}}$. $\square$

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[21] $\boxed{!}$ Is $\mathcal{G}[G, (G/N)^{\text{aff}}]_c$ connected?

[22] The last requirement is in terms of the embedding $N\backslash\mathcal{T} = (G/N)^{\mathcal{T}} \subseteq [(G/N)^{\text{aff}}]^{\mathcal{T}}$.

Corollary. (a) The reduced part $[\mathcal{G}(G)_0]_{red}$ of the trivial connected component $\mathcal{G}(G)_0$, as well as S_0 and $\overline{S_0}$ are the same for G and $G_{\text{ad}} \stackrel{\text{def}}{=} G/Z(G)$.

(b) The T-fixed part $\overline{S_0}^T$ is isomorphic to the Hilbert scheme $\mathcal{H}_{d\times I}$ and the $T\mathcal{R}$-fixed part $\overline{S_0}^{T\mathcal{R}} = \overline{S_0} \cap X_*(T)$ is $-\check{Q}^+$.

(c) $\overline{S_0}$ has a stratification by orbits S_α with $\alpha \leq 0$.

Proof. (a) S_0 and $\overline{S_0}$ are formed inside the reduced paart $[\mathcal{G}(G)_0]_{red}$ of the trivial connected component $\mathcal{G}(G)_0$ which is equal to $\mathcal{G}[(G_{\text{ad}})_{sc}]$. $\square$[23]$\blacksquare$

(b) The fixed points $\overline{S_0}^T = \mathcal{G}(G,(G/N)^{\text{aff}})^T = \mathcal{G}(G,(G/N)^{\text{aff}}) \cap \mathcal{G}(T)$ have been identified in 2.3.2.c with $\mathcal{G}(T,\overline{H}) \cong \mathcal{G}(H,\overline{H})$ where $\overline{H}$ is the closure of $H = B/N$ in $(G/N)^{\text{aff}}$. We know that $\mathcal{G}(G_m,\overline{G_m})$ is $\mathcal{H}_d$ (lemma 2.2.3). When G is adjoint then $\overline{H}$ is described as $\overline{G_m}^I$ in the lemma 1.4 so we get identification $\overline{S_0}^T \cong \mathcal{H}_{d\times I}$ in this case. According to (a) this gives an identification for any G.

According to the lemma 2.3.4.b, the point L_λ in $\mathcal{G}(T)_c$ (for $\lambda \in X_*(T)$), lies in $\mathcal{G}(G,(G/N)^{\text{aff}})_c$ iff $-\lambda : G_m \to T$ extends to $\overline{G_m} \to \overline{H}$. When G is simply connected $\square$[24]$\blacksquare$ then $\mathcal{O}(\overline{H}) = \Bbbk[\omega_i,\ i \in I]$ (lemma 1.4.b), so the condition is that $\omega_i \circ (-\lambda) : G_m \to G_m \subseteq \mathbb{A}^1$ extends across $0 \in \overline{G_m}$. In other words, that $\langle \omega_i, -\lambda \rangle \geq 0$, i.e., $-\lambda \in \check{Q}^+ = \oplus_{i\in I}\ \mathbb{N}\check{\alpha}_i$.

(c) $\overline{S_0}$ only depends on G_{ad} (corollary 6.4)), so we can assume that G is semisimple. Then $\mathcal{G}(G)$is reduced and has a stratification by $N_\mathcal{K}$-orbits S_λ, $\lambda \in X_*(T)$. Now, $\overline{S_0}$ has a stratification by S_λ such that L_λ lies in $\overline{S_0}^T$, so one can use part (b) of the lemma. $\square$

Remark. The proposition shows that we can extend the spaces $S_0 \subseteq \overline{S_0} \subseteq \mathcal{G}(G)_c$ to factorization spaces over $\mathcal{R}_C$ given by $\mathcal{G}(G,G/N) \subseteq \mathcal{G}(G,(G/N)^{\text{aff}}) \subseteq \mathcal{G}(G)$.

3.4. **The semi-infinite filtrations $\overline{S_\lambda} \subseteq \overline{\mathcal{S}_\lambda}$ of $\mathcal{G}(G)$ and the $\mathcal{V}(G)$-space $(G/N)^{\text{aff}}$.**

3.4.1. *The Vinberg group $\mathcal{V} = \mathcal{V}(G)$.* For a reductive group G we define its *Vinberg group*

$$\mathcal{V} = \mathcal{V}(G) \stackrel{\text{def}}{=} G \times_{Z(G)} H.$$

The map $\mathcal{V} \to \mathcal{V}/Z(G) \cong G_{\text{ad}} \times H_{\text{ad}}$ is $\mathcal{V} \to \mathcal{V}/H \times \mathcal{V}/G$.

We also denote by $G_{\text{ad}} \stackrel{\text{def}}{=} G/Z(G)$ its adjoint form and by G_{sc} the simply connecred cover of G_{ad}. Similarly, we move the data $T \subseteq B \twoheadrightarrow H$ for G to the same kind of data $T_{\text{ad}}, B_{\text{ad}}, H_{\text{ad}}$ for G_{ad} and $T_{\text{sc}}, B_{\text{sc}}, H_{\text{sc}}$ for G_{sc}.[(25)]

[23] $\boxed{!}$reference for “$[\mathcal{G}(G)_0]_{red}$ equal to $\mathcal{G}([G_{\text{ad}}]_{sc})$”? Take quotients by the largest central torus Z and then by the finite $Z(G)$.

[24] $\boxed{!}$When not?

[25] So, $X^*(H_{\text{ad}}) = \check{Q}$ and $X_*(H_{\text{ad}}) = \oplus_{i\in I}\ \mathbb{Z}\check{\omega}_i$ while $X^*(H_{\text{sc}}) = \oplus_{i\in I}\ \mathbb{Z}\omega_i$ and $X_*(H_{\text{sc}}) = \check{Q}$.

We denote by $\underline{\Delta} : B \to \mathcal{V}$ the diagonal map $\underline{\Delta}_b \stackrel{\text{def}}{=} (b, bN)\cdot Z(G)$. So, $\underline{\Delta}_B \cong B/Z(G) = B_{\text{ad}}$ and for $B = N \ltimes T$ we have $N \xrightarrow{\cong} \underline{\Delta}_N$ and $\underline{\Delta}_T \cong T_{\text{ad}}$.

The Vinberg group acts on G/N by $(g,h)\cdot xN = gxNh^{-1}$, hence also on $(G/N)^{\text{aff}}$. The stabilizer in $\mathcal{V}$ of the origin in G/N is $\underline{\Delta}_B$ hence, as a $\mathcal{V}$-space G/N is $\mathcal{V}/\underline{\Delta}_B$.[26]

$\square$[27]$\blacksquare$ We have defined $\mathcal{V}$ as $(G\times H)/De^{-}_{Z(G)}$. Then $\underline{\Delta}$ should be $\underline{\Delta}^{-}$ in order to agree!

3.4.2. *Loop Grasmannians of* $G_{\text{sc}}, \mathcal{V}_{\text{sc}}, H_{\text{ad}}$. Here Gwill usually be simply connected.

Lemma. (a) The map $\mathcal{G}(\mathcal{V}_{\text{sc}}) \to \mathcal{G}(H_{\text{ad}})$ is an isomorphism on π_0. It is a $\mathcal{G}(G_{\text{sc}})$-bundle and

$$\mathcal{G}(\mathcal{V}_{\text{sc}})_{red} = \mathcal{G}(\mathcal{V}_{\text{sc}})\times_{\mathcal{G}(H_{\text{ad}})} X_*(H_{\text{ad}}).$$

(b) For any semisimple G the map $\mathcal{G}(\mathcal{V}) \to \mathcal{G}[\mathcal{V}/Z(G)] \cong \mathcal{G}(G_{\text{ad}})\times\mathcal{G}(H_{\text{ad}})$ is an open and closed embeding.

(c) Also, the embedding $\mathcal{G}(\mathcal{V}, G/N) \hookrightarrow \mathcal{G}(\mathcal{V}) \hookrightarrow \mathcal{G}[\mathcal{V}/Z(G)]$ is the diagonal map $\mathcal{G}(B_{\text{ad}}) \stackrel{\delta}{\hookrightarrow} \mathcal{G}(G_{\text{ad}})\times\mathcal{G}(H_{\text{ad}})$.

(d) $\mathcal{G}[\mathcal{V}_{\text{sc}}, G_{\text{sc}}/N]$ is open and dense in $\mathcal{G}[\mathcal{V}_{\text{sc}}, (G_{\text{sc}}/N)^{\text{aff}}]$ which is in turn closed in $\mathcal{G}(\mathcal{V}_{\text{sc}})$.

Proof. (a) is the lemma 6.4.2 applied to the exact sequence $0 \to G_{\text{sc}} \to \mathcal{V}_{\text{sc}} \to H_{\text{ad}} \to 0$.

(b) Since $Z(G)$ is finite and $\mathcal{V}$ is connected, $\mathcal{G}(\mathcal{V})$ is open and closed in $\mathcal{G}[\mathcal{V}/Z(G)]$ by the lemma 6.4. Also, $\mathcal{V}/Z(G) \xrightarrow{\cong} G_{\text{ad}}\times H_{\text{ad}}$, gives $\mathcal{G}[\mathcal{V}/Z(G)] \cong \mathcal{G}(G_{\text{ad}})\times\mathcal{G})H_{\text{ad}})$.

(c) First, $\mathcal{G}(\mathcal{V}, G/N)$ is indeed $\mathcal{G}(B_{\text{ad}})$ since

$$\mathcal{G}(\mathcal{V}, G/N) = \mathcal{G}(\mathcal{V}/H, G/N/H) = \mathcal{G}(G_{\text{ad}}, G_{\text{ad}}/B_{\text{ad}}) = \mathcal{G}(B_{\text{ad}}).$$

Now, the following diagram commutes because all maps are canonical

$$\begin{array}{ccccc}
\mathcal{G}(\mathcal{V}, \mathcal{V}/\underline{\Delta}_B) & \xrightarrow{\subseteq} & \mathcal{G}(\mathcal{V}) & \xrightarrow{\subseteq} & \mathcal{G}(\mathcal{V}/Z(G)) \\
\cong\big\downarrow & & & & =\big\downarrow \\
\mathcal{G}(\underline{\Delta}_B) & \xrightarrow{=} & \mathcal{G}(B_{\text{ad}}) & \xrightarrow{\subseteq} & \mathcal{G}(G_{\text{ad}})\times\mathcal{G}(H_{\text{ad}})
\end{array}$$

(d) is a case of lemmas 2.2.2.c and 2.3.2,d. (because $\mathcal{G}(G, (G/N)^{\text{aff}})$ is connected $\square$[28]$\blacksquare$). $\square$

3.4.3. *Closures of subindschemes* $S_\lambda \subseteq \mathcal{S}_\lambda$. Again, here $G = G_{\text{sc}}$ is simply connected and $H = H_{\text{sc}}, \mathcal{V} = \mathcal{V}_{\text{sc}}$ are its Cartan and Viberg groups.

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[26] $(b,h) \in B\times H$ fixes $1_G N$ if $b\cdot N/N\cdot h^{-1} = N/N\cdot(bN\cdot h^{-1})$ equals N/N, i.e., $h = bN$ in $H = B/N$.

[27] $\boxed{!}$

[28] $\boxed{!}$ as we will prove in the preceding subsection.

Proposition. (a) The following maps are isomorphisms on π_0

$$\mathcal{G}(\mathcal{V}_{\text{sc}}, G_{\text{sc}}/N) \underset{\subseteq}{\xrightarrow{u_1}} \mathcal{G}(\mathcal{V}_{\text{sc}}, (G_{\text{sc}}/N)^{\text{aff}}) \xrightarrow{u_2} \mathcal{G}(\mathcal{V}_{\text{sc}}) \xrightarrow{u_3} \mathcal{G}(H_{\text{ad}}) \underset{\supseteq}{\xleftarrow{u_4}} X_*(H_{\text{ad}}).$$

(We will denote the component corresponding to $\lambda \in X_*(H_{\text{ad}})$ by the index λ.)

(a') For instance,

$$\mathcal{G}(\mathcal{V}_{\text{sc}}) \underset{\cong}{\leftarrow} \mathcal{G}(Hsc)_0 \times \mathcal{G}(\mathcal{V}_{\text{sc}})_{red} \quad \text{and} \quad \mathcal{G}(\mathcal{V}_{\text{sc}})_{red} \cong \mathcal{G}(G_{\text{ad}}).$$

Also,

$$[\mathcal{G}(\mathcal{V}_{\text{sc}})_\lambda]_{red} \cong \mathcal{G}(G_{\text{ad}})_\lambda] \quad \text{and} \quad \mathcal{G}(Hsc)_0 \times \mathcal{G}(\mathcal{V}_{\text{sc}})_{red} \xrightarrow{\cong} \mathcal{G}(\mathcal{V}_{\text{sc}}).$$

(b) $\mathcal{G}(\mathcal{V}_{\text{sc}}, (G_{\text{sc}}/N)^{\text{aff}})$ is the filtration $\sqcup_{\lambda \in X_*(H_{\text{ad}})} \overline{\mathcal{S}_\lambda}$ of $\mathcal{G}(G_{\text{ad}})$. More precisely, the map $\mathcal{G}(\mathcal{V}_{\text{sc}}) \to \mathcal{G}(G_{\text{ad}})$ induces

$$\mathcal{G}(\mathcal{V}_{\text{sc}}, (G/N)^{\text{aff}})_\lambda \xrightarrow{\cong} \overline{\mathcal{S}_\lambda} \quad \text{and} \quad \mathcal{G}(\mathcal{V}_{\text{sc}}, (G/N)^{\text{aff}}) \times_{\mathcal{G}(H_{\text{ad}})} L_\lambda^{H_{\text{ad}}} = [\mathcal{G}(\mathcal{V}_{\text{sc}}, (G/N)^{\text{aff}})_\lambda]_{red} \xrightarrow{\cong} \overline{\mathcal{S}_\lambda}.$$

Also, $[\mathcal{G}(\mathcal{V}_{\text{sc}}, (G/N)^{\text{aff}})_\lambda]_{red} = \mathcal{G}(\mathcal{V}_{\text{sc}}, (G_{\text{sc}}/N)^{\text{aff}}) \times_{\mathcal{G}(H_{\text{ad}})} L_\lambda^{H_{\text{ad}}}$ is identified with $\overline{\mathcal{S}_\lambda}$.

Proof. (a) We have $G/N = \mathcal{V}/\underline{\Delta}_B$, so the first object is $\mathcal{G}(\mathcal{V}.\mathcal{V}/uDe_B) \cong \mathcal{G}(\underline{\Delta}_B) \cong \mathcal{G}(B_{\text{ad}})$ and the composition $u_3u_2u_1$ is $\mathcal{G}(B_{\text{ad}}) \to \mathcal{G}(H_{\text{ad}})$. so $\pi_0(u_3u_2u_1)$ is an isomorphism since so is $\pi_0(B_{\text{ad}}) \to \pi_0(H_{\text{ad}})$. Also, $\pi_0(u_3)$ is a bijection by the lemma 3.4.2.a. The same calim for the fourth map is just $\pi_0(\mathcal{G}(H)) = \pi_1(H)$.

By now we know that $\pi_0(u_2u_1)$ is bijective so $\pi_0(u_1)$ is injective. However, for $G = G_{\text{sc}}$ the inclusion $u_1 : \mathcal{G}(\mathcal{V}, \mathcal{V}/\underline{\Delta}_B) \subseteq \mathcal{G}[\mathcal{V}, (\mathcal{V}/\underline{\Delta}_B)^{\text{aff}}]$ is dense **??**, so $\pi_0(u_1)$ is also surjective. NO??? $\square$[29] $\blacksquare$. ???

xxx

A. *The connected components* $\mathcal{G}(\mathcal{V}, (G/N)^{\text{aff}})_\lambda$ *of* $\mathcal{G}(\mathcal{V}, (G/N)^{\text{aff}})$. First, recall that the embedding $\mathcal{G}(\mathcal{V}, G_{\text{sc}}/N) \hookrightarrow \mathcal{G}(\mathcal{V}) \hookrightarrow \mathcal{G}[(\mathcal{V}/Z(G)]$ is the diagonal $\mathcal{G}(B_{\text{ad}}) \overset{\delta}{\hookrightarrow} \mathcal{G}(G_{\text{ad}}) \times \mathcal{G}(H_{\text{ad}})$ (lemma 3.4.2.b). In particular, $\mathcal{G}(\mathcal{V}, G/N) \subseteq \mathcal{G}(\mathcal{V})$ embedds as $\mathcal{G}(B_{\text{ad}}) \subseteq \mathcal{G}(G_{\text{ad}})$ composed with the diagonal $\mathcal{G}(G_{\text{ad}}) \overset{id \times 1}{\hookrightarrow} \mathcal{G}(G_{\text{ad}}) \times \mathcal{G}(H_{\text{ad}})$.

The ideal $\mathfrak{p}_- \overset{\text{def}}{=} z^{-1}\mathcal{O}_-$ in $\mathcal{O}_- \overset{\text{def}}{=} \Bbbk[z^{-1}]$ defines the negative congrunce subgroups of loop groups $A_{\mathfrak{p}_-} \overset{\text{def}}{=} \text{Ker}(A_{\mathcal{O}_-} \to A_{\mathcal{O}_-/\mathfrak{p}_-} \subseteq A_{\mathcal{O}_-} \subseteq A_{\mathcal{K}}$. In particular, we have $T_{\mathfrak{p}_-} \xrightarrow{\cong} \mathcal{G}(T)_0$. Now, we can write the connected component of $\mathcal{G}(B_{\text{ad}})$ corresponding to $\lambda \in X_*(H)$ as

$$\mathcal{G}(B_{\text{ad}})_\lambda \;=\; \mathcal{G}(T_{\text{ad}})_\lambda \cdot \mathcal{G}(N) = (T_{\text{ad}})_{\mathfrak{p}_-} \cdot z^{-\lambda} \cdot S_0^{G_{\text{ad}}}.$$

So, inside $\mathcal{G}(G_{\text{ad}}) \times \mathcal{G}(H_{\text{ad}})$ we have

$$\delta[\mathcal{G}(B_{\text{ad}})_\lambda] \;=\; (\underline{\Delta}_T)_{\mathfrak{p}_-} \cdot (z^{-\lambda}, z^{-\lambda}N) \cdot (S_0^{G_{\text{ad}}} \times L_0^{H_{\text{ad}}}] \;=\; (\underline{\Delta}_T)_{\mathfrak{p}_-} \cdot (S_\lambda^{G_{\text{ad}}} \times L_\lambda^{H_{\text{ad}}}).$$

[29] $\boxed{!}$?

Now, $\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})$ is the closure of $\mathcal{G}(\mathcal{V},G/N)$ inside $\mathcal{G}(G_{\mathrm{ad}})\times\mathcal{G}(H_{\mathrm{ad}})$ (lemma 3.4.2). Since $(\underline{\Delta}_T)_{\mathfrak{p}_-}$ is an indfinite indscheme, the closure of $\mathcal{G}(\mathcal{V},G/N)_\lambda=(\underline{\Delta}_T)_{\mathfrak{p}_-}\cdot(S^{G_{\mathrm{ad}}}_\lambda\times L^{H_{\mathrm{ad}}}_\lambda)$ inside $\mathcal{G}(G_{\mathrm{ad}})\times\mathcal{G}(H_{\mathrm{ad}})$ is

$$\overline{\mathcal{G}(\mathcal{V},G/N)_\lambda}=\ (\underline{\Delta}_T)_{\mathfrak{p}_-}\cdot\overline{(S^{G_{\mathrm{ad}}}_\lambda\times L^{H_{\mathrm{ad}}}_\lambda)}=\ (\underline{\Delta}_T)_{\mathfrak{p}_-}\cdot(\overline{S^{G_{\mathrm{ad}}}_\lambda}\times L^{H_{\mathrm{ad}}}_\lambda).$$

The multiplication here is clearly free, i.e., this is isomorphic to the product $(\underline{\Delta}_T)_{\mathfrak{p}_-}\times\overline{S^{G_{\mathrm{ad}}}_\lambda}$. Since all $\overline{\mathcal{G}(\mathcal{V},G/N)_\lambda}$ are disjoint we see that $\overline{\mathcal{G}(\mathcal{V},G/N)_\lambda}$ is a connected component of $\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})$, we denote it $\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})_\lambda$.

B. We see that

$$\begin{aligned}\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})_\lambda\ &=\ \mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})\cap[\mathcal{G}(G_{\mathrm{ad}})\times\mathcal{G}(H_{\mathrm{ad}})_\lambda]\ =\ (\underline{\Delta}_T)\mathfrak{p}_-\cdot(\overline{S^{G_{\mathrm{ad}}}_\lambda}\times L^{H_{\mathrm{ad}}}_\lambda)\\ &=\ (\underline{\Delta}_T)\mathfrak{p}_-\cdot\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})_\lambda\xleftarrow[\cong]{}\mathcal{G}(\underline{\Delta}_T)_0\times\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})_\lambda.\end{aligned}$$

Also,

$$\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})\cap[\mathcal{G}(G_{\mathrm{ad}})\times L^{H_{\mathrm{ad}}}_\lambda]\ =\ \overline{S^{G_{\mathrm{ad}}}_\lambda}\times L^{H_{\mathrm{ad}}}_\lambda\ \cong\ \mathcal{G}(G,(G/N)^{\mathrm{aff}})_\lambda.$$

Clearly the map $pr_1:\mathcal{G}(G_{\mathrm{ad}})\times\mathcal{G}(H_{\mathrm{ad}})\to\mathcal{G}(G_{\mathrm{ad}})$ identifies $\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})_\lambda$ with $\overline{\mathcal{S}^{G_{\mathrm{ad}}}_\lambda}$ which has the corresponding presentation as $T_{\mathfrak{p}_-}\cdot\overline{S^{G_{\mathrm{ad}}}_\lambda}\xrightarrow{\cong}\mathcal{G}(T)_0\times\overline{S^{G_{\mathrm{ad}}}_\lambda}$ (since $T_{\mathfrak{p}_-}$ is indfinite and $\mathcal{G}(B)\xleftarrow[\cong]{}\mathcal{G}(T)\times\mathcal{G}(N)$). □

3.5. **Closures of $G_{\mathcal{O}}$-orbits and the Vinberg semigroup $\overline{\mathcal{V}}$.**

3.5.1. **Compare with Joel's "reduceness" conjecture.** ?

3.5.2. *The Vinberg semigroup $\overline{\mathcal{V}}$.* □[30]■

Lemma. (a) The largest commutative quotients of $\mathcal{V}$ and $\overline{\mathcal{V}}$ are H_{ad} and $\overline{H}_{ad}$. So, there is a canonical map of pairs which we call the determinant map

$$(\overline{\mathcal{V}},\mathcal{V})\xrightarrow{\det}(\overline{H}_{ad},H_{\mathrm{ad}}).$$

Remark. When $G=SL(U)$ then $(\overline{\mathcal{V}},\mathcal{V})\xrightarrow{\det}(\overline{H}_{ad},H_{\mathrm{ad}})$ is really the determinnat map $(\mathrm{End}(U),GL(U))\to(\mathbb{A}^1,G_m)$ (see 3.12).

Proof. The claim for $\mathcal{V}$ is obvious since $H_{\mathrm{ad}}=\mathcal{V}/G_{\mathrm{sc}}$ and G_{sc} is semisimple. The map $\overline{\mathcal{V}}\to\overline{H}_{ad}$ is □[31]■ still to be constructed.

□

3.5.3. *The indscheme $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$.*

[30] $\boxed{!}$ Some basic facts should be written here from section 1 The following lemma is in 8.5
[31] $\boxed{!}$?

Lemma. (a1) $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$ is closed in $\mathcal{G}(\mathcal{V})$ hence also in $\mathcal{G}[\mathcal{V}/Z(G)]=\mathcal{G}(G_{\rm ad})\times\mathcal{G}(H_{\rm ad})$.

(a2) $\mathcal{G}(\mathcal{V},\mathcal{V})=$ pt is open in $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$ (however, it is not dense).

(b) $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})\subseteq\mathcal{G}(\mathcal{V})$ is $(G_{\rm ad})_{\mathcal{O}}$-invariant.

Proof. (a1) The first claim is because $\overline{\mathcal{V}}$ is affine. The second because $\mathcal{G}(\mathcal{V})$ is closed in $\mathcal{G}[\mathcal{V}/Z(G)]$ (it is also open, i.e., a union of connected components).

(a2) $\mathcal{G}(\mathcal{V},\mathcal{V})=$ pt is open in $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$ It is not dense because the boundaru of $\mathcal{V}$ in $\overline{\mathcal{V}}$ is not in codimension two.

(b) The open $\mathcal{V}$-orbit $Y^o=\mathcal{V}/A$ in $Y=\overline{\mathcal{V}}$ is $\mathcal{V}$ itself, hence $A=1$ and therefore $\dot{A}=\mathcal{V}$ and $\ddot{A}=\mathcal{V}_{\mathcal{O}}$. However, $H_{\mathcal{O}}$ acts trivially on $\mathcal{G}(\mathcal{V})$ so this is really just the invariance under the quotient $(G_{\rm ad})_{\mathcal{O}}$ of $\mathcal{V}_{\mathcal{O}}$. $\square$

3.5.4. *The toric semigroup $\overline{\mathcal{T}}=\overline{\mathcal{T}}$.* A Cartan T of G defines a Cartan $\mathcal{T}\overset{\rm def}{=}T{\cdot}H$ of $\mathcal{V}$ and $\mathcal{T}\cong\ (T\times H)/Z(G)$.

Lemma. (a) $\mathcal{G}(\mathcal{V})^T=\ \mathcal{G}(T{\cdot}H)$ contains $\mathcal{G}(\mathcal{V})^{T\mathcal{R}}=\ X_*(TH)\cong\ \check{P}\times_{\check{P}/X_*(H)}\check{P}$. The map from $\mathcal{G}(\mathcal{V})^{T\mathcal{R}}$ to $\pi_0[\mathcal{G}(\mathcal{V})]\xrightarrow{\cong}\pi_0[\mathcal{G}(H_{\rm ad})]\cong\check{P}$ is the second projection $(L^G_\mu,L^H_\lambda)\mapsto\lambda$.

(b) A cocharacter $(\mu,\lambda)\in\check{P}\times_{\check{P}/\check{Q}}\check{P}=X_*(TH)$ extends to $\overline{G_m}\to\overline{TH}$ iff[32]

$$-\lambda \text{ is dominant and } -\lambda\geq w(-\mu) \text{ for } w\in W.$$

Proof. (a) Since H acts trivially on $\mathcal{G}(\mathcal{V})$ the T-fixed points are the same as $\mathcal{G}(\mathcal{V})^{T\cdot H}=\mathcal{G}(T{\cdot}H)$. Also, $\mathcal{G}(TH)^{\mathcal{R}}=\ X_*(TH)$ maps injectively to $X_*TH/Z(G)=X_*(T_{\rm ad}\times H_{\rm ad})=\check{P}\times\check{P}$ and the image is given by the requirement that the two components have the same image in $Z(G)=\check{P}/X_*(H)$. (If G is simply connected this is $\check{P}\times_{\check{P}/\check{Q}}\check{P}$.)

The last claim is because $\mathcal{G}(TH)\to\mathcal{G}(H_{\rm ad})$ is the second projection.

(b) Since $\overline{\mathcal{V}}$ is $\mathrm{End}_{\mathbb{H}}((G/N)^{\rm aff})$, a cocharacters (μ,λ) of $TH\subseteq\mathcal{V}$ extends $\overline{G_m}\to\overline{\mathcal{V}}$ iff the action of G_m on G/N corresponding to (μ,λ) extends to an action of $\overline{G_m}$ on $(G/N)^{\rm aff}$.

For a choice of a frame v_i of $(V^G_{\omega_i})^N$, and $\eta\in X_*(T)$, te element $\eta(s)*1_GN$ of G/N corresponds to a collection of vectors $\eta(s)v_i=\omega_i(\eta(s)){\cdot}v_is^{\langle\omega_i,\eta\rangle}){\cdot}v_i$ for $i\in I$. This extends to an orbit of $\overline{G_m}$ iff $\langle\omega_i,\eta\rangle\geq 0$, i.e., iff $\eta\in\check{Q}^+$.

For a cocharacter (μ,λ) let us choose $v\in W$ so that μ contracts $N'=^v N$, For $w\in W$ an element of $N'w^{-1}B/N$ can be written as uxN with $u\in N'$ and $x\in N_G(T)$ a representative of w^{-1}. Then

$$(\mu,\lambda)(s)*uxN=\ \mu(s){\cdot}uxN\lambda(s)^{-1}=\ {}^{\mu(s)}u{\cdot}x{\cdot}(w\mu)(s){\cdot}\lambda(s)^{-1}N.$$

[32] This funny statement is what is needed in order to match the convention $L_\lambda\overset{\rm def}{=}z^{-\lambda}G_{\mathcal{O}}$ from (1.3 for parametrizing $\mathcal{G}(G)^{T\mathcal{R}}$.

In order to get rid of the minus we would have to consider the (H,G)-action on $N\backslash G$.

As $s\to 0$, we have ${}^{\mu(s)}u\to 1$, hence $(\mu,\lambda)(s)*uxN$ converges in $(G/N)^{\text{aff}}$ if $(w\mu-\lambda)(s)\cdot N$ does, i.e., iff $w\mu-\lambda\in\check{Q}_+$.[33] Therefore, the action of G_m via (μ,λ) extends iff $\lambda\le w\mu$ for all $w\in W$. $\square$

3.5.5. *Connected components of* $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$.

Proposition. (a) The connected components of $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$ are paremeterized by dominant $\lambda\in X_*(H_{\text{ad}})$ by

$$\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})_\lambda \overset{\text{def}}{=} \mathcal{G}(\mathcal{V},\overline{\mathcal{V}})\times_{\mathcal{G}(H_{\text{ad}})}\mathcal{G}(H_{\text{ad}})_\lambda \;=\; \mathcal{G}(\mathcal{V},\overline{\mathcal{V}})\cap\;\mathcal{G}(\mathcal{V})_\lambda.$$

Their reduced parts are identified via the map $\mathcal{G}(\mathcal{V})\to\mathcal{G}(G_{\text{ad}})$ as

$$[\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})_\lambda]_{red}\xrightarrow{\cong}\overline{\mathcal{G}_\lambda(G_{\text{ad}})}.$$

Proof. (a1) Remember that the map $\pi_0[\mathcal{G}(\mathcal{V})]\xrightarrow{det}\pi_0[\mathcal{G}(H_{\text{ad}})]=X_*(H_{\text{ad}})=\check{P}$ is a bijection. The extension to a map $(\overline{\mathcal{V}},\mathcal{V})\xrightarrow{\text{det}}(\overline{H}_{ad},H_{\text{ad}})$ shows that $\overline{\mathcal{V}}$ lies above $\overline{H}_{ad}\subseteq H_{\text{ad}}$, hence in the union of all connected components $\mathcal{G}(\mathcal{V})_\lambda$ of $\mathcal{G}(\mathcal{V})$ such that λ is dominant.

(a2) According to the lemma 3.5.4.b, $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})^{T\mathcal{R}}=\mathcal{G}(TH,\overline{\mathcal{T}})^{\mathcal{R}}$ consists of all (L^G_μ,L^H_λ) such that λ is dominant and $\lambda\ge w\mu,\ w\in W$.

The first projection $\mathcal{G}(G_{\text{ad}})\times_{\mathcal{G}}(H_{\text{ad}})\to\mathcal{G}(G_{\text{ad}})$ identifies $\mathcal{G}(\mathcal{V})\times_{\mathcal{G}(H_{\text{ad}})}L^{H_{\text{ad}}}_\lambda$ with a connected component of $\mathcal{G}(G_{\text{ad}})$. We know that for λ dominant $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})\times_{\mathcal{G}(H_{\text{ad}})}L^{H_{\text{ad}}}_\lambda\hookrightarrow\mathcal{G}(G_{\text{ad}})_{\overline{\lambda}}$ is closed and $G_{\mathcal{O}}$-invariant (lemma 3.5.3) . So, from the description of its $T\mathcal{R}$-fixed points we see that its reduced part is $\overline{\mathcal{G}_\lambda(G_{\text{ad}})}$. So,

$$[\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})_\lambda]_{red}\;=\;\overline{\mathcal{G}_\lambda(G_{\text{ad}})}\times L^{H_{\text{ad}}}_\lambda.$$

$\square$

3.5.6. *Langlands self-duality of the Vinberg (semi)group constructions.* The Langlands duality $G\leftrightarrow G^\vee$ for reductive groups in its Satake form extends trivially to a class of reductive semigroups $M\leftrightarrow\check{M}^\vee$ (whose invertible parts are G and $\check{G}$). For this one just replaces $\mathcal{G}(\mathcal{G})$ with $\mathcal{G}(G,M)\subseteq\mathcal{G}(G)$ (which we think of as the “Loop Grassmannian of M”) and the category of perverse sheaves $\mathcal{P}_{G_{\mathcal{O}}}[\mathcal{G}(G)]$ with $\mathcal{P}_{G_{\mathcal{O}}}[\mathcal{G}(G,M)]$.

[33] In terms of $(G/N)^{\text{aff}}\subseteq\mathbb{V}\overset{\text{def}}{=}\oplus_{i\in I}V^G_{\omega_i}$, the image of $\lim_{s\to 0}(\mu,\lambda)(s)*uxN$ in $V^G_{\omega_i}$ is $\lim_{s\to 0}(\mu,\lambda)(s)*uxv_i=xv_i$ if $\mu-\lambda\perp\omega^i$ and it is zero otherwise. So, the limit is a projector to an extremal weight space of $V^G_{\omega_i}$,

Corollary. For a semisimple Lie algebra $\mathfrak{g}$ and its Langlands dual $\check{\mathfrak{g}}$ the Vinberg groups $\mathcal{V}_{\mathfrak{g}}$ and $\mathcal{V}_{\check{\mathfrak{g}}}$ are Langlands dual. The same holds for the Vinberg semigroups $\overline{\mathcal{V}}_{\mathfrak{g}}$ and $\overline{\mathcal{V}}_{\check{\mathfrak{g}}}$.

Proof. First, the tori $TH_{\mathfrak{g}}$ and $TH_{\check{\mathfrak{g}}}$ are dual. For this we know that $X = X_*(TH)$ is $\check{P}\times_{\check{P}/\check{Q}}\check{P}$ so it appears in $0 \to X \to \check{P}\oplus\check{P} \stackrel{-}{\to} \check{P}/\check{Q} \to 0$. This dualizes to $0 \to (\check{P}\oplus\check{P})^* \stackrel{-}{\to} X^* \to (\check{P}/\check{Q})^* \to 0$, i.e., $0 \to (Q\oplus Q \stackrel{-}{\to} X^* \to P/Q \to 0$ Now we can see that X^* is $P\times_{P/Q}P = X_*(TH_{\check{\mathfrak{g}}})$ and then also the duality claim for Vinberg groups.

The standard representations of the dual of $\overline{\mathcal{V}}_{\mathfrak{g}}$ are according to the proposition parameterized by all pairs of dominant (μ,λ) in $\check{P}\times_{\check{P}/\check{Q}}\check{P}$ such that $\mu \leq \lambda$. This is precisely the description of $\overline{\mathcal{V}}_{\check{\mathfrak{g}}}$. $\square$

3.5.7. *The nonreduced directions of $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$ [Junk].*

Questions. (a) (?) For a dominant $\lambda \in X_*(G)$, the map $p_* : \mathcal{G}(\mathcal{V},\overline{\mathcal{V}}) \to \mathcal{G}(G_{\mathrm{ad}})$ induces

$$\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})_\lambda \to \overline{\mathcal{G}(G)_\lambda}.$$

(b) $\mathcal{G}(\overline{\mathcal{V}})$ is not reduced. $\square$[34] $\blacksquare$

So, for λ dominant $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}}) \cap \mathcal{G}(\mathcal{V})_\lambda = \mathcal{G}(\mathcal{V},\overline{\mathcal{V}})\times_{\mathcal{G}(H_{\mathrm{ad}})} \mathcal{G}(H_{\mathrm{ad}})_\lambda$ contains

So, for λ dominant $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}}) \cap \mathcal{G}(\mathcal{V})_\lambda = \mathcal{G}(\mathcal{V},\overline{\mathcal{V}})\times_{\mathcal{G}(H_{\mathrm{ad}})} \mathcal{G}(H_{\mathrm{ad}})_\lambda$ contains

(c) We <u>know</u> that $\mathcal{G}(\mathcal{V}) \to \mathcal{G}(G_{\mathrm{ad}})$ induces for any $\lambda \in X_*(H_{\mathrm{ad}}) = \pi_0(\mathcal{V})$ and its image $\overline{\lambda} \in X_*(H_{\mathrm{ad}})/X_*(H) = \pi_0[\mathcal{G}(G_{\mathrm{ad}})]$, the isomorphism $[\mathcal{G}(\mathcal{V})_\lambda]_{red} \xrightarrow{\cong} \mathcal{G}(G_{\mathrm{ad}})_{\overline{\lambda}}$.

(d) ??? So, $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}}) \cap [\mathcal{G}(\mathcal{V})\times_{\mathcal{G}(H_{\mathrm{ad}})} L_\lambda]$ is the union of $G_{\mathcal{O}}$-orbits $\mathcal{G}_\mu = G_{\mathcal{O}}\cdot L_\mu \subseteq \mathcal{G}(G_{\mathrm{ad}})_{\overline{\lambda}}$ such that L_μ is contained in $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$. ...

Remark. Now positivity conventions agree for all parts of the theory and for the $SL2$ example, while the $SL3$ example currently has the opposite convention.

3.6. **The loop Grassmannian $\mathcal{G}(G_{\mathrm{ad}},\overline{G_{\mathrm{ad}}})$ of the wonderful compactification $\overline{G_{\mathrm{ad}}}$.**

3.6.1. *The loop Grassmannian $\mathcal{G}(G_m,\mathbb{P}^1)$.*

Lemma. $\mathcal{G}(G_m,\mathbb{P}^1)$ is the union of two copies of the semigroup $\mathcal{H}_d$ (at $0,\infty \in \mathbb{P}^1$) where the zeros in semigroups are identified. In particular $Z \xrightarrow{\cong} \mathcal{G}(G_m,\mathbb{P}^1)_{red} \xrightarrow{\cong} \pi_0[\mathcal{G}(G_m,\mathbb{P}^1)]$.

Proof. Clearly, $\mathcal{G}(G_m,\mathbb{A}^1)$ and $\mathcal{G}(G_m,\mathbb{P}^1 - 0)$ cover $\mathcal{G}(G_m,\mathbb{P}^1)$. Each is isomorphic to $\mathcal{H}_d$ and they meet at $\mathcal{G}(G_m,G_m) = \mathrm{pt}$ which is the zero in both semigroups. $\square$

[34] $\boxed{!}$

Remark. For any local space such as $\mathcal{G} * G_m, \mathbb{P}^1)$ there is fusion in homology and in K-homology. Notice that these only see the reduced part $\mathbb{Z}$.

Here, the fusion does not work on the level of sets or categories (such as the coherent sheaves).[35]

3.6.2. *Vinberg semigroup and the wonderful compactification.* Let G be semisimple and $\overline{G}$ be the wonderful compactification of G_{ad}.

Lemma. $\overline{G}$ carries a canonical $\mathcal{V}$-equivariant H_{sc}-torsor $\overline{\mathcal{V}}^0$ which is quasiaffine. Its affinization is the Vinberg semigroup $\overline{\mathcal{V}}$.

Proof. The $G\times G$-stratification of $\overline{G}$ is given by intersections of smooth irreducible G-invariant divisors $D_i,\ i \in I$. These define G_m-torsors $\mathcal{E}_i$ over $\overline{G}$ and an H_{sc}-torsor $\overline{\mathcal{V}}^o \to \overline{G}$ which is the product $(\prod/\overline{G})_{i\in I}\ \mathcal{E}_i$. Since the divisors D_i are invariant under $\mathcal{V}$, so is the H_{sc}-torsor $\overline{\mathcal{V}}^o$ over $\overline{G}$.

... □

3.6.3. $\mathcal{G}(G, \overline{G})$ *for the wonderful compactification* $\overline{G}$. This will be calculateed using the results for $\mathcal{G}(\mathcal{V}, \overline{\mathcal{V}})$.

Lemma. (a) There is a canonical map $\mathcal{G}(G_{\text{ad}}, \overline{G}) \to \mathcal{G}(H_{\text{ad}}, \overline{H_{\text{ad}}})$. This is an isomorphism on sets of connected components $\pi_0[\mathcal{G}(G_{\text{ad}}, \overline{G}) \xrightarrow{\cong} \pi_0[\mathcal{G}(H_{\text{ad}}, \overline{H_{\text{ad}}})] \cong X_*(H_{\text{ad}}) \underset{\mathcal{S}et}{/} W$.

(b) The reduced part of the connected component $\mathcal{G}(G, \overline{G})_\lambda$ corresponding to the orbit $W\lambda$ in $X_*(H_{\text{ad}})$ is the $G_{\mathcal{O}}$-orbit $\mathcal{G}_\lambda(G)$.

(c) The map $\mathcal{G}(G, \overline{G}) \to \mathcal{G}(G) \times \mathcal{G}(H_{\text{ad}}, \overline{H_{\text{ad}}})$ is a locally closed embedding. For a dominant coweight $\lambda = \sum \lambda_i \check{\omega}_i$, the connected component $\mathcal{G}(G, \overline{G})_\lambda$ is a product of $\mathcal{G}_\lambda(G)$ and $\mathcal{G}(H_{\text{ad}}, \overline{H_{\text{ad}}})_\lambda \cong \prod_{i\in I} \mathcal{H}^{\lambda_i}_{d\times I}$.

Proof. Since $\mathcal{G}(G, \overline{G}) = \mathcal{G}(G_{\text{ad}}, \overline{G}) \cap_{\mathcal{G}(G_{\text{ad}}} \mathcal{G}(G)$ we can suppose that $G = G_{\text{ad}}$.

(a) We can rewrite $\mathcal{G}(G, \overline{G})$ in terms of the extension $\mathcal{V}$ of $G = G_{\text{ad}}$ and the $\mathcal{V}$-equivariant H_{sc}-torsor $\overline{\mathcal{V}}^o$ over $\overline{G}$ (lemma 3.6.2):

$$\mathcal{G}(\mathcal{V}, \overline{\mathcal{V}}^o) = \mathcal{G}(\mathcal{V}/H_{\text{sc}}, \overline{\mathcal{V}}^o/H_{\text{sc}}) = \mathcal{G}(G_{\text{ad}}, \overline{G}).$$

Now, $\overline{\mathcal{V}}^o$ is open in its affinization $\overline{\mathcal{V}}$ and there is a canonical map (lemma 8.0.1)

$$\mathcal{G}(\mathcal{V}, \overline{\mathcal{V}}) \to \mathcal{G}(H_{\text{ad}}, \overline{H_{\text{ad}}})$$

such that ....

[35] For "sets", if $s, u \in \Bbbk$ are nilpotent then at $\{x, 0\} \in \mathcal{H}_{\mathbb{A}^1}$ we have a map $f_x : (C, C - \{0, x\}) \to (\mathbb{P}^1, G_m)$ such that in the coordinates u, v on $\mathbb{P}^1$, at 0 and ∞ (related by $uv = 1$), it is given by $u = z - x + s$ near x and $v = z + u$ near 0. Then as $x \to 0$ in $\mathbb{A}^1$, the map f_x does not converge in $\mathcal{G}(G_m, \mathbb{P}^1)$. □[36] ■

This gives a map $\mathcal{G}(G_{\mathrm{ad}},\overline{G}) = \mathcal{G}(\mathcal{V},\overline{\mathcal{V}}^o)\subseteq = \mathcal{G}(\mathcal{V},\overline{\mathcal{V}}) \to\mathcal{G}(\mathcal{V}/H_{\mathrm{ad}},\overline{\mathcal{V}}^o/H_{\mathrm{ad}})$.

$\overline{G} \to\mathrm{pt}/H_{\mathrm{sc}}$. which gives The H_{sc}-torsor $\overline{\mathcal{V}}^o$ over $\overline{G}$ gives a map $\overline{G} \to\mathrm{pt}/H_{\mathrm{sc}}$. which gives $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}}) \to\mathcal{G}(H_{\mathrm{sc}},\overline{H_{\mathrm{sc}}})$. $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}}^o) =$
$GG(\mathcal{V}/H_{\mathrm{sc}},\overline{\mathcal{V}}^o/H_{\mathrm{sc}}) = \mathcal{G}(G_{\mathrm{ad}},\overline{G}) \to\mathcal{G}(H_{\mathrm{sc}},\overline{H_{\mathrm{sc}}})$. gives □

Corollary. (b) For a Cartan T of G, the connected components of the fixed point set $\mathcal{G}(G,\overline{G})^T$ are parameterized by $X_*(H)$ so that each L_λ ($\lambda \in X_*(T)$) lies in a single connected component $[\mathcal{G}(G,\overline{G})^T]_\lambda$. It lies in the connected component $\mathcal{G}(G,\overline{G})_\lambda$. of $\mathcal{G}(G,\overline{G})$.

(c) If $\lambda = \sum_{i\in I} l_i\sigma_i\check{\omega}_i$ with $l_i \in \mathbb{N}$ and $\sigma \in \{\pm1\}$ then $[\mathcal{G}(G,\overline{G})^T]_\lambda$ is canonically isomorphich to the connected component $\mathcal{H}_{d\times I})^l$ of the colored Hilbert scheme $\mathcal{H}_{d\times I}$. So, for each Weyl chamber $\mathcal{C}$ in $X_*(H_{\mathrm{ad}})$, the part $[\mathcal{G}(G,\overline{G})^T]_{\mathcal{C}}$ is isomorphic to $\mathcal{H}_{d\times I}$ and has a structure of a semigroup.

Proof. (b$_1$) We have $\mathcal{G}(G,\overline{G})^T = \mathcal{G}(T,\overline{G}_T)$ where $\overline{G}_T$ is the closure of the T-orbit $T{\cdot}a = T$ in $\overline{G}$. For the wonderful compactification the factorization $T \xrightarrow{\prod\ \omega_i\cong} \prod_{i\in I}\ G_m$ induces $\overline{G}_T \overset{\cong}{\to} (\mathbb{P}^1)^I$. So, the claim follows from the lemma 3.6.1.

(a$_1$) We already know that $X_*(T)\subseteq\mathcal{G}(G,\overline{G})$. Also, $\mathcal{G}(\overline{G})$ is invariant under $\ddot{A}$ for the stabilizer A of the point a in Y^o, Here $a = 1 \in G = Y^o$, so $A = 1$ and $\ddot{A} = G_{\mathcal{O}}$. Therefore, $\mathcal{G}(G,\overline{G})$ contains each $G_{\mathcal{O}}$-orbit $G_{\mathcal{O}}{\cdot}L_\lambda = \mathcal{G}_\lambda$.

(c) There is a canonical map □

Example. $G = SL_2$. The SL_2 example is done later in details in 3.12 and that of matrices in 3.13 !

Besides $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$ and $\mathcal{G}(G,\overline{G})$ I am also wondering about $\mathcal{G}(GL(V),V)$? However, for SL_2. V is just $(G/N)^{\mathrm{aff}}$

□[37] ■

Remark. $\mathcal{G}(H_{\mathrm{ad}},\overline{H_{\mathrm{ad}}}) \cong \mathcal{H}_{C\times I}$.

3.7. **Quasimaps.** These are $\mathcal{G}(G,\overline{\mathcal{B}}) = \mathcal{G}(G{\times}H,(G/N)^{\mathrm{aff}})$ [Beilinson]. [How do they depend on $Z(G)$? In the adjoint case can they be written using the Vinberg group?]

3.8. **Closures of orbits of $\ddot{U} = U_{\mathcal{K}} \ltimes L_{\mathcal{O}}$.** For $P = B$ we use $(G/N)^{\mathrm{aff}}$ and for $P = G$ we use $\overline{\mathcal{V}} = \mathrm{End}_H((G/N)^{\mathrm{aff}})$.

In general we have a torsor $G/U \to G/P'$ for $P'/U = (P/U)'$, We factor it as $G/U\twoheadrightarrow G/N\twoheadrightarrow G/P'$ with fibers N/U and P'/N, then the relative affinization $(G/U)^{\mathrm{aff}}_{G/N}$

[37] $\boxed{!}$Maybe in 8.5 ? or in section 1

has fibers endomorphism bundle $\mathrm{End}[(G/U)^{\mathrm{aff}}_{G/N}]$ which is the Vinberg semigroup of P'/U ??? , then the

3.8.1. $\ddot{U}$ acts on $\mathcal{G}(G,G/U)=\mathcal{G}(U)\subseteq\mathcal{G}(G)$. Here, $L_{\mathcal{O}}$ acts by conjugation.

This suffices for $P=B$. However, we want to act on a larger subspace which is an orbit of the connected component $(P_{\mathcal{K}})_0=(L_{\mathcal{K}})_0U_{\mathcal{K}}$ of $P_{\mathcal{K}}$. or of orbit of

3.8.2. The first step may be to get to closres of orbits of $(P_{\mathcal{K}})_0$ since these contain the closures of orbits of $\ddot{U}$.

$\mathcal{G}(P)=\mathcal{G}(G,G/P)=\mathcal{G}(\mathcal{V},\mathcal{V}/P)$. For the cloure thewe are candidates for $\overline{\mathcal{P}}$ as

$$(G/P')^{\mathrm{aff}}/(P/P') and (G/U)^{\mathrm{aff}}/(P/U).$$

One has $G/U\to G/P'$ and $P/U\to P/P'$, hence

$$\overline{\mathcal{P}}_{\bullet}\overset{\mathrm{def}}{=}(G/U)^{\mathrm{aff}}/(P/U)\ \to G/P')^{\mathrm{aff}}/(P/P')\overset{\mathrm{def}}{=}\ \overline{\mathcal{P}}_{\cdot}.$$

3.9. **Intersections.**

3.9.1. *Orbits of a subgroup $\mathfrak{V}\subseteq G_{\mathcal{K}}$ on $\mathcal{G}(G)$.* For a subgroup $\mathfrak{V}$ of $G_{\mathcal{K}}$ we would like to realize the (closures) of orbits of $\mathfrak{V}$ on $\mathcal{G}(G)$ in terms of some mapping space $\mathcal{G}(\mathcal{U},Y)$ related to G and $\mathfrak{V}$.

Here, we choose the group $\mathcal{U}$ by combining G with a torus H' such that $X_*(H')$ parametrizes the $\mathfrak{V}$-orbits. This gives a Vinberg type group $\mathcal{U}'=G\times_{Z(G)}H'$.

Next, its semigroup closure $\overline{\mathcal{U}}'$ acts on some space Y'. For each $\nu\in X_*(H')$ we get a connected component $\mathcal{G}(\mathcal{U}',Y')_\nu$ of $\mathcal{G}(\mathcal{U}',Y')$ and its reduced part should be (the closure of) the orbit $\mathfrak{V}\cdot L^G_\nu$ in $\mathcal{G}(G)$.

Notice that $\mathcal{U}'/H'=G_{\mathrm{ad}}$.

3.9.2. *Intersections.* The intersections in $\mathcal{G}(G_{\mathrm{ad}})$ of several kinds of orbits or their closures (say $\cap_{i\in I}\ \mathfrak{V}_iL_{\nu_i}$), are then given by

$$\prod_i\ [\mathcal{G}(\mathcal{U}_i,Y_i)\to\mathcal{G}(G_{\mathrm{ad}})]\ =\ \mathcal{G}(\prod_i\ \mathcal{U}_i/G_{\mathrm{ad}},\prod_i\ Y_i/G_{\mathrm{ad}})$$

(see lemma 2.2.1.b2. Notice that $\prod_i\ \mathcal{U}_i/G_{\mathrm{ad}}$ has a Cartan $T\times_{Z(G)}\prod\ H_i$.

Remark. So, the interesting objects are now the orbit $(\mathcal{U}'\times_{G_{\mathrm{ad}}}\mathcal{U}'')a$ for $a=(a',a'')\in Y'\times Y''$ and its closure, as well as the same for $TH'H''a$ for the Cartan subgroup of $\mathcal{U}'\times G_{\mathrm{ad}}\mathcal{U}'$.

Example. The space $\sqcup_{\lambda,\nu}\ \overline{\mathcal{G}_\lambda}\cap\overline{S_\nu}$ has two parameters λ and ν in $X_*(H)$. So, in order to build its classifying space we replace $\mathcal{U}$ by $\mathcal{U}_2=\mathcal{U}\times_{G_{\mathrm{ad}}}\mathcal{U}$ and then the above space of intersections:

$$\overline{\mathcal{G}_\lambda}\cap\overline{S_\nu}\ =\ \mathcal{G}(\mathcal{U}_2,\overline{\mathcal{V}}\times(G/N)^{\mathrm{aff}})\times_{\mathcal{G}({H_{\mathrm{ad}}}^2)}\ L_{\mu,\nu}.$$

Remark. The same for $\sqcup\ \overline{S^+_\lambda}\cap\overline{S^-\mu}$. However, when we fix $\lambda=0$ then we have hust one parameter ν and $\mathcal{U}$ suffices.

3.10. **Zastavas.** This is Drinfeld's description of zastava spaces as moduli of maps from a curve

3.10.1. *Zastava spaces 1.* $\overline{S_0}\cap\overline{\mathcal{S}^-_\mu}\ \supseteq\ \overline{S_0}\cap\overline{S^-_\mu}$. Loop Grassmannian $\mathcal{G}(G\times H)$ embedds into $\mathcal{G}[\Delta_{Z(G)}\backslash(G\times H)]=\ \mathcal{G}(\mathcal{V})$ as a union of some of the connected components. Here, the map $\pi_1(G\times H)\to\pi_1(\mathcal{V})$, i.e., $\pi_1(H)\to\pi_1(H_{\mathrm{ad}})$, is the inclusion $\check{Q}\subseteq\check{P}$.

Let $G\times H$ act on $(G/N)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}}$ by $(g,h)(x,y)=(gx,gyh^{-1})$.

Lemma. For this action of $G\times H$

$$\overline{S_0}\cap[\sqcup_{\mu\in X_*(H)}\ \overline{\mathcal{S}^-_\mu}]\ =\ \mathcal{G}[G\times H,(G/N)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}}]\ =\ \mathcal{G}[G,(G/N)^{\mathrm{aff}}\times\overline{(G/N^-)^{\mathrm{aff}}}/H].$$

$$\overline{S_0}\cap\overline{S^-_\mu}=\ \mathcal{G}(G\times H,(G/N)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}})\times_{\mathcal{G}(H)}L^H_\mu.$$

Proof. The intersection in $\mathcal{G}(G)$ of $\overline{S_0}$ and $\sqcup_{\mu\in X_*(H)}\ \overline{\mathcal{S}^-_\mu}$ is

$$\mathcal{G}(G,(G/N)^{\mathrm{aff}})\ \times_{\mathcal{G}(G,\mathrm{pt})}\ \mathcal{G}(G\times H,(G/N^-)^{\mathrm{aff}})$$

$$\cong\ \mathcal{G}(G\times_G(G\times H),(G/N)^{\mathrm{aff}}\times_{\mathrm{pt}}(G/N^-)^{\mathrm{aff}})\ =\ \mathcal{G}(G\times H,(G/N)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}})$$

where the action of $G\times H=G\times_G(G\times H)$ on $(G/N)^{\mathrm{aff}}\times_{(G/N^-)^{\mathrm{aff}}})$ is just as stated above. One can also write $\mathcal{G}(G\times H,(G/N)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}})$ as

$$\mathcal{G}([G\times H]/H,(G/N)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}}/H)=\ \mathcal{G}(G,(G/N)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}}/H).$$

Finally, for $\mu\in\check{Q}$, $\overline{S_0}\cap\overline{S^-_\mu}$ equals

$$\mathcal{G}(G,(G/N)^{\mathrm{aff}})\times_{\mathcal{G}(G,\mathrm{pt})}[\mathcal{G}(G\times H,(G/N^-)^{\mathrm{aff}})\times_{\mathcal{G}(H)}L^H_\mu]=\ \mathcal{G}(G\times H,(G/N)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}})\times_{\mathcal{G}(H)}L^H_\mu.$$

$\square$

xxx

3.10.2. *Zastava spaces 2.* $\overline{S_0}\cap\overline{\mathcal{S}^-_\mu}\ \supseteq\ \overline{S_0}\cap\overline{S^-_\mu}$. Let $\mathcal{V}$ act on $(G/N)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}}$ by $(g,h)(x,y)=(gx,gyh^{-1})$.

Lemma. For this action of $\mathcal{V}$

$\overline{S_0}\cap[\sqcup_{\mu\in X_*(H_{\text{ad}})}\overline{\mathcal{S}_\mu^-}] \;=\; \mathcal{G}(\mathcal{V},(G_{\text{ad}}/N)^{\text{aff}}\times(G/N^-)^{\text{aff}}) \;=\; \mathcal{G}(G_{\text{ad}},(G_{\text{ad}}/N)^{\text{aff}}\times\overline{(G/N^-)^{\text{aff}}}/H).$

$$\overline{S_0}\cap\overline{S_\mu^-} = \;\mathcal{G}(\mathcal{V},(G/N)^{\text{aff}}\times(G/N^-)^{\text{aff}})\times_{\mathcal{G}(H_{\text{ad}})}L^H ad_\nu.$$

Proof. The intersection in $\mathcal{G}(G_{\text{ad}})$ of $\overline{S_0}$ and $\sqcup_{\mu\in X_*(H_{\text{ad}})}\overline{\mathcal{S}_\mu^-}$ is

$$\mathcal{G}(G_{\text{ad}},(G_{\text{ad}}/N)^{\text{aff}})\times_{\mathcal{G}(G_{\text{ad}},\text{pt})}\mathcal{G}(\mathcal{V},(G/N^-)^{\text{aff}})$$

$$\cong\;\mathcal{G}(G_{\text{ad}}\times_{G_{\text{ad}}}\mathcal{V},(G_{\text{ad}}/N)^{\text{aff}}\times_{\text{pt}}(G/N^-)^{\text{aff}}) \;=\; \mathcal{G}(\mathcal{V},(G_{\text{ad}}/N)^{\text{aff}}\times(G/N^-)^{\text{aff}})$$

where the action of $\mathcal{V}=G_{\text{ad}}\times_{G_{\text{ad}}}\mathcal{V}$ on $(G_{\text{ad}}/N)^{\text{aff}}\times_{(G/N^-)^{\text{aff}}})$ is just as stated above. This can be written as $\mathcal{G}(\mathcal{V},(G_{\text{ad}}/N)^{\text{aff}}\times(G/N^-)^{\text{aff}}) = \mathcal{G}(\mathcal{V}/H,(G_{\text{ad}}/N)^{\text{aff}}\times(G/N^-)^{\text{aff}}/H) = \mathcal{G}(G_{\text{ad}},(G_{\text{ad}}/N)^{\text{aff}}\times(G/N^-)^{\text{aff}}/H)$.

Then

$\overline{S_0}\cap\overline{S_\mu^-} = \mathcal{G}(G_{\text{ad}},(G/N)^{\text{aff}})\times_{\mathcal{G}(G_{\text{ad}},\text{pt})}\mathcal{G}(\mathcal{V},(G/N^-)^{\text{aff}})\times_{\mathcal{G}(H_{\text{ad}})}L^H ad_\nu = \mathcal{G}(\mathcal{V},(G/N)^{\text{aff}}\times(G/N^-)^{\text{aff}})\times_{\mathcal{G}(H_{\text{ad}})}L$

$\square$

Remark. $\square$[38] $\blacksquare$ $\overline{(G/N^-)^{\text{aff}}}/H)$ differs from $\overline{(G_{\text{ad}}/N^-)^{\text{aff}}}/H_{\text{ad}})$ since $Z(G)$ does not act freely on $(G/N^-)^{\text{aff}}$.

3.10.3. *The filtration of $\overline{S_0}$ by intersections $\overline{S_0}\cap\overline{S_\alpha^-}$, $\alpha\le 0$.*

Theorem. [Drinfeld] $\sqcup_{\alpha\le 0}\overline{S_0}\cap\overline{S_\alpha^-}$ is given by the classifying space...

Proof. The space $\cup_{\alpha\in\check{Q}^+}\overline{S_\alpha}\cap\overline{S_0}$ is the fibered product over $\mathcal{G}(G)=Map[(d,d^*),(BG,\text{pt})]$ of $Map[(d,d^*),(G\backslash(G/N)^{\text{aff}},\text{pt})$ and $Map[(d,d^*),(\mathcal{V}\backslash(G/N)^{\text{aff}},\text{pt})]_{red}$, i.e., the space of maps from (d,d^*) to the fibered product

$$G\backslash(G/N)^{\text{aff}}\times_{G\backslash\text{pt}}G\backslash[(G/N)^{\text{aff}}/H] \;=\; G\backslash[(G/N)^{\text{aff}}\times_{\text{pt}}(G/N)^{\text{aff}}/H].$$

$\square$.

3.10.4. *Corollary.* The zastava spaces $Z(G,N,N^-)$ is the moduli $\mathcal{G}(G,(G/N^+)^{\text{aff}}\times(G/N^-)^{\text{aff}}/H)$.

This is the same as the moduli $\mathcal{M}_\mathcal{Y}$ for Drinfeld's compactification of a point

$$\mathcal{Y}\;\overset{\text{def}}{=}\;G\backslash[(G/N^+)^{\text{aff}}\times(G/N^-)^{\text{aff}}/H].$$

This global modul extends the intersections $\overline{S_0}\cap\overline{S_\alpha^-}$, $\alpha\le 0$.

3.10.5. *Orbits in $G/N^+\times G/N^-$.* The space $G/N^+\times G/N^-$ has the origin $a=(N,N^-)$. Let $\mathcal{V}$ act on $G/N^+\times G/N^-$ by $(g,h)*(\alpha,\beta)=(g\alpha,g\beta h^{-1})$. Let us also recall the Bruhat decomposition of $N^+\backslash G/N^-$ into "cells" $K_w\overset{\text{def}}{=}N^+\backslash(N^+wB^-/N^-,\ w\in W(T)$.

[38] $\boxed{!}$ CHECK which one should appear in $\overline{\mathcal{S}_\mu}$

Lemma. (a) The orbits of $\mathcal{V}$ in $G/N^+{\times}G/N^-$ are indexed by W via $w{\mapsto}\,\mathcal{O}_w \stackrel{\text{def}}{=} \mathcal{V}{\cdot}(N, wN^-)$.

The isomorphismm $G\backslash[G/N^+{\times}G/N^-]\xrightarrow{\cong}N^+\backslash G/N^-$ (by $(\alpha,\beta)\mapsto\alpha^{-1}\beta$) identifies the quotient $G\backslash\mathcal{O}_w$ with the Bruhat cell $K_w \stackrel{\text{def}}{=} N^+\backslash(N^+wB^-/N^-$. The open orbits correspond to $w=1$.

(b) The $\mathcal{V}$-orbit $\mathcal{V}{\cdot}a{\subseteq}G/N^+{\times}G/N^-$ consists of all $(\alpha,\beta)\in G/N^+{\times}G/N^-$ such that αH, βH meet (equivalently, α meets βH or αH meets β. This is open in $G/N^+{\times}G/N^-$.

The G-orbit $G{\cdot}a{\subseteq}G/N^+{\times}G/N^-$ consists of (α,β) such that α and β meet. It is a G-torsor.

(c) The boundary $\partial\mathcal{Y}_G^o$ of the dense point in $\mathcal{Y}_G$ is a divisor D with irreducible components D_i parameterized by the vertices I of the Dynkin diagram.

Proof. (c) follows from (a) and the same statement for the Bruhat cells in $N^+\backslash G/N^-$. Here, $D_i=\mathcal{O}_{s_i}$. □

Corollary. Zastava space $Z_C(G)$ has a canonical structure of a local space over C. The structure map $\pi: Z_C(G)\to\mathcal{H}_{C,I}$ to the Hilbert scheme of I-colored points of C is

$$\pi(f) \ \stackrel{\text{def}}{=}\ f^{-1}(\partial\mathcal{Y}^o).$$

Proof. If a map $f:C\to\mathcal{Y}_G$ visits $\mathcal{Y}_G^o$ then the pull back $\pi(f)\ \stackrel{\text{def}}{=}\ f^{-1}\partial\mathcal{Y}_G^o$ is an effective divisor D in the curve C. Moreover, it is I-colored, i.e., a system $f^{-1}D_i$ of finite subscheme of C indexed by $i\in I$.

The locality structure for a disjoint union $D = D'{\sqcup}D''$ is the gluing map $Z_C(G)_{D'}{\times}Z_C(G)_{D''}\to Z_C(G)_D$ which takes maps f',f'' to unique f such that $f=f''$ off D' and $f=f'$ off D''. □

3.10.6. *Appendix. Some objects related to zastavas.* On the global level each parabolic P defines a partial compactification $Bun_C(G,P,M)$ of $Bun_C(P)$, defined through the diagram $G{\supseteq}P{\twoheadrightarrow}M$ for the reductive group $M=P/P_1$.

For any subgroup $V{\subseteq}G$ we consider the normalizer $N_G(V)$, its quotient $M_V=N_G(V)/V$ and its center $\mathcal{Z}_V$. Then $G{\times}_{Z(G)}M_V$ maps to $\mathcal{V}_V \stackrel{\text{def}}{=} \mathrm{Aut}_{\mathcal{Z}_V}(G/V)$. If G/V is quasi-affine we also get the semigroup $\overline{\mathcal{V}}_V \stackrel{\text{def}}{=} \mathrm{End}_{\mathcal{Z}_V}((G/V)^{\mathrm{aff}})$ whose invertible part is $\mathcal{V}_V = \mathrm{Aut}_{\mathcal{Z}_V}(G/V)=\mathrm{Aut}_{\mathcal{Z}_V}((G/V)^{\mathrm{aff}})$.

This construction appears in [**?**] when V is related to some parabolic $P=U\ltimes U$. Here, V is either U or P', hence its normalized if P and $M=P/V$. When P is a Borel B and $V=N$ we get the Vibnerg semigroup $\mathcal{V}$.

3.10.7. *Appendix. Twisting by a P/V-torsor.* Furthermore, a torsor Q for P/V over C, defines a twisted form $[(G/V)^{\mathrm{aff}}{\times}(G/V^-)^{\mathrm{aff}}/H_-]^Q$ over C. The corresponding zastava space

$$\mathcal{G}(G,[(G/V)^{\text{aff}}\times(G/V^{-})^{\text{aff}}/H_{-}]^{Q}$$

is the space of sections of $[(G/V)^{\text{aff}}\times(G/V^{-})^{\text{aff}}/H_{-}]^{Q}$ over C.

3.10.8. *Appendix. The paper* [**?**]. This setting was studied in [**?**] when V is either the unipotent radical U of P or its derived subgroup P'. In the first case the stalks of intersection homology sheaves were computed. That paper use the Pluecker description ("coordinate description") of zastava spaces.

The case $\mathcal{V}=U$ also appears below. The affinizations of G/U and $G/[P,P]$ are studied in the notes on affinization 6.

3.11. **Intersections $\overline{\mathcal{G}_{\lambda}}\cap\overline{S_{\nu}}$.**

3.12. **Example $G=SL_2$.** Let U be a two dimensional vector space and $G=SL(U)$.

We consider the structural results in 3.12.1–3.12.3. We start woth the notation (3.12.2) and the action of $B\subseteq G$ on $(G/N)^{\text{aff}}\cong U$ (3.12.3).

For the Vinberg group $\mathcal{V}=\mathcal{V}_{\text{sc}}=GL(U)$ we calculate $\mathcal{G}(\mathcal{V},(G/N)^{\text{aff}})$ and $\mathcal{G}(G\times H,(G/N)^{\text{aff}})$ in 3.12.4.

For U of any dimension and the pair $\mathcal{V}\subseteq\overline{\mathcal{V}}$ given by $GL(U)\subseteq\text{End}(U)$ we calculate $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$ in 3.13.

3.12.1. $G_{\text{sc}}=SL(U)$ *and* $G_{\text{ad}}=PGL(U)$. $\square$[39]$\blacksquare$

3.12.2. *Notation.* We choose a basis $(e_+,e_-)=(e,f)$ so that $G\cong\ SL_2\supseteq B=\left(\begin{smallmatrix}*&*\\0&*\end{smallmatrix}\right)=N{\cdot}T$ for $T=\left(\begin{smallmatrix}*&0\\0&*\end{smallmatrix}\right)$ and $N=\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)$. Let $\Delta^+=\{\alpha\}$ correspond to $\Delta_T(\mathfrak{n})$ via $T\xrightarrow{\cong}H$ defined by $T\subseteq B$. For $\rho=\alpha/2$ we have $\rho[\left(\begin{smallmatrix}a&0\\0&a^{-1}\end{smallmatrix}\right)N]=a$.

3.12.3. *Actions of $B\subseteq G$ on $(G/N)^{\text{aff}}\cong U$.* Group $G\times H$ acts on G/N by $(g,tN)*xN\overset{\text{def}}{=}g{\cdot}xN{\cdot}(tN)^{-1}=gxt^{-1}N$. This descends to an action of the Vinberg group $\mathcal{V}\overset{\text{def}}{=}Z(G)\backslash(G\times H)$ since the diagonal $\Delta_{Z(G)}\subseteq(ZG)^2$ acts trivially.

Lemma. (a) The affine closure of G/N is identified with U via the orbit map $\iota:G/N\to U,\ gN\mapsto\ g{\cdot}e$.

(b) The conjugation action of B on G/N gives a new structure of a B-module on U, isomorphic to the B-module $\mathfrak{g}/\mathfrak{n}$.

(c) The action of $\mathcal{V}$ on $(G/N)^{\text{aff}}$ gives $\mathcal{V}\xrightarrow{\cong}GL(V)$ which is identity on G and H is identified with $Z[GL(V)]\cong G_m$ by $\rho^{-1}:H\xrightarrow{\cong}G_m$.

---

[39] $\boxed{!}$ Need to see the difference in the map $\mathcal{G}[SL(U),U]\to\mathcal{G}[PGL(U),\mathcal{N}_2]$ to check general claims.

(d) The image $\underline{\Delta}_T$ of T in $\mathcal{V}$ by $T \ni t \mapsto (t, tN)$ is canonically isomorphic to T_{ad}. In terms of $GL(U)$ the group $\underline{\Delta}_T$ is $S \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} 1 & 0 \\ 0 & * \end{smallmatrix}\right)$. As a $\mathcal{V} \cong GL(V)$-space $G/N = U - 0$ is $\mathcal{V}/N\underline{\Delta}_T \cong GL(V)/NS$.

Proof. (a) The stabilizer of $e = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is $G_e = N$, so $\iota : G/N \to U - \{0\}$ is an isomorphism of G-spaces.

(b) The claim is that the action of G on G/N becomes the standard G-action on U and the H-action becomes via $\rho^{-1} : H \xrightarrow{\cong} G_m$ the standard action of G_m on a vector space.

Since $s = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) N \in H$ acts on G/N by $s * xN = x \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)^{-1} N$, So, the transported action on $u = xe \in U - 0$ is by

$$s * u = s * xe = x \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)^{-1} e = a^{-1} \cdot xe = a^{-1} \cdot u = \rho^{-1}(s) \cdot u.$$

(c) The new action of $b \in B$ on $u = xe \in U$ is $b \underset{new}{\cdot} xe \overset{\mathrm{def}}{=} ({}^b x)e = b \cdot x \cdot b^{-1} \cdot e = \rho_B(b)^{-1} \cdot bxe$. So, the new B-action is $U \otimes \rho_B{}^{-1}$, and this is non-canonically isomorphic to $\mathfrak{g}/\mathfrak{n}$ (both are indecomposable B-modules with the same weights $-2, 0$).

(d) The stabilizers of $1 \cdot N \in G/N$ in $\mathcal{V}$ and of $e \in U$ in $GL(U)$ are clearly $N\underline{\Delta}_T$ and NS. The map $T \to \underline{\Delta}_T \cong S$ sends $t = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in T$ to the image of (t, tN) in $GL(U)$ and this is $\rho^{-1}(t) \cdot t = a^{-1} \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a^{-2} \end{smallmatrix}\right)$. $\square$

3.12.4. *The spaces* $\mathcal{G}(G \times H, (G/N)^{\mathrm{aff}}) \subseteq \mathcal{G}(\mathcal{V}, (G/N)^{\mathrm{aff}}) = \mathcal{G}[GL(U), U]$. We have

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{V}, (G/N)^{\mathrm{aff}}) & \xrightarrow[\cong]{} & \mathcal{G}(G_{\mathrm{ad}}, (G/N)^{\mathrm{aff}}/H) & \xrightarrow{\overset{\mathrm{def}}{=}} & \mathcal{G}(G_{\mathrm{ad}}, \overline{\mathcal{B}}) \\ {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\ \mathcal{G}[GL(U), U] & \xrightarrow[\cong]{} & \mathcal{G}[PGL(U), G_m \backslash U]. & \xrightarrow{\overset{\mathrm{def}}{=}} & \mathcal{G}[PGL(U), \overline{\mathbb{P}^1}]. \end{array}$$

since $\mathcal{G}(\mathcal{V}, (G/N)^{\mathrm{aff}}) = \mathcal{G}(H \backslash \mathcal{V}, H \backslash (G/N)^{\mathrm{aff}}) = \mathcal{G}(G_{\mathrm{ad}}, (G/N)^{\mathrm{aff}}/H)$.

Lemma. (a) $\mathcal{G}(\mathcal{V}, (G/N)^{\mathrm{aff}}) = \mathcal{G}[GL(U), U]$ is the submoduli of $\mathcal{G}(\mathcal{V}) = \mathcal{G}(GL(U))$ consisting of all lattices $\mathcal{U}$ in the sheaf $U \otimes \mathcal{O}_{d^*}$ over d, such that $\mathcal{U}$ contains the constant section $e \in U$. (For $\mathcal{U} \in \mathcal{G}(\mathcal{V}, G/N)$ the condition is that $\mathcal{U} \cap \mathcal{K}e = \mathcal{O}e$.)

(b) Then $\mathcal{G}(G, (G/N)^{\mathrm{aff}}) = \mathcal{G}[SL(U), U]$ is the connected component $\mathcal{G}[GL(U), U]_0$, i.e., the submoduli given by $\mathcal{U} \in \mathcal{G}[SL(U)]$, i.e., by asking that the volume of the lattice $\mathcal{U}$ is zero.

(c) In $\mathcal{G}(SL(U))$ the closrues of semiinfinite orbits $\overline{S_{l\check{\rho}}}$ are the moduli of lattices L of volume zero such that $L \ni z^{-l}e$.

(d) Also, $\overline{S_0} \cap \overline{S^-_{m\check{\rho}}}$ is the moduli of L such that $L \ni e, z^m f$. This is $Gr_m[z^{-m}\mathcal{O}/\mathcal{O}e \oplus \mathcal{O}/z^m\mathcal{O}f]^z$.

Proof. (a) The elements of $\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})$ are pairs $(\mathcal{T},\tau)\in\mathcal{G}(\mathcal{V})$ such that for the stabilizer $A=NS\subseteq\mathcal{V}\cong GL(U)$ of the origin $1N=e$ in $Y=(G/N)^{\mathrm{aff}}\cong U$ the section $A\tau\in\Gamma(d^*,\mathcal{T})$ extends to $\Gamma(d,Y^{\mathcal{T}})$.

Here, $\mathcal{Y}^{\mathcal{T}}=\mathcal{T}^{-1}\times_{\mathcal{V}}U$ is a vector bundle $\mathcal{U}$ over d and $(Y^o)^{\mathcal{T}}$ is $\mathcal{U}-0$. One recovers $\mathcal{T}$ as the space of trivializations $Isom(U,\mathcal{U})$.

Now, NS acts on $\mathcal{T}=Isom(U,\mathcal{U})$ and the restriction of isomorphisms to $e\in U$ gives $NS\backslash Isom(U,\mathcal{U})\xrightarrow{\cong}\mathcal{U}-0$. We can think of τ as an element of $Isom_{d^*}(U,\mathcal{U})$ and of $\overline{\tau}=NS\tau$ as the section τe of $\mathcal{U}$ on d^* (which is invertible in the sense that $\mathcal{O}\overline{\tau}\subseteq\mathcal{U}$ is a line subbundle). Then the condition is that $\overline{\tau}$ extends to a section of $\mathcal{U}$ on d.

If we think of τ as the embedding $\tau^{-1}:\ \mathcal{U}\hookrightarrow U\otimes\mathcal{O}_{d^*}$ then the data $(\mathcal{T},\tau)$ are a lattice $\mathcal{U}$ in $U\otimes\mathcal{O}_{d^*}$ (a vector subbundle which is generically everything) and the condition is that $\mathcal{U}$ contains the section e (i.e., the subsheaf $\mathcal{O}e$).

(b) For a subgroup G' of G $\mathcal{G}(G,Y)\cap\mathcal{G}(G')=\mathcal{G}(G',Y_{\overline{G'}})$ (proposition 2.3.2.b). So, if $\overline{G'a}=\overline{Ga}$ then $\overline{G'a}=\overline{Ga}$, this is $\mathcal{G}(G,Y)\cap\mathcal{G}(G')=\mathcal{G}(G',Y)$,

Now, for the origin $a\in(G/N)^{\mathrm{aff}}$ we have $\mathcal{V}{\cdot}a=(G/N)^{\mathrm{aff}}=G{\cdot}a$, hence $\mathcal{G}(G,(G/N)^{\mathrm{aff}})=\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})\cap\mathcal{G}(G)$, i.e., the elements of $\mathcal{G}(G,(G/N)^{\mathrm{aff}})$ are just the elements of $\mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}})$ which lie in $GG(G)=\mathcal{G}(V)_0$, i.e., such that the lattice $\mathcal{U}$ has volume zero.

(c) The description of $\overline{S_{l\check{\rho}}}$ for $l=0$ is already in (a-b). The general l follows by ta shift $S_{l\check{\rho}}=z^{-l}S_0$.

(d) By conjugating with $s\in W-1$, we get that $\overline{S_{m\check{\rho}}}$ is given by $L\ni z^{-m}f$.

SL_2 fixes the standard symplectic structure ω on U. Using ω we see that $L\ni e,z^mf\Leftrightarrow L\supseteq\mathcal{O}e\oplus\mathcal{O}z^mf\Leftrightarrow L=L^{\perp}\subseteq(\mathcal{O}e\oplus\mathcal{O}z^mf)^{\perp}=\mathcal{O}f\oplus\mathcal{O}z^{-m}e$. So, $\overline{S_0}\cap\overline{S^-_{-m\check{\rho}}}$ consists of all subspaces $M=L/(\mathcal{O}e\oplus z^m\mathcal{O}f)$ of $z^{-m}\mathcal{O}/\mathcal{O}e\oplus\mathcal{O}/z^m\mathcal{O}f$ which are z-invariant and of dimension m. $\square$

3.12.5. *Zastavas.* $\square$[40]$\blacksquare$

3.13. **Example: Matrices.**

3.13.1. $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$.

Lemma. For any vector space U consider the action of $G=GL(U)$ on $Y=\mathrm{End}(U)$ by left multiplication. Then

$$\mathcal{G}[GL(U),\mathrm{End}(U)]\subseteq\mathcal{G}[GL(U)]$$

[40] $\boxed{!}$

is invariant under G_{in} (the global version of the group $G_{\mathcal{O}}$). It is the moduli of lattices $\mathcal{U}$ in $U\otimes\mathcal{O}_{\eta_C}$ which lie in the trivial lattice $\mathcal{O}_C$. (For $\dim(U)=2$ and $G=SL(U)$ this is $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$.)

Proof. For $(\mathcal{T},\tau)\in\mathcal{G}(G)$ let $\mathcal{U}=\mathcal{T}^{-1}\times_G U$ be the corresponding vector bundle. We have $Y^{\mathcal{T}}=\operatorname{Hom}(U,U)^{\mathcal{T}}=\operatorname{Hom}(U,U^{\mathcal{T}})=\operatorname{Hom}(U,\mathcal{U})$ and $(Y^o)^{\mathcal{T}}=Isom(U,\mathcal{U})$. The stabilizer A of the origin $1\in\operatorname{End}(U)$ is trivial and so $\ddot{A}=G_{\mathcal{O}}$ acts on $\mathcal{G}(Y,G)$.

So, $(\mathcal{T},\tau)$ is in $\mathcal{G}(G,Y)$ if the trivialization $\tau\in\Gamma[d^*,Isom(U,\mathcal{U})]$ extends to a d-section of $\operatorname{Hom}(U,\mathcal{U})$. In terms of $\tau^{-1}:\mathcal{U}\hookrightarrow U\otimes\mathcal{O}_{d^*}$ this means that $\mathcal{U}$ contains □[41]■ $U\otimes\mathcal{O}_d$. □

Question. □[42]■ Does this say that $\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})$ is in this case reduced – it seems to be literally a union of $G_{\mathcal{O}}$-orbits?

3.14. **Functions on asymptotic cones as intersection homology.** The G-geometry, i.e., the moduli $\mathcal{G}(G_1,Y_1)$ that corresponds to a given $\check{G}$-variety $\check{Y}$ seems easy to find. However, the problem is that its connected components should really be viewed as a *filtration* and the perverse sheaves on these components should than be viewed as an indsystem that really geometrizes $\mathcal{O}(\check{Y})$.

The second difficulty is that the appropriate sheaf may be a projective rather than just an IC sheaf (as wee see in the case $\check{Y}=\check{G}$).

Question. Does this prevent constructions through a small resolution or a perverse correspondence transform?

3.14.1. $\mathcal{G}(G,Y)$ *when* Y *is a semigroup closure of* G. In this casse the intersection homloogy of $\mathcal{G}(G,Y)$ has a commutative ring structure. Its spectrum is an affine $\check{G}$-variety.

Question. Is this the affine closure of a homogeneous space of $\check{G}$? □

Theorem. The intersection cohomology $IH[\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})]$ is the ring of functions on $\check{G}/\check{N}$, so its spectrum is $(\check{G}/\check{N})^{\mathrm{aff}}$.

Remark. So, affine G-variety $Y=(G/N)^{\mathrm{aff}}$ gives Vinberg group and semigroup $\mathcal{V}_Y\subseteq\overline{\mathcal{V}}_Y$. Then $IH[\mathcal{G}(\mathcal{V}_Y,\overline{\mathcal{V}}_Y)]$ is $\mathcal{O}(\check{Y})$ for $\check{Y}=(\check{G}/\check{N})^{\mathrm{aff}}$.

Question. This extends the L-duality of semigroup closures to L-duality of closures of homogenous spaces? Maybe of affine spherical varieties? Such Y defines its Vinberg pair $\mathcal{V}_Y\subseteq\overline{\mathcal{V}}_Y$, then $IH[\mathcal{G}(\mathcal{V}_Y,\overline{\mathcal{V}}_Y)]$ is $\mathcal{O}(\check{Y})$.

3.14.2. $\boldsymbol{a}\check{G}$. Let $\boldsymbol{a}\check{G}$ be the asymptotic cone of $\check{G}$.

[41] $\boxed{!}$check direction
[42] $\boxed{!}$

Lemma. $\mathcal{O}(\boldsymbol{a}\check{G})$ is approximately

$$IH[\mathcal{G}(\mathcal{V}\times_{H_{\rm ad}}\mathcal{V},\overline{\mathcal{V}}\times\overline{\mathcal{V}})].$$

[Yes as a vector space but not as convolution object in sheaves. In order to get algebra structure on $IH[\mathcal{G}(\mathcal{V}\times_{H_{\rm ad}}\mathcal{V},\overline{\mathcal{V}}\times\overline{\mathcal{V}})]$ We need to combine the convolution of $IC(\overline{\mathcal{G}_\lambda})$ and $IC(\overline{\mathcal{G}_\mu})$ with the projection to the quotient $IC(\overline{\mathcal{G}_{\lambda+\mu}})$.

Problems:

(1) $\mathcal{O}(\check{G})$ is an injective object in $Rep(\check{G})$. The correct construction is as in [MV] exvept that it shopuld be interpreted as coinduction.
(2) Since $\mathcal{O}(\check{G})$ is indeomposable ober integers (?), ot can not be naturally constructed on a disjoint union such as $\mathcal{G}(\mathcal{V}^2,Y)$.

So, the construction of functions on $\check{G}$-varieties in the form approximately $\mathcal{G}(K,\overline{K})$ only works for the $\check{G}$-varieties that are "asymptotic cones".

Question. How does one deform the asymtotic objects to the correct objects? (Asymptotic cones are correct in characteristic zero.)

3.15. **Appendix. Satake induction procedures.** This has not yet been written. An example is the construction of $\mathcal{O}(\check{G})$ as induced from $\mathcal{O}(cH)$ realized as $(X_*(H)\hookrightarrow\mathcal{G}(H))_*\Bbbk_{X_*(H)}$. Then one pulls back along $q:\mathcal{G}(B)\to\mathcal{G}(H)$. The final step from B to G uses $\gamma_{B_\mathcal{O}}^{G_\mathcal{O}}$.

(The induction needs to be in $K_\mathcal{O}$-equivariant sheaves. This is miraculously satisfied for q^* and it is exactly the construction for the step from B to G.)

Also, the the line bundle and the Poisson bivector should be induced from a Cartan. Interestingly, the passage from H to B seems weak but everything is recovered once one get to G.

3.16. **Appendix I. Several kinds of induction.** .

A. Induction of conditions (Y,a)**.** For $\iota:G'\to G$ one has the pullback functor $\iota^*:Sp_G\to Sp(G')$ and two direct image functors $\iota_*,\iota_*^{\rm aff}:Sp'_G\to Sp(G)$. Here, $\iota^*(Y,a)=(Y,a)$ where Y becomes a G'-space, while $\iota_*(Y',a')\stackrel{\rm def}{=}(G\times_{G'}Y',(1_G,a')$ and $\iota_*^{\rm aff}(Y',a')\ \stackrel{\rm def}{=}\ \big((G\times_{G'}Y')^{\rm aff},(1_G,a')\big)$. Here,

$$\mathcal{G}(G',Y')\ \cong\ \mathcal{G}(G,\iota_*(Y'))\subseteq\mathcal{G}(G,\iota_*^{\rm aff}(Y')).$$

For a parabolic $P=UL$ one has $G\stackrel{i}{\hookleftarrow}P\stackrel{q}{\twoheadrightarrow}\overline{P}=P/U\cong L$. The parabolic inductions functors are

$$_P{\rm Ind}_L^G\ \stackrel{\rm def}{=}\ i_*q^*\quad\text{and}\quad {_P{\rm AInd}_L^G}\ \stackrel{\rm def}{=}\ i_*^{\rm aff}q^*.$$

Example. We can view G/N as $G\times_N\mathrm{pt} \cong\ G\times_B(B/N)$, hence

$$G/N \ =\ {}_B\mathrm{Ind}_H^G(H) \quad\text{and}\quad (G/N)^{\mathrm{aff}} \ =\ {}_B\mathrm{AInd}_H^G(H).$$

This is used for a construction of the disjoint union of closures of $(B_{\mathcal{K}})_0$-orbits

$$\sqcup_\lambda\ \overline{\mathcal{S}_\lambda} \ =\ \mathcal{G}(\mathcal{V},(G/N)^{\mathrm{aff}}) \ =\ \mathcal{G}[\mathrm{Aut}_H((G/N)^{\mathrm{aff}}),\mathrm{End}_H((G/N)^{\mathrm{aff}})],$$

i.e.,

$$\sqcup_\lambda\ \overline{(B_{\mathcal{K}})_0{\cdot}L_\lambda} \ =\ \mathcal{G}\big(\mathrm{Aut}_H[{}_B\mathrm{AInd}_H^G(H)],\mathrm{End}_H[{}_B\mathrm{AInd}_H^G(H)]\big).$$

B. Construction of $\mathcal{O}(\check{G}/\check{N})=Coind_{\check{N}}^{\check{G}}[\mathcal{O}(\mathbf{pt})]=Coind_{\check{B}}^{\check{G}}[\mathcal{O}(H)]$ in terms of induction of conditions We have

$$\mathcal{O}(\check{G}/\check{N}) \ =\ IH[\mathcal{G}(\mathcal{V},\overline{\mathcal{V}})].$$

Here, $\mathcal{V}$ should appear because we want to separate closures of $G_{\mathcal{O}}$-orbits. Then the construction $\overline{\mathcal{V}}=\mathrm{End}_H((G/N)^{\mathrm{aff}})$ si a "natural way to restrict" the $\mathcal{V}$-Grassmannian.

However, for me it is still mysterious on the G-side, i.e., from the point of view of constructing the closures $G_{\mathcal{O}}$-orbits. On the other hand it seems natural from the $\check{G}$-side since "closures of $G_{\mathcal{O}}$-orbits" here means exactly the representations of $\check{\overline{\mathcal{V}}}$ (the perverse sheaves that appear on $\overline{\mathcal{G}_\lambda}$ are the representations of $\check{G}$ whose weights are dominated by λ). Then one notices that the representations of $\check{\mathbb{H}}\overline{\mathcal{V}}$ should be realized on the Grassmannian of the L-dual of $\check{\overline{\mathcal{V}}}$, i.e., of $\overline{\mathcal{V}}$.

Remark. Formally, $\overline{\mathcal{V}}$ is also a double centralizer of $\mathcal{V}$ in the action on $(G/N)^{\mathrm{aff}} = {}_B\mathrm{Ind}_H^G(H)$, i.e., in $\mathrm{End}((G/N)^{\mathrm{aff}})$. As it should be, $\mathcal{V}$ is dense in its double centralizer.

C. Satake constructions of functions of $\check{G},\check{G}/\check{N}$ and $\check{N}$. Here, $\mathcal{O}(\check{G})=Coind_1^{\check{G}}(\Bbbk)$ and $\mathcal{O}(\check{G}/\check{N})=Coind_{\check{N}}^{\check{G}}(\Bbbk)$.

Question. Can one combine the construction of $\mathcal{O}(\check{N})$ of Jared and of $Coind_{\check{N}}^{\check{G}}$ in terms of induxction of conditions to reconstruct $\mathcal{O}(\check{G})$?

D. The GS constructions of (parameters for) canonical bases. This one is "different" in the sense that???

E. Induction for conditions and Satake construction of (parts of) $\check{G}$. Induction of conditions should clarify the Satake constructions? In particular, it should suffice for closures of $\ddot{U}$-orbits.

F. IC shaf of $\mathcal{G}(G,Y^{\mathrm{aff}})$? The interesting spaces are the closures of orbits. The closure of $\mathcal{G}(G,Y){\subseteq}\mathcal{G}(G)$ seems to be $\mathcal{G}(G,Y^{\mathrm{aff}})$. So, one ends up computing the IC sheaf of $\mathcal{G}(G,Y^{\mathrm{aff}})$.

This may clarify the idea of the interseting IC sheaves being produced together with the spaces?

G. *From G to $\mathcal{V}$.* Fo a subgroup $\mathcal{C}$ of $G_{\mathcal{K}}$, in order to realize closures of orbits $\mathcal{C}{\cdot}L_\lambda{\subseteq}\mathcal{G}(G)$ for $\lambda \in X_*(H)$ we need to replace the group G with $G{\times}H$ (or more economically by $\mathcal{V} \overset{\text{def}}{=} G{\times}_{Z(G)}H$). ince the connected componnets $\mathcal{G}(H)_\lambda$ of $\mathcal{G}(H)$ are indexed by $\lambda \in X_*(H)$, The role of the factor $\mathcal{G}(H)$ in $\mathcal{G}(G{\times}H)$ is that it separates the closures of $\overline{\mathcal{C}{\cdot}L_\lambda}$ for different λ's in the sense that $\mathcal{G}(G){\times}H) = \sqcup_\lambda\ \mathcal{G}(G){\times}\mathcal{G}(H)_\lambda$ and one can realize $\overline{\mathcal{C}{\cdot}L_\lambda}{\subseteq}\mathcal{G}(G)$ in the copy $\mathcal{G}(G){\times}L^H_\lambda \subseteq \mathcal{G}(G){\times}\mathcal{G}(H)_\lambda$ of $\mathcal{G}(G)$.

3.16.1. *The closures of $S^P_\lambda = \ddot{U}{\cdot}L_\lambda$ for the unipotent radical U of a parabolic P.* When $P = B = TN$ then $U = N$ and $\ddot{N} = T_{\mathcal{O}}N_{\mathcal{K}}$ is the reduced part of $B(\mathcal{K})_0$. For the orbits $\mathcal{S}_\lambda$ of $(B_{\mathcal{K}})_0$ we have

$$\sqcup_{\lambda\in X_*(H)}\ \overline{\mathcal{S}^B_\lambda}\ =\ \mathcal{V}[\mathcal{V}, (G/N)^{\text{aff}}].$$

When $P = G$ then $U = 1$ and $\ddot{U} = G_{\mathcal{O}}$. We have

$$\sqcup_{\lambda\in X_*(H)_+}\ \overline{\mathcal{G}_\lambda}\ =\ \mathcal{G}(\mathcal{V}, \overline{\mathcal{V}}).$$

Therefore,

4. MV cycles and their T-fixed points

4.1. **Inclusion $\overline{\cap_{\tau_i}^{w_i}\ S_{\lambda_i}^{w_i}}\cap_{\tau_i}^{w_i}\ \overline{S_{\lambda_i}^{w_i}}$ may be proper [Zhijie].** Here $\underline{s} = (s_L, , ..., s_1)$ is a reduced decomposition of w_0 and $w_p = s_p \cdots s_1$, while $\lambda_\bullet$ is a Lusztig walk.

Example. In A_2 consider $\lambda = i + j$ and $\underline{s} = (s_1, s_2, s_1)$. We consider two Lusztig walks to λ corresponding to two irreducible components of $S_0 \cap\ S_\lambda^-$ $\alpha = (0, i, i, i + j)$ and $\beta = (0, 0, i + j, i + j)$. The corresponding MV-polytopes are the two triangles $P_\alpha = conv(0, i, i + j)$ and $P_\beta = conv(0, i + j, j)$.

(1) The Kamnitzer cycles for α, β are

$$S_\alpha^{\underline{s}} = \ S_0^1 \cap\ S_i^{s_1} \cap\ S_i^{s_2 s_1} \cap\ S_{i+j}^{s_1 s_2 s_1} \quad \text{and} \quad S_\beta^{\underline{s}} = \ S_0^1 \cap\ S_0^{s_1} \cap\ S_{i+j}^{s_2 s_1} \cap\ S_{i+j}^{s_1 s_2 s_1}.$$

He proves that they are open in MV-cycles C_α, C_β.

(2) On the other hand, the intersection of closures

$$\overline{S}_\alpha^{\underline{s}} = \ \overline{S}_0^1 \cap\ \overline{S}_i^{s_1} \cap\ \overline{S}_i^{s_2 s_1} \cap\ \overline{S}_{i+j}^{s_1 s_2 s_1}$$

is clearly larger than C_α since it contains the point j.

(3) Actually, this intersection of closures $\overline{S}_\alpha^{\underline{s}}$ seems to be the union $C_\alpha \cup C_\beta$, i.e., it is the same as the intersection of the first and last term $\overline{S}_0^1 \cap\ \overline{S}_{i+j}^{s_1 s_2 s_1}$?

4.2. $\overline{\mathcal{G}_\lambda}^T$**.** Denote the Cartan $T \times_{Z(G)} H$ in $\mathcal{V}$ by $\mathcal{T}$. Its closure in $\overline{\mathcal{V}}$ is a semigroup that we denote $\overline{\mathcal{T}}$.

Lemma. For any dominant $\lambda \in X_*(H)$ the T-fixed points in $\overline{\mathcal{G}_\lambda}$ are

$$\overline{\mathcal{G}_\lambda}^T \ = \ \mathcal{G}(\mathcal{T}, \overline{\mathcal{T}}) \times_{\mathcal{G}(H_{\mathrm{ad}})} L_\lambda.$$

Proof. We know that $\sqcup\ \overline{\mathcal{G}_\lambda}^T$ is (lemma **??**)

$$[\mathcal{G}(\mathcal{V}, \overline{\mathcal{V}})_{red}]^T \ = \ [\mathcal{G}(\mathcal{V}, \overline{\mathcal{V}}) \times_{\mathcal{G}(H_{\mathrm{ad}})} \mathcal{G}(H_{\mathrm{ad}})_{red}]^T \ = \ \mathcal{G}(\mathcal{V}, \overline{\mathcal{V}})^T \times_{\mathcal{G}(H_{\mathrm{ad}})} \mathcal{G}(H_{\mathrm{ad}})_{red}$$

and $\mathcal{G}(\mathcal{V}, \overline{\mathcal{V}})^T$ is $\mathcal{G}(Z_\mathcal{V}(T), \overline{\mathcal{V}})$ (by **??**). Since $Z_\mathcal{V}(T) = \mathcal{T}$ we have

$$\mathcal{G}(\mathcal{V}, \overline{\mathcal{V}})^T \ = \ \mathcal{G}(\mathcal{T}, \overline{\mathcal{V}}) \ = \ \mathcal{G}(\mathcal{T}, \overline{\mathcal{T} \cdot y})$$

where the base point in $\overline{\mathcal{V}}$ is $y = 1_{\overline{\mathcal{V}}}$, so the orbit $\mathcal{T} \cdot y$ is just $\mathcal{T}$ and its closure in $\overline{\mathcal{V}}$ is the semigroup $\overline{\mathcal{T}}$. Finally, $\mathcal{G}(H_{\mathrm{ad}})_{red} = X_*(H)$ gives $\sqcup\ \overline{\mathcal{G}_\lambda}^T \ = \ \mathcal{G}(\mathcal{T}, \overline{\mathcal{T}}) \times_{\mathcal{G}(H_{\mathrm{ad}})} X_*(H) = .$ $\square$

4.2.1. *T-fixed points in MV cycles.* Let C be an MV cycle with Lusztig coorinates $(\underline{s}, \underline{\lambda})$ and AK coordinates $\lambda_\bullet = (\lambda_w)_{w \in W}$.

Conjecture. C^T is described in terms of $\lambda_\bullet$ the "same" as $\overline{\mathcal{G}_\lambda}^T$ in terms of $W \cdot \lambda$.

4.2.2. Recall that for adjoint(?) G its Cartan group H has a semigroup closure $\overline{H} \cong (\overline{G_m})^I$ (with $H \cong (G_m)^I$).

Define the semigroup $H^2_{\leq}$ by asking that $x^{-1}y$ lies in $\overline{H}$. This means that ....

For each $w \in W$ we get $H^2_w = H^2_{\leq w}$ and then $\overline{H^2} \stackrel{\text{def}}{=} \cap_{w\in W}\ H^2_{\leq w}$. Notice that we can also define $\overline{T{\times}H}$ by using $\iota_B : T \xrightarrow{\cong} H$ given by (any) Borel B that contains T.

Lemma. The $\overline{\mathcal{T}{\times}H}$ defined here is a $Z(G)$-torsor over $\overline{\mathcal{T}}$ defined above. $\square$

We are really interested in a larger object based on $T{\times}H^{\mathcal{B}^T}$ for the T-fixed points torsor $\mathcal{B}^T$ for $W_T \stackrel{\text{def}}{=} N_G(T)/T$.

4.3. **Questions.**

5. **Odds**

5.1. **The "I-colored divisor map" $\mathbf{div}_I:\ \Gamma[(C,\eta_C),(\overline{P},P)]\ \to\ \mathcal{H}_{C\times I}$.** Any H-torsor P has an extension $\overline{P}\overset{\text{def}}{=}P\times_H\overline{H}$. When G is simply connected then the identification $\overline{H}\overset{\cong}{\to}\overline{G_m}^I$ defines the "I-colored divisor map"

$$\mathrm{div}_I:\ \Gamma[(C,\eta_C),(\overline{P},P)]\ \to \mathcal{H}_{C\times I}.$$

(The boundary $\partial P\overset{\text{def}}{=}\overline{P}-P$ is a divisor with normal crossings with irreducible components $\partial_i P$ indexed by $i\in I$.)

The divisor of a rational sections $\phi\in\Gamma(\eta_C,P)$ has less structure, i.e., $\mathrm{div}_I(\phi)$ is defined to lie in $\mathbb{Z}[|C|\times I]$. Here, ϕ extends to a regular section of $\overline{P}$ iff $\mathrm{div}_I(\phi)$ is effective.

5.2. **Supports.** A pointed stack $(\mathcal{Y},\mathrm{pt})$ will give a machinery (an algebro-geometric "sigma model") that associates to each source space X with a family Φ of *supports* the space $Map_\Phi(X,\mathcal{Y})$ of maps from X to $\mathcal{Y}$ supported in Φ. This is $\lim\limits_{\to\ S\in\Phi}\ Map_S(X,\mathcal{Y})$ where

$$Map_S(X,\mathcal{Y})\ \overset{\text{def}}{=}\ Map[(X,X-S),(\mathcal{Y},\mathrm{pt})].$$

Example. If Φ consists of all proper closed subvarieties then maps with Φ-support are the same as as *generically trivialized* maps $Map[(X,\eta_X),(\mathcal{Y},\mathrm{pt})]$.

If X is also a curve then it is the same as *finitely supported* or *generically trivialized.*

5.3. **The $\mathcal{R}_X$-family $\ddot{\mathcal{M}}^X_{\mathcal{Y}}$ of Hecke groupoids over** $Map(X,\mathcal{Y})$**.** For a space $\mathcal{Y}$ let $\ddot{\mathcal{M}}^X_{\mathcal{Y}}\to\mathcal{R}_C$ with fibers at $E\in\mathcal{R}_X$

$$\ddot{\mathcal{M}}^X(\mathcal{Y},\mathrm{pt})_E\ \overset{\text{def}}{=}\ Map(X,\mathcal{Y})\times_{Map(X-E,\mathcal{Y})}Map(X,\mathcal{Y}).$$

This is an $\mathcal{R}_X$-space familiy of groupoids over $Map(X,\mathcal{Y})$.

A point $\mathrm{pt}\to\mathcal{Y}$ defines the constant map $\mathrm{pt}_X:X\to\mathcal{Y}$ and then also the subfunctor $\mathcal{M}^X(\mathcal{Y},\mathrm{pt})$ with

$$\mathcal{M}^X(\mathcal{Y},\mathrm{pt})_E\ \overset{\text{def}}{=}\ Map(X,\mathcal{Y})\times_{Map(X-E,\mathcal{Y})}\mathrm{pt}_X\ =\ Map[(X,E-X),(\mathcal{Y}),\mathrm{pt})].$$

The locality of $\mathcal{M}^X_{\mathcal{Y}}\to\mathcal{R}_X$ means that the original groupoid is local in k^{th} variable for $k=1m2$, i.e., for $E^1,E^2\in\mathcal{R}_X$ disjoint we have (using $pr_k:\ddot{\mathcal{M}}^X_{\mathcal{Y},\mathrm{pt}}\times_{Map(X,\mathcal{Y})}$ each time)

$$\ddot{\mathcal{M}}^X(\mathcal{Y},\mathrm{pt})_{E^1}\times_{Map(X,\mathcal{Y})}\ddot{\mathcal{M}}^X_(\mathcal{Y},\mathrm{pt})_{E^2}\ \overset{\cong}{\to}\ \ddot{\mathcal{M}}^X_(\mathcal{Y},\mathrm{pt})_{E^1\sqcup E^2}\ .$$

Remark. The idea is that in order to reconstruct the group G, i.e., the representing object $(\mathbb{B}(G), \mathrm{pt})$ for $\mathcal{G}(G, \mathrm{pt})$, maube one should replace $\mathcal{G}(G, \mathrm{pt})$ with the corresponding Hecke groupoid $\ddot{\mathcal{G}}(G, \mathrm{pt})$?

Then the origin $L_0 \to \ddot{\mathcal{G}}(G)$ gives $\mathbb{B}(G_{\mathcal{O}})$ at least. Now one could take the part invariant under $\mathrm{Aut}(d)$ to get $\mathbb{B}(G)$?

Sounds good?

5.4. **Check 2.1.1(2).**

5.5.

Question. I would also like to see the corresponding constructions for global curves.

5.6. Improve the text. For instance for the stabilizer A of $a \in Y$ in G, we will relate in 2.3.4 the orbits of $N_G(A)_{\mathcal{O}} \cdot A_{\mathcal{K}} \subseteq G_{\mathcal{K}}$ in $\mathcal{G}(G, Y) \subseteq \mathcal{G}(G)$ to the Cartan fixed points.

5.7.

Example. The standard zastava spaces are not projective and $Y = (G/N)^{\mathrm{aff}} \times G/N$ is not affine (but Y^o is still $G \times H$). , i.e., $\mathrm{pt} \to \mathcal{Y}$ is open. (So, $A = 1$ and one is in the Drinfeld setting, i.e., in the local space setting.)

Remark. When $A = 1$ then $\mathcal{G}(G, Y)_E$ is $Map[(C, C - E), (Y, Y^o)]/G_{\mathcal{O}}$, i.e., $\widetilde{\mathcal{G}}(G, Y)$ is $Map[(C, \eta_C), (Y, Y^o)]/G_{\mathcal{O}}$. So, we can think of generalizing it by

- modifying the target (Y, Y^o) to $'Y,' Y^o$
- replacing the group G acting on the target by a groupoid G on $'Y$, compatible with Y^o.

Question. We know that $Y = U_+ \times U_-$ contains $\mathcal{V}$ as the open orbit. Does it contain $\overline{\mathcal{V}}$? $\mathcal{V}$ acts on Y, does $\overline{\mathcal{V}}$ act on Y, does it embedd into Y?

Example. In SL_2, is Y a torsor copy of the semigroup $\overline{\mathcal{V}}$? Here, a is a frame (e, f) of U and $Y^0 = \mathcal{V}a = GL(U)(e, f)$ is exactly the space of all frames of U.

5.7.1. *The maps $d\to Y=(G/N^+)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}}$.* These will have to be divided by $G_{\mathcal{O}}$ but for a moment we consider the maps themselves.

We consider the cases (1) when we exclude U_- and (2) when we keep U_- and allow D to go to ∞c.

[Here, (1) is different from (2) since $A=\Delta_B\neq 1$, so the map is not the only part of the data – we also have a torsor.]

1. When we exclude U_- we get all $u\in U_{\mathcal{O}}$ that can be completed to a frame of $U_{\mathcal{O}}$, i.e., such that $\mathcal{O}u\subseteq U_{\mathcal{O}}$ is a summand of $U_{\mathcal{O}}$.]

2. When we allow D to go to ∞c then we get all frames (u,v) of $U_{\mathcal{K}}$ with u as above.

So, when one divides by $G_{\mathcal{O}}$ one gets

2. all Lagrangian lattices that contain e (?)

1. the obvious part is $\mathcal{O}u$ but remember theat there is also a torsor $\mathcal{T}$.

5.7.2. *Twisting.* The zastava space is $Z=\mathcal{G}(G\times H,Y$ for $Y=(G/N^+)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}}$, with H acting only on the second factor. It can be viewed as $\mathcal{G}(G,\underline{Y})$ where $\underline{Y}$ lives over $\mathcal{H}_{C\times I}\times C$ and at $D\in\mathcal{H}_{C\times I}$ the fiber is

$$\underline{Y}_D\ =\ (G/N^+)^{\mathrm{aff}}\times(G/N^-)^{\mathrm{aff}}(D)/H.$$

Something like that?

6. **Appendix Generalities on loop Grassmannians**

6.0. **Summary.** We will describe $N_{\mathcal{K}}^{\pm}$-orbits $S_\alpha^\pm$ in $\mathcal{G}$, their closures $\overline{S_\alpha^\pm}$, the intersections $\overline{S_\alpha^+}\cap\overline{S_\beta^-}$ and the the T-fixed points in each of these (i.e., the intersections with the the loop Grassmannian $\mathcal{G}(T)$ of a Cartan).

6.0.1. *The loop Grassmannian of the multiplicative group G_m.* For the smooth formal disc d the commutative monoid S_d freely generated by the disc d is its Hilbert scheme $(\mathcal{H}_d,+)$ with the operation $+=\cup_d$ of the schematic union, i.e., the addition of divisors. The commutative monoid A_d freely generated by d is the loop Grassmannian group $\mathcal{G}(G_m)=A_d$. The map $S_d\to A_d$ is a close inclusion $\mathcal{H}_d\hookrightarrow\mathbb{G}(G_m)$. When we replace d with the pair (d,a) (where a is the center of d) then $S_{d,a}=A_{d,a}$ is the quotient of $\mathcal{G}(G_m)$ by $\mathcal{G}(G_m)^{\mathcal{R}}\cong\mathbb{Z}$, and the quotient map has a splitting which is the connected component $\mathcal{G}_0(G_m)$ of $\mathcal{G}(G_m)$.[43]

The embedding $\mathcal{H}_d\hookrightarrow\mathcal{G}(G_m)$ is by $\mathcal{H}_d\ni D\mapsto\mathcal{O}_d(D)\in\mathcal{G}(G_m)$. The image is the submonoid $\mathcal{G}(G_m,\overline{G_m})$ for the semigroup closure $\overline{G_m}=(\mathbb{A}^1,\cdot)$ of G_m. The inverse map $\mathcal{G}(G_m),\overline{G_m}\to\mathcal{H}_d$ is the divisor map div.[44]

6.0.2. *Inclusions $\mathcal{G}(N)\subseteq\mathcal{G}(B)\subseteq\mathcal{G}(G)$.* $\mathrm{div}_T:\mathcal{G}(T)\to\mathbb{Z}[I]=X_*(T)$. Notice that $\mathrm{div}_T(L_\lambda)=-\lambda$ since $L_0=(T\times d,1)$ and $L_\lambda\overset{\text{def}}{=}z^{-\lambda}L_0=(T\times d,z^{-\lambda})$.

The induction embeds $\mathcal{G}(B)$ as a subfunctor $\mathcal{G}(G,B)$ of $\mathcal{G}(G)$. It consists of all $x\in\mathcal{G}(G)$ which have a reduction x_B to B (such reduction is unique). We denote $\mathrm{div}_B(x)=\mathrm{div}_H(x_H)$ for the image x_H of x_B in $\mathcal{G}(H)$. Then the connected components of $\mathcal{G}(B)$ are $\mathcal{G}_\alpha(B)=\{\mathrm{div}_B=\alpha\}$.

The action of $\mathcal{G}(T)$ on $\mathcal{G}(B)$ gives identifications

$$\mathcal{G}(T)\times\mathcal{G}(N)\xrightarrow{\cong}\mathcal{G}(B)\quad\text{hence}\quad\mathcal{G}_\alpha(T)\times\mathcal{G}(N)\xrightarrow{\cong}\mathcal{G}_\alpha(B).$$

Here, $\mathcal{G}(N)$ is reduced and $\mathcal{G}_\alpha(T)_{red}$ is the point L_α.

6.0.3. *$N_{\mathcal{K}}^{\pm}$-orbits and their closures.* We denote $S_\alpha^\pm\overset{\text{def}}{=}N_{\mathcal{K}}^{\pm}\cdot L_\alpha$ and $N=N^+$, hence $S_\alpha=N_{\mathcal{K}}\cdot L_\alpha$.

The $N_{\mathcal{K}}$-orbits S_α form a stratification of $\mathcal{G}$. We have $\sqcup_\alpha S_\alpha\subseteq\mathcal{G}(G,B)\subseteq\mathcal{G}(G)$ and these are equalities for points over a field. Inside $\mathcal{G}(G,B)\cong\mathcal{G}(B)$, S_α is given by all x with $x_H=L_\alpha^H$.

The closures $\overline{S_\alpha}$ form a filtration of $\mathcal{G}$. We will only describe the closure $\overline{S_\alpha}$ inside $\mathcal{G}(G,B)$, here $\overline{S_\alpha}\cap\mathcal{G}(G,B)=\sqcup_{\beta\le\alpha}S_\beta$.

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43 So, $\mathcal{G}_0(G_m)$ is obtained by stabilizing the Hilbert scheme $(\mathcal{H}_d,+)$ with respect to a.

44 The divisor map is defined for any torus T as $\mathrm{div}_T:\mathcal{G}(T)\to X_*(T)$. It indexes the connected components $\mathcal{G}_\alpha(T)=\{\mathrm{div}_T=\alpha\}$ by $\alpha\in X_(T)$.

6.0.4. *T-fixed points in $\overline{S_\alpha}$, i.e., filtration of $\mathcal{G}(T)$ by $\overline{S_\alpha}$.* First, $\mathcal{G}(T)$ lies inside $\mathcal{G}(B) \cong \mathcal{G}(G,B)$. While $\mathcal{G}(T) \cap \cup_\alpha S_\alpha$ is just $\mathcal{G}(T)^{\mathcal{R}} = X_*(T)$, the exhaustive filtration $\overline{S_\alpha}$ of $\mathcal{G}(G)$ induces one on $\mathcal{G}(T)$.

6.0.5. *Intersections $\overline{S_\alpha^+} \cap \overline{S_\beta^-}$ and their T-fixed points.*

6.1. **Loop Grassmannian of G_m.** Inside $\mathcal{G}(G_m)$ we consider a generating submonoid $(\mathcal{H}_d, \cup_d)$. and for the center a of d we interpret the submonoid $z^{\mathbb{N}} \times \mathcal{G}_0(G_m) \subseteq \mathcal{G}(G_m)$ as a product of $\mathcal{H} \subseteq \mathcal{H}_d$ and of the "a-stabilized Hilbert scheme $\mathcal{H}_{d,a}$" of the disc.

Lemma. (0) The monoid $(\mathcal{H}_d, \cup_d)$ embeds into $\mathcal{G}(G_m)$ by a closed embedding $\mathcal{H}_d^n \ni D \mapsto \mathcal{O}_d(D) \in \mathcal{G}(G_m)$.

(1) The image is the submoduli $\mathcal{G}(G_m, \overline{G_m})$ consisting of all $(\mathsf{S}, \sigma, D) \in \mathcal{G}(G_m)$ such that σ extends to a section of $\mathsf{S} \times_{G_m} \overline{G_m}$. The inverse map $\operatorname{div} : \mathcal{G}^+(G_m) \to \mathcal{H}_d$ is

$$\operatorname{div}(\mathsf{S}, \sigma) \stackrel{\text{def}}{=} s_i{}^{-1} 0 \ \in \ \mathcal{H}_C \ \text{ for } \ (\mathsf{S}, \sigma, D) \in \mathcal{G}_C^+(G_m).$$

(2) This embedding identifies the submonoid $\mathbb{N}[a] \subseteq \mathcal{H}_d$ with the submonoid $z^{\mathbb{N}} \subseteq \mathcal{G}(G_m)$.[45]

(3) The connected component $\mathcal{G}_0(G_m)$ (viewed as a quotient of $\mathcal{G}(G_m)$), is identified with the "stable Hilbert scheme" of the disc

$$\mathcal{H}_{(d,a)} \stackrel{\text{def}}{=} \lim_{\to} \mathcal{H}^n(d)$$

where inclusions are given by adding a multiple of the center c of the disc.

(4) For $\mathcal{G}^+ \stackrel{\text{def}}{=} \mathcal{H}_d \subseteq \mathcal{G}(G_m)$, the difference map $(a,b) \mapsto ab^{-1}$ gives an isomorphism

$$\mathcal{G}^+ \times_{\mathcal{G}^+} \mathcal{G}^+ \ \cong \ \mathcal{G}.$$

Proof. (3) The factorization of $\mathcal{G}(G_m)$ as $z^{\mathbb{Z}} \times K_-$ for the negative congruence subgroup $K_- \subseteq G_{m,\mathcal{K}}$, identifies the connected component $\mathcal{G}_0(G_m)$ with K_-. In these terms $D \in \mathcal{H}_d$ is the same as a monic polynomial $\chi_D(z) \in \Bbbk[z]$ and this corresponds to $\check{\chi}_D(z^{-1}) \in K_- = \Gamma(\mathbb{P}^1 - 0, \infty; G_m)$ by $\chi_D(z) = z^{l(D)} \check{\chi}_D(z^{-1})$.

(4) A subsemigroup A^+ of a commutative group A acts freely on $A^+ \times A^+$ and the quotient is the quotient by the equivalence relation generated by $(ag, b) \sim (a, gb)$, $a, b, g \in A^+$. Then the quotient is is the subgroup of A generated by A^+.

□

Remark. We define $\operatorname{div} : \mathcal{G}^+(G_m) \to \mathbb{N}$ as the composition of the Div map defined in the lemma with the degree $\operatorname{div}(\mathsf{S}, \sigma) \stackrel{\text{def}}{=} \deg[\operatorname{Div}(\mathsf{S}, \sigma)] \in \mathbb{N}$.

[45] $z \in \mathcal{G}(G_m)$ is well defined since local parameters form a torsor for $G_{m,\mathcal{O}}$.

6.2. **Loop Grassmannians for parabolic subgroups $\mathcal{G}(P) = \mathcal{G}(G,P)\subseteq\mathcal{G}(G)$.** Here we set up the notation. Let $P\subseteq G$ be a parabolic subgroup with the unipotent radical U and Levi group $\overline{P} \stackrel{\text{def}}{=} P/U$.

Let $\mathcal{G}(G,P)\subseteq\mathcal{G}(G)$ be the subfunctor consisting of all $(\mathsf{P},\tau) \in \mathcal{G} = \mathcal{G}(G)$ such that the closure of $P\tau\subseteq\mathsf{P}|_{d^*}$ in P is a P-torsor (i.e., a reduction of P from G to P). We denote this reduction by $\mathsf{P}_{\tau,P}$. It comes with a meromorphic section τ_P and $(\mathsf{P},\tau)_P \stackrel{\text{def}}{=} (\mathsf{P}_\tau,\tau_P) \in \mathcal{G}(P)$ is the unique reduction of (P,τ) to the parabolic P.

For $(\mathsf{P},\tau_P) \in \mathcal{G}_{P,G}$ $(\mathsf{P}_{P,\tau},\tau_P) \in \mathcal{G}_{P,G}$ Composing with the image under $P \to \overline{P}$ gives the functor $\mathcal{G}(P,G) \to \mathcal{G}(\overline{P})$ which sends $(\mathsf{P},\tau_P) \in \mathcal{G}_{P,G}$ to $(\mathsf{P},\tau)_{P,\overline{P}} \stackrel{\text{def}}{=} (\mathsf{P}_{P,\tau},\tau_P)_{\overline{P}}$. This is a pair of the $\overline{P}$-torsor $\mathsf{P}_{\tau,P,\overline{P}} \stackrel{\text{def}}{=} \mathsf{P}_{\tau,P}/U$ and the image $\tau_{P,\overline{P}}$ of τ_P.

Lemma. (a) Ind_P^G is an embedding of functors $\mathcal{G}(P)\hookrightarrow\mathcal{G}(G)$.[46] The image is $\mathcal{G}(G,P)$ and the inverse map is the above map $(\mathsf{P},\tau) \mapsto (\mathsf{P},\tau)_P \stackrel{\text{def}}{=} (\mathsf{P}_{P,\tau},\tau_P)$.

(b) For a Levi subgroup L of P the functor Ind_L^P commutes with the action of $L_\mathcal{K}$. Also, the functors Ind_B^G, $\mathcal{G}(G,P) \to \mathcal{G}(P)$ and $(P \to \overline{P})_*$ commute with the action of $P_\mathcal{K}$.

(c) The composition of $\mathcal{G}(L) \xrightarrow{Ind_L^G} \mathcal{G}(G,P) \xrightarrow{x\mapsto x_{P,\overline{P}}} \mathcal{G}(\overline{P})$ is the isomorphism given by $L\xrightarrow{\cong}\overline{P}$. For instance, for $T\subseteq B\subseteq G$ and $\alpha \in X_*(T)$, we have

$$L_\alpha^G = Ind_T^G(L_\alpha^T) \quad \text{and} \quad (L_\alpha^G)_{B,H} = L_\alpha^H.$$

(d) Over any field F, the inclusion of F-points for $\mathcal{G}(G,P)\subseteq\mathcal{G}(G)$ is equality.

Proof. (a) is a case of lemma 2.3.2.

(c) According to (a) it suffices to prove the same statement for $\mathcal{G}(L) \xrightarrow{Ind_L^P} \mathcal{G}(P) \xrightarrow{(P\to\overline{P})_*} \mathcal{G}(\overline{P})$, but $(P \to \overline{P})_* Ind_L^P(\mathsf{S},\sigma) = (P \to \overline{P})_*(\mathsf{S}\times_L P,\sigma) = \big((\mathsf{S}\times_L P)/U,\sigma\big)$ returns us back to (S,σ) (up to $L\xrightarrow{\cong}\overline{P}$).

(d) holds since G/P is complete.[47] □

6.3. **The orbits $S_\alpha^\pm$ of $N_\mathcal{K}^\pm$, their intersections and the T-fixed points.** We describe $\overline{S_\alpha} \cap \mathcal{G}(G,B)$ and $\overline{S_\alpha} \cap \mathcal{G}(T)$. For $\overline{S_\alpha}$ itself see 3.3.

6.3.1. *The semiinfinite orbits $S_\alpha\subseteq\mathcal{S}_\alpha$ and their closures.* For $S_\alpha \stackrel{\text{def}}{=} N_\mathcal{K}{\cdot}L_\alpha = (T_\mathcal{O}N_\mathcal{K}){\cdot}L_\alpha$ and $\mathcal{S}_\alpha \stackrel{\text{def}}{=} (B_\mathcal{K})_0{\cdot}L_\alpha$. Then S_α is the reduced part of $\mathcal{S}_\alpha \cong \mathcal{G}_0(T)\times S_\alpha$ and the inclusion $S_\alpha\subseteq\mathcal{S}_\alpha$ is equality for points over fields.

[46] This embedding is not closed.

[47] The difference of $\mathcal{G}(P)\subseteq\mathcal{G}(G)$ is 'therefore 'schematic" – it is given by all $(T,\tau) \in \mathcal{G}(G)$ such that $\lim_{z\to 0} P\tau$ does not exist in P_0.

Lemma. (a) $\mathcal{G}(B)\xrightarrow{\cong}\mathcal{G}(G,B) = \sqcup_\alpha \mathcal{S}_\alpha$.

The subfunctors $S_\alpha\subseteq\mathcal{S}_\alpha$ of $\mathcal{G}(G,B)$ consist of all x such that $x_{B,H} = L^T_\alpha$, resp. such that $x_{B,H} \in \mathcal{G}_\alpha(T)$, i.e., that the divisor $\mathrm{div}_H(\tau_{B,H})$ of the section $\tau_{B,H}$ is $-\alpha$. [(48)]

(b) $\mathcal{G}(T)$ is a group and it acts on $\mathcal{G}(B)$. This gives $\mathcal{G}(T)\times\mathcal{G}(N)\xrightarrow{\cong}\mathcal{G}(B)$. The connected components are the isomorphisms $\mathcal{G}(T)_\lambda\times S_0\xrightarrow{\cong}\mathcal{S}_\lambda$, $\lambda \in X_*(T)$. This extends to an isomorphism of indschemes $\mathcal{G}(T)_\lambda\times\overline{S_0}\xrightarrow{\cong}\overline{\mathcal{S}_\lambda}$.

(c) The following are equivalent: (0) $\overline{S_\alpha} \ni \beta$, (i) $\overline{S_\alpha}\supseteq S_\beta$, (ii) S_α meets S^-_β, (iii) $\alpha \geq \beta$.

Proof. (a) The map $m : B_\mathcal{K}\times_{B_\mathcal{O}} G_\mathcal{O}\xrightarrow{\cong}G_\mathcal{K}$ is an injective immersion (on the level of tangent spaces this is $\mathfrak{b}_\mathcal{K} +_{\mathfrak{b}_\mathcal{O}} \mathfrak{g}_\mathcal{O}\hookrightarrow\mathfrak{g}_\mathcal{K}$).

Then $\mathcal{G}(G,B) = (B_\mathcal{K}\times_{B_\mathcal{O}} G_\mathcal{O})/G_\mathcal{O} \cong B_\mathcal{K}/B_\mathcal{O}) = \mathcal{G}(B)$ and $B_\mathcal{K} \cong (B_\mathcal{K})_0 \ltimes X_*(T)$ gives $\mathcal{G}(B) = B_\mathcal{K}/B_\mathcal{O} = X_*(T) \rtimes (B_\mathcal{K})_0/B_\mathcal{O}$. This is the decomposition $\mathcal{G}(G,B) = \sqcup_\alpha \mathcal{S}_\lambda$.

Recall that Ind^B_T commutes with $T_\mathcal{K}$, and $(B \to H)_*$, Ind^G_B, $?_B$ commute with $(B_\mathcal{K})$. This implies that the points x of $\mathcal{S}_\alpha$ satisfy $x_{B,H} \in \mathcal{G}_\alpha(T)$ and then $\mathcal{G}(G,B) = \sqcup_\alpha \mathcal{S}_\alpha$ implies that this precisely describes $\mathcal{S}_\alpha$.

Now, $(L^G_\alpha)_{B,H} = L^T_\alpha$ implies the characterization of S_α since $N_\mathcal{K}$ commutes with $(B \to H)_*$, Ind^G_B, $?_B$.

(b) The extension to closures follows because the indscheme $\mathcal{G}(T)_\lambda$ is indfinite.

(c) is well known. $\square$

Corollary. $T_{\mathfrak{p}_-}$ acts freely on $\mathcal{G}(G)$.[(49)]

Proof. $T_{\mathfrak{p}_-}$ acts freely on each $\mathcal{S}_\lambda\subseteq\mathcal{G}(G)$ by part (b) of the lemma. ($\mathcal{S}_\lambda$ is isomorphic to a connected component of $\mathcal{G}(B)$ on which $T_{\mathfrak{p}_-}$ acts as $\mathcal{G}(T)_0$ and we have $\mathcal{G}(T)\times\mathcal{G}(N)\xrightarrow{\cong}\mathcal{G}(B)$,)

So, the restriction of the stabilizer scheme $\mathsf{S} = \{(t,x) \in T_{\mathfrak{p}_-}\times\mathcal{G}(G);\ t\cdot x = x\}$ to the strata S_λ of $\mathcal{G}(GG)$ is trivial. This implies that S is trivial.[(50)] $\square$

Remark. The map $m : B_\mathcal{K}\times_{B_\mathcal{O}} G_\mathcal{O}\xrightarrow{\cong}G_\mathcal{K}$ is an injective immersion and a bijection on the level of points over a field. However, m is not an isomorphism – it is a proper embedding on the level of tangent spaces and $\pi_0(B_\mathcal{K}) = X_*(T)$ is larger the $\pi_0(G_\mathcal{K})$.

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48 So, $x \in \mathcal{G}(G)$ is in $\mathcal{S}_\alpha$ (resp. S_α) iff it is induced from (unique) $y \in \mathcal{G}(B)$. and the connected component of y in $\mathcal{G}(B)$ is $\mathcal{G}_\alpha(B)$, i.e., $(B \to H)_*y \in \mathcal{G}_\alpha(T)$. Also, x is in S_α iff it is induced from (unique) $y \in \mathcal{G}(B)$ and $(B \to H)_*y \in \mathcal{G}_\alpha(T) \cong L^T_\alpha$.

49 This is not true for the action of $N_{\mathfrak{p}_-}$ on $\mathcal{G}(G)$ (the stabilizer is nontrivial at S_α for $\alpha < 0$).

50 The locus $X\subseteq\mathcal{G}(G)$ where the fiber of S is nontrivial is a closed) subindscheme which is $T\mathcal{R}$-invaraint. The same then holds for $X_{red}\subseteq X$ and this implies that if $X \neq \emptyset$ then X_{red} would contain some fixed point L_λ which is impossible.

6.3.2. *The T-fixed part $\overline{S_0}^T$ as the Hilbert scheme $\mathcal{H}_{d\times I}$.* Recall that for a choice of a local parameter z we embed $\mathcal{H}_d$ into $G_{m,\mathcal{K}}$ by interpreting $S\in\mathcal{H}_d^n$ as a degree n monic polynomial $\chi_S\in\Bbbk[z]$, with nilpotent coefficients. This gives $\mathcal{H}_{d\times I}=(\mathcal{H}_d)^I\overset{\iota}{\hookrightarrow}(G_{m,\mathcal{K}})^I\underset{\cong}{\leftarrow}T_{\mathcal{K}}$. Also, recall that our convention for $X_*(T)\hookrightarrow\mathcal{G}(T)\subseteq\mathcal{G}(G)$ is $\lambda\mapsto L_\lambda\overset{\text{def}}{=}\lambda^{-1}L_0$.

Lemma. (a) The closed embedding $\mathcal{H}_{d\times I}\ni P\mapsto\iota(P)\cdot L_0\in\mathcal{G}(T)$ gives the Y-fixed part:

$$\mathcal{H}_{d\times I}\xrightarrow{\cong}(\overline{S_0})^T\ =\ \overline{S_0}\cap\mathcal{G}(T).$$

The connected components for $\alpha\in\mathbb{N}[I]$ are

$$\overline{S_0}\cap\mathcal{G}_{-\alpha}(T)\underset{\cong}{\leftarrow}\mathcal{H}^\alpha_{d\times I}\ =\ \prod_{i\in I}\mathcal{H}_d^{\alpha_i}.$$

(b) Equivalently, the filtration of $\mathcal{G}_0(T)$ induced by $\overline{S_\alpha}$, $\alpha\in\mathbb{N}[I]$, is the filtration of the a-stable Hilbert scheme $\mathcal{H}_{(d,a)\times I}$ by subschemes $\mathcal{H}^\alpha_{d\times I}\overset{\text{def}}{=}\prod_{i\in I}\mathcal{H}_d^{\alpha_i}$.

Proof. (a) The factorization

$$\overline{S_0}\cap\mathcal{G}_{-\alpha}(T)\ \cong\ \prod_{i\in I}\overline{S_0}\cap\mathcal{G}_{-\alpha_i i}(T)$$

reduces the claim to SL_2.

Here, $\alpha=n\cdot\check{i}$ for the simple coroot $\check{i}=\check{\varepsilon}_1-\check{\varepsilon}_2$ and $n\in\mathbb{N}$. Then $L_\alpha=L_{n\check{i}}$ is the lattice generated by two vectors $\langle z^ne_1,z^{-n}e_2\rangle$.

Let us start with $S_\alpha=N_{\mathcal{K}}L_\alpha$. For $u=\begin{pmatrix}1&x\\0&1\end{pmatrix}\in N_{\mathcal{K}}$, we have $uL_\alpha=u\langle z^ne_1,z^{-n}e_2\rangle=\langle z^ne_1,z^{-n}(e_2+xe_1)\rangle$. So, S_α consists of all lattices $\mathcal{L}\in\mathcal{G}_0(SL_2)$ such that $\mathcal{L}\cap\mathcal{K}e_1=z^n\mathcal{O}e_1$ and $\overline{S_\alpha}$ consists of $\mathcal{L}\in\mathcal{G}_0(SL_2)$ such that $\mathcal{L}$ contains $z^n\mathcal{O}e_1$ or equivalently, $\mathcal{L}$ lies in $\mathcal{K}e_1\oplus\mathcal{O}z^{-n}e_2$.

The $\Bbbk$-points of the negative congruence subgroup $K_-(G_m)\subseteq G_{m,\mathcal{K}}$ are the comonic polynomials $Q(z^{-1})=1+a_1z^{-1}+\cdots+a_nz^{-n}$ in z^{-1} that are invertible in $\Bbbk[z^{-1}]$. Consider the filtration $K_-(G_m)_{\le n}$ where powers of z^{-1} in Q are restricted to $\le n$. Recall that $K_-(G_m)_{\le n}$ identifies by $P(z)=z^nQ(z^{-1})$ with the Hilbert scheme $\mathcal{H}_d^n$ which is the moduli of monic polynomials $P(z)=z_n+a_1z^{n-1}+\cdots+a_n$ in z, of degree n and with nilpotent coefficients.

We choose $\phi:K_-(G_m)\xrightarrow{\cong}\mathcal{G}_0(T)$ by $\phi(Q)=\ \check{i}^{-1}(Q)L_0=\ \langle Q^{-1}e_1,Qe_2\rangle$. This lies in $\overline{S_n}$ iff $Qe_2\in\mathcal{O}z^{-n}e_2$, i.e., iff Q is in $K_-(G_m)_{\le n}$. $\square$

Corollary. This identifies the subscheme $(\overline{S^+_\alpha}\cap\overline{S^-_0})^T\hookrightarrow\overline{S^+_\alpha}^T$ with the Hilbert scheme $\mathcal{H}_{\alpha\cdot a}\subseteq\mathcal{H}_d$ of the finite colored scheme $\alpha\cdot a\overset{\text{def}}{=}\sqcup_{i\in I}\ \alpha_i\cdot a\cdot i\ \in\ \mathcal{H}_d^\alpha$. The connected components are

$$\mathcal{H}^\beta_{\alpha\cdot a}\xrightarrow{\cong}(\overline{S^+_\alpha}\cap\overline{S^-_0})\cap\mathcal{G}_\beta(T).$$

Proof. If $\overline{S_\alpha^+}\cap\mathcal{G}_\beta(T)\neq\emptyset$ and $\overline{S_0^-}^T\cap\mathcal{G}_\beta(T)\neq\emptyset$ then $0\geq-\beta$ and $-\beta\geq-\alpha$, i.e., $\alpha\geq\beta\geq 0$. Then $\overline{S_0^+}\cap\mathcal{G}_{-\beta}(T)=\ \mathcal{H}_{d\times I}^{\alpha-\beta}\cdot L_\beta$. Moreover,a point $x\in\mathcal{H}_{d\times I}^{\beta}\cdot L_\beta$ lies in $\overline{S_0^-}^T$ iff $\overline{N_\mathcal{K}^+\cdot x}$ contains L_0, i.e., iff $L_0\in\mathcal{H}^\beta\cdot x$.

So, points x of $(\overline{S_\alpha^+}\cap\overline{S_0^-})\cap\mathcal{G}_\beta(T)$ correspond to pairs $A\in\mathcal{H}_d^{\beta-\alpha}$ and $B\in\mathcal{H}_d^{\beta}$ such that $\iota(\chi_B\chi_c)L_\alpha=L_0$, i.e., $A\cup_C B=\alpha\cdot a$. This is equivalent to $A\in\mathcal{H}_d^{\beta-\alpha}$ which lie inside D_α, i.e., to $A\in\mathcal{H}_{\alpha\cdot a}^{\beta-\alpha}$. □

Remarks. (0) We can think of a point of $(\overline{S_\alpha^+}\cap\overline{S_0^-})^T$ as corresponding to a pair of effective I-divisors $\alpha_\pm$ with $\alpha_++\alpha_-=\alpha$, or a pair of colored subschemes $D^\pm$ of C with D^++D^- equal to the subscheme $D\subseteq C$ of type α. Then $D^\pm$ are complementary subschemes of D so one determines the other by the "complementation" procedure $\mathcal{H}(D)\ni D'\mapsto D-D'\in\mathcal{H}_D$.

So for a fixed D, the moduli of data (D',D'') is identified with $Gr(D)$ (in two ways corresponding to $N^\pm$).

(1) Here, we are taking the $B^\pm$-divisors of a section σ of a T-torsor. Notice that this is a single information, i.e., $\mathrm{Div}_{B^-}(s)=-\mathrm{Div}_B(s)$. For $T=G_m$ the difference is just which one of positions $0,\infty\in\partial G_m$ is regarded as positive.

6.3.3. *Intersections* $\overline{S_\alpha}\cap\mathcal{S}_\beta$.

Lemma. (a) $\overline{S_\alpha}\cap\mathcal{S}_\beta=\ N_\mathcal{K}\cdot[\overline{S_\alpha}\cap\mathcal{G}_\beta(T)]$ (and it is nonempty iff $\beta\leq\alpha$, i.e., $\overline{S_\alpha}\supseteq S_\beta$). In other words,

$$\overline{S_\alpha}\cap\ \mathcal{G}(G,B)\ =\ N_\mathcal{K}\ \cdot\ \sqcup_{\beta\leq\alpha}\ \overline{S_\alpha}\cap\mathcal{G}_\beta(T)\ \underset{\cong}{\leftarrow}\ \sqcup_{\beta\leq\alpha}\ [\overline{S_\alpha}\cap\mathcal{G}_\beta(T)]\times S_\beta.$$

Remark. $\overline{S_\alpha}\cap\mathcal{G}_\beta(T)$ has been calculated above.

Proof. (a) First, $\overline{S_\alpha}$ meets $\mathcal{S}_\beta$ iff $\overline{S_\alpha}$ contains S_β, i.e., iff $\beta\leq\alpha$. (If $\overline{S_\alpha}$ meets $\mathcal{S}_\beta$ then it meets $(\mathcal{S}_\beta)_{red}=S_\beta$ and then $\overline{S_\alpha}\supseteq S_\beta$.) Now,

$$\overline{S_\alpha}\cap\mathcal{S}_\beta=[\overline{S_\alpha}\ N_\mathcal{K}\cdot(T_\mathcal{K})_0\cdot L_\beta]=N_\mathcal{K}\cdot[\overline{S_\alpha}\cap(T_\mathcal{K})_0\cdot L_\beta]=\ N_\mathcal{K}\cdot[\overline{S_\alpha}\cap\mathcal{G}_\beta(T)].$$

In other words,

$$\overline{S_\alpha}\cap\ \mathcal{G}(G,B)\ =\ \sqcup_{\beta\leq\alpha}\ N_\mathcal{K}\cdot\overline{S_\alpha}\cap\mathcal{G}_\beta(T).$$

Finally, denote by $K_-(T)_\beta$ the pull back of $\overline{S_\alpha}\cap\mathcal{G}_\beta(T)$ under $K_-(T)\xrightarrow{\cong}\mathcal{G}_\beta(T)$, then

$$N_\mathcal{K}\cdot\overline{S_\alpha}\cap\mathcal{G}_\beta(T)=\ N_\mathcal{K}\cdot K_-(T)_\beta\cdot L_\beta=\ K_-(T)_\beta\cdot N_\mathcal{K}\cdot L_\beta=\ K_-(T)_\beta\ cd\ S_\beta$$

and this is isomorphic to $K_-(T)_\beta\times S_\beta\cong\overline{S_\alpha}\cap\mathcal{G}_\beta(T)\times S_\beta$. □

6.3.4. *The closure* $\overline{\mathcal{S}_0^-}$ *of the non-reduced ind-subscheme* $\overline{\mathcal{S}_0^-}$*?* We would like to know $\overline{S_\alpha}\cap\overline{\mathcal{S}_0^-}$.

Example. The reduced part of $\overline{S_i^+}\cap\overline{\mathcal{S}_0^-}$ is $\overline{S_i^+}\cap\overline{S_0^-}\cong\mathbb{P}^1$. The non-reduced part is a double point at L_0.

6.4. **Central extensions of reductive groups.** We consider the effect of a central extension of a reductive group on its loop Grassmannian.

Consider an exact sequence of reductive groups $0\to S\to G\to\overline{G}\xrightarrow{\beta}0$ such that S is central in G which is connected.

Lemma. (1) $\mathcal{G}(S)$ is a group and it acts on $\mathcal{G}(G)$.

(2) If S is a torus then

- (a) $\mathcal{G}(G)\to\mathcal{G}(\overline{G})$ is a $\mathcal{G}(S)$-torsor.
- (b) The groups of connected components of loop Grassmannians fit into an exact sequence $0\to\pi_0[\mathcal{G}(S)]\to\pi_0[\mathcal{G}(G)]\xrightarrow{\pi_1(\beta)}\pi_0[\mathcal{G}(\overline{G})]\to 0$.
- (c) Certain simplifications of $\mathcal{G}(G)$ are covers of $\overline{G}$-objects:
$$\mathcal{G}(G)/\mathcal{G}(S)_0\ \cong\ \mathcal{G}(\overline{G})\times_{\pi_1(\overline{G})}\pi_1(G)\quad\text{and}\quad\mathcal{G}(G)_{red}\ \cong\ \mathcal{G}(\overline{G})_{red}\times_{\pi_1(\overline{G})}\pi_1(G).$$
- (c') If we denote the map $\pi_1(G)\overset{\pi_1(\beta)}{\twoheadrightarrow}\pi_1(\overline{G})$ by $x\mapsto\overline{x}=x+X_*(S)$ then
$$\mathcal{G}(G)_x/\mathcal{G}(S)_0\cong\mathcal{G}(\overline{G})_{\overline{x}}\quad\text{and}\quad[\mathcal{G}(G)_x]_{red}\ \cong\ [\mathcal{G}(\overline{G})_{\overline{x}}]_{red}.$$

(3) If Z is etale we have $0\to\pi_0[\mathcal{G}(G)]\to\pi_0[\mathcal{G}(\overline{G})]\to Z\to 0$ and $\mathcal{G}(G)_x\xrightarrow{\cong}\mathcal{G}(\overline{G})_x$ for any $x\in\pi_1(G)\subseteq\pi_1(\overline{G})$.

Proof. (1) is obvious.

(2a) First, for $X=d$ or $X=d^*$ the maps $G(X)\to\overline{G}(X)$ are surjective since $H^1(X,G_m)=0$. Now, $0\to S\xrightarrow{\alpha}G\xrightarrow{\beta}\overline{G}\to 0$ gives maps
$$S_{\mathcal{K}}/S_{\mathcal{O}}\xrightarrow{\overline{\alpha}}G_{\mathcal{K}}/G_{\mathcal{O}}\xrightarrow{\overline{\beta}}\overline{G}_{\mathcal{K}}/\overline{G}_{\mathcal{O}}$$
Map $\overline{\alpha}$ is injective since $G_{\mathcal{O}}\cap_{G_{\mathcal{K}}}S_{\mathcal{K}}=S_{\mathcal{O}}$ and $\overline{\beta}$ is surjective since $G_{\mathcal{K}}\xrightarrow{\beta}\overline{G}_{\mathcal{K}}$ is surjective.

Since $S\subseteq G$ is central the subgroup $S_{\mathcal{O}}\subseteq G_{\mathcal{K}}$ acts trivially on $G_{\mathcal{K}}/G_{\mathcal{O}}$, hence $\mathcal{G}(S)$ acts on $G_{\mathcal{K}}/G_{\mathcal{O}}$. Then, for $v_i\in\mathcal{V}_{\mathcal{K}}$, $\overline{\beta}(v_1\mathcal{V}_{\mathcal{O}})=\overline{\beta}(v_2\mathcal{V}_{\mathcal{O}})$ means that $\beta(v_1)\overline{G}_{\mathcal{O}}=\beta(v_2)\overline{G}_{\mathcal{O}}$, i.e., for $v={v_2}^{-1}v_1$ we have $\beta(v)\in\overline{G}_{\mathcal{O}}=\ \beta(\mathcal{V}_{\mathcal{O}})$, i.e., for some $x\in\mathcal{V}_{\mathcal{O}}$ we have $\beta(v)=\beta(x)$, i.e., $vx^{-1}=\alpha(y)$ with $y\in S_{\mathcal{K}}$. Now, $v_1\mathcal{V}_{\mathcal{O}}=v_2g\mathcal{V}_{\mathcal{O}}=v_2\alpha(y)x\mathcal{V}_{\mathcal{O}}=v_2\alpha(y)\mathcal{V}_{\mathcal{O}}=\ {}^{v_2}y{\cdot}v_2\overline{G}_{\mathcal{O}}$ is in $\mathcal{G}(S){\cdot}v_2\mathcal{V}_{\mathcal{O}}$.

(2b) The set of connected components $\pi_0[\mathcal{G}(\mathsf{G})]$ of the loop Grassmannian of G is $\pi_1(\mathsf{G})$, so it is a group. The above exact sequence of groups gives $0\to\pi_1(S)\to\pi_1(G)\to\pi_1(\overline{G})\to 0$ since $\pi_2(\overline{G})=0=\pi_0(S)$.

(2c') The first claim follows from (a) and then the description of $[\mathcal{G}(G)_x]_{red}$ follows because the reduced part of the connected component of $\mathcal{G}(S)$ is trivial.

Finally, (2c) follows from (2c') and (2b).

(3) From $\pi_1(Z)=0=\pi_0(G')$ we have $0\to\pi_1(G)\to\pi_1\mathcal{G}(\overline{G})\to\pi_0(Z)\to 0$. So, the first claim follows from $\pi_0(Z)=Z$. □

Corollary. The following spaces are canonically the same for G and $\overline{G}$: $[\mathcal{G}(G)_0]_{red}\supseteq\overline{S_0}\subseteq S_0$ and $\mathcal{G}(G,(G/N)^{\mathrm{aff}})$.

Proof. $\mathcal{G}(G,(G/N)^{\mathrm{aff}})$ is the closure of $\mathcal{G}(G,G/N)=S_0$ in $[\mathcal{G}(G)_0]_{red}$ (since $S_0\subseteq[\mathcal{G}(G)_0]_{red}$ and $[\mathcal{G}(G)_0]_{red}\subseteq\mathcal{G}(G)$ is closed). □

6.4.1. *$\mathcal{G}(G)_{red}$ as a cover of $\mathcal{G}(G_{\mathrm{ss}})$.* A reductive group G has the largest central torus Z and the quotient G_{ss} is semisimple.

Lemma. (a) The map $\mathcal{G}(G)\to\ \mathcal{G}(G_{\mathrm{ss}})$ is an ismorphism on connected components[51] :

$$\mathcal{G}(G)_{red}\ \cong\ \mathcal{G}(G_{\mathrm{ss}})\times_{\pi_1(G_{\mathrm{ss}})}\pi_1(G).$$

(b) $\mathcal{G}(G)_{red}$ splits the quotient $\mathcal{G}(G)\to\mathcal{G}(G)/\mathcal{G}(Z)_0$:

$$\mathcal{G}(G)=\ \mathcal{G}(G)_{red}\cdot\mathcal{G}(Z)_0\ \underset{\cong}{\leftarrow}\mathcal{G}(G)_{red}\times\mathcal{G}(Z)_0.$$

So, for any Cartan $T\subseteq G$ one has $\mathcal{G}(G)=\mathcal{G}(G)_{red}\cdot\mathcal{G}(T)_0$.

Proof. (a) Since $\mathcal{G}(G_{\mathrm{ss}})$ is reduced, this is a case of the lemma 6.4.c.

(b) Observe that the set π_0 of connected components is the same for all three object. So, we need for each $x\in\pi_0[\mathcal{G}(G)]$ that the maps $[\mathcal{G}(G)_x]_{red}\times\mathcal{G}(Z)_0\twoheadrightarrow[\mathcal{G}(G)_x]_{red}\cdot\mathcal{G}(Z)_0\subseteq\mathcal{G}(G)_x$ be isomorphisms.

By 6.4.c' we have $[\mathcal{G}(G)_x]_{red}\cong[\mathcal{G}(G_{\mathrm{ss}})_{\overline{x}}]_{red}=\mathcal{G}(G_{\mathrm{ss}})_{\overline{x}}$ and $\mathcal{G}(G)_x/\mathcal{G}(Z)_0\cong\mathcal{G}(G_{\mathrm{ss}})_{\overline{x}}$. Now, the map $[\mathcal{G}(G)_x]_{red}\to\ \mathcal{G}(G)_x/\mathcal{G}(Z)_0$ is the same as $[\mathcal{G}(G)_x]_{red}\to\mathcal{G}(G_{\mathrm{ss}})_{\overline{x}}$, so it is an isomorphism,

Finally, the last claim follows since any Cartan T of G contains Z. □

Remark. The derived subgroup G' of G is semisimple and the cocenter $C\overset{\mathrm{def}}{=}G/G'$ is a torus. Then $G/Z=(G/Z)'$ is the image of G'. So, $G/Z\cong G'/G'\cap Z$, i.e., $G=G'{\cdot}Z$ and therefore $Z\to C$ is surjective and $C=Z/Z\cap G'$.

Notice that $\mathcal{G}(G')\cdot\mathcal{G}(Z)$ is in general only a union of connected components of $\mathcal{G}(G)$. (Because the same is true for the image of $\mathcal{G}(G')$ in $\mathcal{G}(G)/\mathcal{G}(Z)=\mathcal{G}(G/Z)=\mathcal{G}(G'/Z')$ for a finite central subgroup $Z'=G'\cap Z$ in G'.)

6.4.2. *Torus quotients.* Here we consider exact sequences of the form $0\to G_1\to G\to C\to 0$ where G,G_1 are connected reductive and C is a torus.

[51] So, from the point of view of individual connected components, $\mathcal{G}(G)_{red}$ is itself a loop Grassmannian.

Lemma. (a) If G_1 is simply connected then $\mathcal{G}(G) \to \mathcal{G}(C)$ is an isomorphism on π_0. In general one has $0 \to \pi_0\mathcal{G}(G_1) \to \pi_0\mathcal{G}(G) \to X_*(C) \to 0$.

(b) $\mathcal{G}(G) \to \mathcal{G}(C)$ is a $\mathcal{G}(G_1)$-bundle.

(c) If G_1 is semisimple then the nonreduced part of $\mathcal{G}(G)$ comes from $\mathcal{G}(C)$ in the sense that $\mathcal{G}(G)_{red} = \mathcal{G}(G)\times_{\mathcal{G}(C)}\mathcal{G}(C)_{red}$.

$\square$[52]$\blacksquare$ *Proof.* (a) The exact sequence $\pi_2(C) \to \pi_1(G_1) \to \pi_1(G) \to \pi_1(C) \to \pi_0(G_1)$ gives $0 \to \pi_1(G_1) \to \pi_1(G) \to \pi_1(C) \to 0$

(b) $\square$[53]$\blacksquare$

(c) follows from (b). $\square$

6.4.3. *Groups $SL\subseteq GL\twoheadrightarrow PGL$.* The π_1 of the sequence $0 \to G_m \to GL(U) \to PGL(U) \to 0$ is $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ for $n = \dim(U)$. We have established

$$\mathcal{G}[PGL(U)] \xrightarrow{\cong} \mathcal{G}(G_m)\backslash\mathcal{G}[GL(U)];$$
$$\mathcal{G}[GL(U)]_{red} \ \cong\ \mathcal{G}[PGL(U)]\times_{\mathbb{Z}/n}\mathbb{Z} \quad \text{and} \quad \mathcal{G}[GL(U)]\underset{\cong}{\leftarrow}\mathcal{G}[GL(U)]_{red} \times \mathcal{G}(\mathbb{G}_m)_0$$

Remarks. (0) A model for $\mathcal{G}[GL(U)]_c$ is provided by the space of lattices $\mathcal{U}$ in $U\otimes\mathcal{O}_{d^*}$. Consequently, $\mathcal{G}[PGL(U)]_c$ is the moduli of $\mathbb{P}^1$-bundles $\mathbb{P}$ over d with a trivialization $\tau \in Isom[\mathbb{P}(U), \mathbb{P}]$. This is also a model for $\mathcal{G}[GL(U)]_{red}$.

A model for $\mathcal{G}[SL(U)]_c$ are the *special* finitely supported $\mathbb{P}^1$-bundles (the ones that lie in the connected component of $\mathcal{G}[PGL(U)]$).

(1) When one is describing $\mathcal{G}[PGL(U)]$ from $\mathcal{G}[GL(U)]$ one needs to consider one connected component $\mathcal{G}[PGL(U)]_x$ at a time, to choose a representative $\Bbbk \in \mathbb{Z}$ of x and to pass to the reduced part $\big(\mathcal{G}[GL(U)]_k\big)_{red}$. This is just a way to describe the quotient $\mathcal{G}[PGL(U)]$ of $\mathcal{G}[GL(U)]$ by $\mathcal{G}(G_m)]$. (The quotient by the discrete part $\mathbb{Z}$ is given by the combinatorics of x and k and the quotient by $\mathcal{G}(G_m)_0$ is performed by modifying lattices $\mathcal{U}$ to $\mathcal{U}(D - l_D c)$ for $D \in \mathcal{H}_d$.[(54)])

[52] $\boxed{!}$Unfinished

[53] $\boxed{!}$

[54] The usual identification of $\mathcal{G}[PGL(U)]_x$ with $\mathcal{G}[GL(U)]_k$ is correct only on the reduced parts, say on the points over a field.

7. Various

Part 1. Vinberg semigroups and wonderful compactifications

8. Vinberg semigroups

8.0.1. *The Vinberg semigroup* $\overline{\mathcal{V}}$.

8.1. Vinberg semigroups $\overline{\mathcal{V}}_{G,P,V}$: ideas.

8.1.1. *The relevant mapping spaces.* $Bun_G(C) \supseteq Bun_C^{fs}(G)$ are $Map(C, \mathbb{B}G)$ and $Map[(C, \eta_C), (\mathbb{B}G, \mathrm{pt})]$. Then $\mathcal{G}(G) \to \mathcal{H}_C$ is a "resolution" (or a more elaborate version) of $Bun_C^{fs}(G)$ with $\mathcal{G}(G)_D = Map[(C, C-D), (\mathbb{B}G, \mathrm{pt})]$, while $\mathcal{G}(G) \to \mathcal{R}_C$ is in between.

8.1.2. *Change the parametrization of Vinberg semigroups.* $\square$[55] $\blacksquare$ Data are given by a subgroup $V \subseteq G$ such that G/V is quasiaffine. Then $G \times N_G(V)/V$ acts on G/V by $(g, uV) * xV \overset{\text{def}}{=} g \cdot xV \cdot uV^{-1}$, hence also on $(G/V)^{\text{aff}}$. Also, $N_G(V)$ acts on $G/V \subseteq (G/V)^{\text{aff}}$ by conjgation.

Then one can define $\mathcal{Z}_{\mathcal{V}}$ as the automorphisms

$$\mathcal{Z}_V \overset{\text{def}}{=} \mathrm{Aut}_{G \times N_G(V)/V}(G/V)$$

and the Vinberg group and semigroup as

$$\mathcal{V}_V \overset{\text{def}}{=} \mathrm{Aut}_{\mathcal{Z}_V}(G/V) \subseteq \overline{\mathcal{V}}_V \overset{\text{def}}{=} \mathrm{End}_{\mathcal{Z}_V}((G/V)^{\text{aff}}).$$

Example. For a parabolic P any normal subgroup V between U and P' satisfies the condition that G/V is quasiaffine and $N_G(V) = P$.[56]

AGAIN: For any subgroup $V \subseteq G$ we consider the normalizer $N_G(V)$, its quotient $M_V = N_G(V)/V$ and its center $\mathcal{Z}_V$. Then $G \times_{Z(G)} M_V$ maps to $\mathcal{V}_V \overset{\text{def}}{=} \mathrm{Aut}_{\mathcal{Z}_V}(G/V)$. If G/V is quasiaffine we also get the semigroup $\overline{\mathcal{V}}_V \overset{\text{def}}{=} \mathrm{End}_{\mathcal{Z}_V}((G/V)^{\text{aff}})$ whose invertible part is $\mathcal{V}_V = \mathrm{Aut}_{\mathcal{Z}_V}(G/V) = \mathrm{Aut}_{\mathcal{Z}_V}((G/V)^{\text{aff}})$.

This construction appears in [**?**] when V is related to some parabolic $P = U \ltimes U$. Here, V is either U or P', hence its normalized if P and $M = P/V$. When P is a Borel B and $V = N$ we get the Vibnerg semigroup $\mathcal{V}$.

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55 $\boxed{!}$

56 The conditions here on $V \ltimes P$ are that P/V is reductive and $P/U \to P/V$ is an isomorphism on commutative quotients $(P/U)^{\text{ab}} \xrightarrow{\cong} (P/V)^{\text{ab}}$.

8.1.3. *Vinberg semigroups* $\overline{\mathcal{V}}_{G,P,V}$. For a parabolic $P=U\ltimes L$ in a (reductive?) semisimple group G we consider a normal subgroup V of P that lies between the unipotent radical U and the derived subgroup P'. Then $G{\times}P/V$ acts on $Y^o_{G,P,V}\overset{\text{def}}{=}G/V$ by $(g,pV)*xV\overset{\text{def}}{=}gxV(pV)^{-1}=gxp^{-1}V$ and therefore als on its affinizations $Y_{G,P,V}\overset{\text{def}}{=}(G/V)^{\text{aff}}$. We will use these two spaces to relate G and P/V. Once we pass to mapping spaces $(G/V)^{\text{aff}}$ will really be a correspondence between mapping spaces associated to groups G and P/V (say, the loop Grassmannians $\mathcal{G}(G)$ and the moduli of G-bundles $Bun_G(C)$).

Remark. I think I have checked that G/V is quasiaffine, hence $G/V{\subseteq}(G/V)^{\text{aff}}$. $\square$

The corresponding *Vinberg group* $\mathcal{V}=\mathcal{V}_{G,P,V}$ and *Vinberg semigroup* $\overline{\mathcal{V}}=\overline{\mathcal{V}}_{G,P,V}$ are defined using the double centralizer and the affine closure. First, we conider $\mathcal{Z}=\mathcal{Z}_{G,P,V}\overset{\text{def}}{=}\ \text{Aut}_{G\times P/V}(G/V)$ and then

$$\mathcal{V}=\text{Aut}_{\mathcal{Z}}(G/V)\quad\text{and}\quad\overline{\mathcal{V}}=\text{End}_{\mathcal{Z}}((G/V)^{\text{aff}}).$$

Example. When $P=G$ then V must be G hence $G/V=\text{pt}$ and $1=\mathcal{Z}=\mathcal{V}=\overline{\mathcal{V}}$.

8.1.4. *When P dos not contain any semisimple normal subgroup of G.* Then

Lemma. $\square$[57]$\blacksquare$ (a) $\mathcal{Z}=Z(P/V)$.

(b) $\mathcal{V}=G{\times}_{Z(G)}P/V$.

(c) $\overline{\mathcal{V}}$ is a semigroup closure of $\mathcal{V}$.

(d) When P is a Borel TN and $V=N$ then $\overline{\mathcal{V}}_{G,B,N}$ is the semigroup introduced by Vinberg.

8.1.5. *The (closures) of orbits of $\ddot{V}$ on $\mathcal{G}(G)$.* These should be realized as (reduced parts of) the connected components of the mapping spaces $\mathcal{G}(\mathcal{V},Y)$ for some $Y=Y_{G,P,V}$.

Remark. Seemingly, in all cases $\mathcal{V}=\mathcal{V}_{G,B,N}$ (rather than say $\mathcal{V}_{G,P,V}$). At least this is true for $P=B$ and $P=G$.

However, Y seems to behave in an unusual way. First, for $U=N$ (the radical of $P=B$), the orbits of $\ddot{U}=N_{\mathcal{K}}\ltimes T_{\mathcal{O}}$ are related to $Y=Y_{G,B,N}\overset{\text{def}}{=}(G/N)^{\text{aff}}$. On the other hand, for $U=1$ (the radical of $P=G$), the orbits of $\ddot{U}=G_{\mathcal{O}}$ are related to $Y=\text{End}_H(Y_{G,B,N})=\text{End}_H((G/N)^{\text{aff}})=\overline{\mathcal{V}}_{G,B,N}$.

[57] $\boxed{!}$Conjecture?

8.1.6. *The doubling from* $(G/N)^{\mathrm{aff}}$ *to* $\overline{\mathcal{V}}$. We have seen that the "degenerate" case $P = G$ appears as a "double" of the "substantial" case $P = B$? (Here, the "doubling" terminology comes fromdim$[\mathrm{End}_H((G/N)^{\mathrm{aff}})] = 2\dim((G/N)^{\mathrm{aff}})$. [So, dimensionwise this is more like $U \mapsto T^*U$ than $U \mapsto \mathrm{End}(U) = U \otimes U^*$ which squares the dimension.]

Question. Why is this so?

Remarks. (0) "G as a double of B" is standard in q-groups. However, in this classical version one does not double B but its "dual" G/N.

(1) This "doubling uses H-action on G/N (dual to $H \subseteq B$?). The doubling of $G/N \subseteq (G/N)^{\mathrm{aff}}$ is $\mathcal{V} \subseteq \overline{\mathcal{V}}$.

Questions. Does it work the same for all $Y_{G,P,V}$ and $\overline{\mathcal{V}}_{G,P,V}$?

8.1.7. *The structure theory of* $\overline{\mathcal{V}}$. The basic feature is the following stratification $\overline{\mathcal{V}}_J$, $J \subseteq I$, of $\overline{\mathcal{V}}$.

The fibers of the stablizer scheme for the $\overline{H}$-action on $\overline{\mathcal{V}}$ (or H-action?) are ideals in $\overline{H}$, I guess that they correspond to subsets $J \subseteq I$. This gives a stratification $\overline{\mathcal{V}}_J$ of $\overline{\mathcal{V}}$ and gives varieties $X_J = \overline{\mathcal{V}}_J/(H/H_J)$. When $J = \emptyset$, i.e., where H acts freely, $X_\emptyset$ is the wondrful compactification of G_{ad}.

Possiby, in generall one gets compactifications of adjoint quotients of Levi factors G_I??? (Seemingly not exactly like that since such object is not canonically defined!)

Question. Is the structure theory of $\overline{\mathcal{V}}$ some kind of the double of such theory for $(G/N)^{\mathrm{aff}}$?

8.2. **Vinberg semigroup as $\mathrm{End}_H((G/N)^{\mathrm{aff}})$.**

Lemma. The action of $\overline{\mathcal{V}}(G)$ on $(G/N)^{\mathrm{aff}}$ yields

$$\overline{\mathcal{V}}(G) \ \cong \ \mathrm{End}_H((G/N)^{\mathrm{aff}}).$$

Proof. A. The semigroup $\mathrm{End}_H((G/N)^{\mathrm{aff}})$ has a zero. First, $(G/N)^{\mathrm{aff}}$ contains the semigroup $\overline{H}$ which has zero, we denote this point z. The action of $\overline{H}$ contracts $(G/N)^{\mathrm{aff}}$ to z. So, any $f \in U = \mathrm{End}_H((G/N)^{\mathrm{aff}})$ fixes the point point z.

Now the constant map $0 : (G/N)^{\mathrm{aff}} \to z \in (G/N)^{\mathrm{aff}}$ is the zero in U since it lies in U and for $f \in U$ one has $0 \circ f = 0 = f \circ 0$.

B. The left and right multiplication by $\overline{H} \subseteq \overline{\mathcal{V}}$ now contract U to the point 0. In particular, U is connected.

C. The action of the Vinberg semigroup $\overline{\mathcal{V}}$ on $(G/N)^{\mathrm{aff}}$ identifies $\overline{\mathcal{V}}$ with the invertible part $\mathrm{Aut}_H((G/N)^{\mathrm{aff}})$ of $\mathrm{End}_H((G/N)^{\mathrm{aff}}) = U$.

C'. The invertible part of U is open hence dense. For this we may want to reduce the situation where U is clearly a scheme (rather then an ind-scheme). This is accomplished by replacing $(G/N)^{\mathrm{aff}}$ by finite infinitesimal neighborhoods of the vertex z. As U fixes z it preserves each of these neighborhoods.

D. For a Cartan T of G, the group $T\times_{Z(G)} H$ is a the Cartan in $\mathcal{V}$ and we know its closures in $\overline{\mathcal{V}}$ and U coincide.

This proves that $\overline{\mathcal{V}} = U$ since for a Cartan $\mathcal{T}$ of the invertible part $\mathcal{G}$ of a reductive semigroup $\overline{\mathcal{G}}$ one knows that $\overline{\mathcal{G}} = \mathcal{G}\cdot\overline{\mathcal{T}}\cdot\mathcal{G}$ for the closure $\overline{\mathcal{T}}$ of $\mathcal{T}$ in $\overline{\mathcal{G}}$ [Kapranov].

8.2.1. *Details.* Vinberg semigroup has been defined in characteristic zero by describing its ring of functions. The proof above uses properties of this definition of $\overline{\mathcal{V}}$.

This provides a geometric definition of the Vinberg semigroup over $\mathbb{Z}$ which (as we have checked) coincides with the usual one in characteristic zero.

One should also check that this notion of Vinberg semigroup is $\mathbb{Z}$-flat. (Again, this should follow when one establishes certain pieces of the structure theory of reductive semigroups over $\mathbb{Z}$.)

Question. Is Kapranov's observation that I use "over $\mathbb{Z}$" for split forms?

8.2.2. *Vinberg semigroup "can" be defined via functions over* $\mathbb{Z}$. Over $\mathbb{Q}$ we can write $\mathcal{O}((G/N)^{\mathrm{aff}}) = \mathcal{O}(G/N) = \mathcal{O}(G)^{1\times N}$ as $\oplus_{\lambda\in X_*(H)^+}\ \mathcal{W}_\lambda\otimes\mathcal{S}_{\check{\lambda}}^N$. (I want to choose $\mathcal{W}$ and $\mathcal{S}$ so that $\mathcal{S}_\mu^N = \mathcal{S}_\mu(\mu)$!)

Over $\mathbb{Z}$ we have a filtration with $Gr[\mathcal{O}(G)] =\ \oplus_{\lambda\in X_*(H)^+}\ \mathcal{W}_\lambda\otimes\mathcal{S}_{\check{\lambda}}$.

That gives some estimate on N-invariants which combines with their H-character(?).

8.3. **Summary of Vinberg semigroup results and conjectures.** Let G be semisimple and $\overline{G}$ be the wonderful compactification of G_{ad}.

8.3.1. $\overline{\mathcal{V}}$ *via* $(G/N)^{\mathrm{aff}}$*.* The following property seems to be a reasonable definition of $\overline{\mathcal{V}}$.

Lemma. $\mathcal{V}\subseteq\overline{\mathcal{V}}$ act on $G/N\subseteq(G/N)^{\mathrm{aff}}$ and

$$\mathcal{V}\xrightarrow{\cong}\mathrm{Aut}_H(G/N) =\ \mathrm{Aut}_H((G/N)^{\mathrm{aff}}) \quad \text{and} \quad \overline{\mathcal{V}} = \mathrm{End}_H((G/N)^{\mathrm{aff}}).$$

8.3.2. $\overline{\mathcal{V}}$ *and the wonderful compactification* $\overline{G_{\mathrm{ad}}}$.

Lemma.

- (a) The H-free part $\overline{\mathcal{V}}^o$ of the Vinberg semigroup $\overline{\mathcal{V}}$ is an H_{sc}-torsor over the wonderful compactification $\overline{G_{\mathrm{ad}}}$ of G_{ad}.
- (b) As an H_{sc}-torsor over $\overline{G_{\mathrm{ad}}}$, $\overline{\mathcal{V}}^o$ is the product of G_m-torsors that correspond to the decomposition of $\partial G_{\mathrm{ad}} = \overline{G} - G_{\mathrm{ad}}$ into irreducible G-invariant divisors $D_i,\ i \in I$.
- (c) $\overline{\mathcal{V}}$ is the affinization of $\overline{\mathcal{V}}^o$.

Proof. (a) The free part $\overline{\mathcal{V}}^o$ of $\overline{\mathcal{V}}$ contains $\mathcal{V}$, so $\overline{\mathcal{V}}^o/H_{\mathrm{sc}}$ contains $\mathcal{V}/H_{\mathrm{sc}} = G_{\mathrm{ad}}$. .... $\square$

Remark. $\overline{G_{\mathrm{ad}}}$ is the geometric invariant theory quotient $\overline{\mathcal{V}}//H_{\mathrm{sc}}$ of $\overline{\mathcal{V}}$ by H_{sc} in the sense that it is the quotient of the free part of the space. So, G_{ad} is open in the G_{ad}-stack $\overline{\mathcal{V}}/H$.

8.3.3. *$\overline{\mathcal{V}}$ via a semigrouoid $\overline{\mathcal{S}}$ over $\mathcal{B}$.* We consider a certain groupoid $\mathcal{S}$ over $\mathcal{B}$ with a semigroupoid closure $\overline{\mathcal{S}}$ (see 19.9.4). The notion of *sections* $\Gamma(\mathcal{B},\mathcal{S})$ *of a groupoid* means the sections of $\mathcal{S} \to \mathcal{B}^2 \xrightarrow{pr_2} \mathcal{B}$. The sections form a semigroup $\Gamma(\mathcal{B},\mathcal{S})$ and its invertible part $\Gamma^*(\mathcal{B},\mathcal{S})$ is a group.

Define the *stable sections* of $\overline{\mathcal{S}}$ as the Hilbert scheme closure of $\Gamma^*(\mathcal{B},\mathcal{S})$ in all sections $\Gamma(\mathcal{B},\overline{\mathcal{S}})$ of $\overline{\mathcal{S}}$.

Lemma. $\mathcal{V}$ is the group of invertible sections $\Gamma^*(\mathcal{B},\mathcal{S})$ of $\mathcal{S}$.

Conjecture. The stable sections of $\overline{\mathcal{S}}$ form precisely the Vinberg semigroup $\overline{\mathcal{V}}$.

8.3.4. *Groupoid $\mathcal{S}$ over $\mathcal{B}$ and its semigroupoid closure $\overline{\mathcal{S}}$.* The action groupoid for the G-action on $\mathcal{B}$ is $\mathcal{G}_0 = \mathcal{B}\times G \to \mathcal{B}^2$ by $(\mathfrak{b}',g)\mapsto({}^g\mathfrak{b}',\mathfrak{b}')$. (We will also think of it as $G\times_B G$ for the conjugation action of B on G.)

Its vertical part $\mathcal{G}_0^{\uparrow} = \mathcal{G}|_{\Delta_{\mathcal{B}}}$ is the group bundle $G\times_B\ B$ (for the conjugation action of B on B). It contains a normal group subbundle $\mathcal{G}_- \overset{\mathrm{def}}{=} G\times_B\ N$. Our groupoid over $\mathcal{B}$ will be

$$\mathcal{S}\ \overset{\mathrm{def}}{=}\ \mathcal{G}_0/\mathcal{G}_-.$$

So, the fiber at $(B'',B') \in \mathcal{B}^2$ is

$$\mathcal{S}_{B'',B'} =\ \{gN' \in G/N';\ {}^gB' = B''\} =\ \{N''g \in G/N';\ {}^gB' = B''\}.$$

Also, the total space of $\mathcal{S}$ is $G\times_B G/N$ and the second projection map $pr_2 : \mathcal{S} \to \mathcal{B}$ is $G\times_B G/N \to G/B = \mathcal{B}$ by $(g,xN)\mapsto\ (g,xN)\mapsto\ g$.

We define $\overline{\mathcal{S}}$ as the relative affinization of $\mathcal{S}$ over $\mathcal{B}$

$$\overline{\mathcal{S}}\ \overset{\mathrm{def}}{=}\ \mathcal{S}^{\mathrm{aff}}_{\mathcal{B}} =\ G\times_B\ (G/N)^{\mathrm{aff}}.$$

Lemma. $\overline{\mathcal{S}}$ is a semigroupoid closure of the groupoid $\mathcal{S}$.

8.3.5. *The canonical resolution* $\widetilde{\widetilde{\overline{\mathcal{V}}}}$. *of* $\overline{\mathcal{V}}$.

- (a) $\overline{\mathcal{V}}$ has a canonical resolution $\widetilde{\widetilde{\overline{\mathcal{V}}}}$.
- (b)$\overline{\mathcal{V}}$ and $\widetilde{\widetilde{\overline{\mathcal{V}}}}$. are stable sections of certain groupoids over $\mathcal{B}$.

8.3.6. *More on semigroupoids.* (a) $\Gamma(\mathcal{G}) = \Gamma^*(\mathcal{G})$ is the Vinberg group $G_{\mathcal{P}} \overset{\text{def}}{=} G \underset{Z(G)}{\times} H_{\mathcal{P}}$.

(b) The closure $\overline{\Gamma(\mathcal{G})}$ of $\Gamma^*(\mathcal{G})$ in $\Gamma(\overline{\mathcal{G}})$ is a semigroup.

(b') If $\mathcal{P} = \mathcal{B}$ this is an extension of the wonderful compactification $\overline{G/Z(G)}$ by $\overline{H_{\mathcal{P}}}$.

(b") An open part $\overline{\Gamma(\mathcal{G})}^0$ is an extension by $H_{\mathcal{P}}$ and $\overline{\Gamma(\mathcal{G})} = \overline{\Gamma(\mathcal{G})}^0 \underset{H_{\mathcal{P}}}{\times} \overline{H_{\mathcal{P}}}$. As an $H_{\mathcal{P}}$-torsor over $\overline{G/Z(G)}$, this open part $\overline{\Gamma(\mathcal{G})}^0$ corresponds to the I-colored divisor which is minus the boundary of bolshaya yacheyka in $G/Z(G)$.

(c) The affinization of $\overline{\Gamma(\mathcal{G})}$ is a semigroup which we **call the Vinberg semigroup** $\overline{G_{\mathcal{P}}}$ associated to the partial flag variety $\mathcal{P}$. In turn, $\overline{\Gamma(\mathcal{G})}$ is a resolution of $\overline{G_{\mathcal{P}}}$. In particular when $\mathcal{P}$ is the flag variety $\mathcal{B}$ and G is simply connected, we get the usual Vinberg semigroup.

Add the formulation for general $\mathcal{P}$.

(d) There should be another statement concerning the action of $\widetilde{\mathcal{G}_{\mathcal{P}}}$ on $G/P' \underset{H_{\mathcal{P}}}{\times} \overline{H_{\mathcal{P}}}$.

Proof. (a) is known for $\mathcal{P} = \mathcal{B}$, and all is known for SL_2.

8.4. $\boxed{\textbf{SL(3)}}$. Let $G = SL(V)$ with $\dim(V) = 3$. Then $(G/N)^{\text{aff}}$ embedds into $\mathbb{V} = V \oplus V^*$ by $gN \mapsto (ge, ge^*)$ where $e \in V^N$ and $e^* \in (V^*)^N$ are bases of these lines. $(G/N)^{\text{aff}}$ is given in $V \oplus V^*$ as the quadric $\langle v, v^* \rangle = 0$. The G-orbits in $(G/N)^{\text{aff}}$ are given by $G/N = \{v, v^* \neq 0\}$ and $G/P_i' = \{v = 0 \neq v^*\}$, $G/P_i' = \{v \neq 0 = v^*\}$ and $G/P_{i,j}' = G/G == \{v = 0 = v^*\}$.

The Vinberg group $\mathcal{V}$ is $G \times H$ acting on $\mathbb{V}$ by the diagonal action of G and $H \subseteq \overline{H} = \overline{G_m}^I$ acting by $(z_i, z_j)(v_i, v_j) = (z_i v_i, z_j v_j)$. The Vinberg semigroup $\overline{\mathcal{V}}$ is the closure in $\text{End}(\mathbb{V})$ of $G \cdot \overline{H}$ (or the algebraic subgroup generated by?).

The Cartan TH is parameterized by $\check{\alpha}_i^B : G_m \to T \subseteq$ and $\check{\alpha}_i : G_m \to H$ (here $\check{\alpha}_i(z) = \check{\alpha}_i^B(Z) \cdot N$), realized via $H \xrightarrow{\cong} {G_m}^I$ as $\check{\alpha}_i$ being the identity map $G_m \to (G_m)^i$.

The standard basis e_1, e_2, e_3 of $V_i = V$ has weights ε_i so that for $a = diag(a_1, a_2, a_3) \in T$, $\varepsilon_i(a) = a_i$; in the dual basis e^i of V^* we have weights $-\varepsilon_i$.

The Cartan TH acts on $e_k \in V_i$ by $a_k z_i$ and on $e^l \in V_j$ by ${a_l}^{-1} z_j$.

A cocharacter $\chi = (\mu, \lambda)$ of TH cabe written by $a_i = s^{\mu_i}$ and $z_p = s^{\lambda_p}$, where $\mu = (\mu_1, ..., \mu_3) \in \mathbb{Z}^3$ and $\lambda = \lambda_i \check{\alpha}_i + \lambda_j \check{\alpha}_j$. It acts on e_k by $s^{\mu_k + \lambda_i}$ and on on e^l by $s^{-\mu_l + \lambda_j}$.

We are interested in $\chi = (\lambda, \mu)$ such that it extends from $G_m \to GL(\mathbb{V})$ to $\overline{G_m} \to \mathrm{End}(\mathbb{V})$, i.e., such that

$$-\lambda_i \ \le\ \mu_k \ \le\ \lambda_j.$$

Now, if $\mu = x_i \check{\alpha}_i^B + x_j \check{\alpha}_i^B = (x_i, x_j - x_i, -x_j)$then the conditions are that

$$-\lambda_i \ \le\ x_i,\ -x_j,\ x_i - x_j \ \le\ \lambda_j.$$

This involves 6 inequalities:

$$\lambda_i + x_i \ge 0 \quad \lambda_j - x_i \ge 0 \quad \lambda_i - x_j \ge 0 \quad \lambda_j + x_j \ge 0 \quad \lambda_i + x_i - x_j \ge 0 \quad \lambda_j + x_j - x_i \ge 0.$$

Say, if $\lambda_i = \lambda_j = p$ then the conditions are that $|x_i|$, $|x_j|$ and $|x_i - x_j|$ are all $\le p$.

Say, in the chamber where $x_i, x_j \ge 0$ the conditions are that $|x_i|, |x_j|, |x_i - x_j \le 1$.

However, the conditions are W-invariant in μ as $\sigma_i(x_i\check{\alpha}_i + x_j\check{\alpha}_j = -x_i\check{\alpha}_i + x_j(\check{\alpha}_i + \check{\alpha}_j) = \check{\alpha}_i(x_j - x_i) + ch\alpha_j x_j$, i.e., $s_i(x_i, x_j) = (x_j - x_i, x_j)$. So, the conditions in all chambers are of the same kind as in the dominant chamber, i.e., the conditions that $w\mu \le \lambda$.

When $p = 1$ we get weights μ with $|\mu_i| \le 1$. If say, $x_i = 0$ then we get three coweights $-\check{\alpha}_j, 0, \check{\alpha}_j$. If $x_j = 0$ we get $-\check{\alpha}_i, 0, \check{\alpha}_i$. When $x_i, x_j \ne 0$ then the last inquality says that $x_i = x_j$ so we also get $\pm(\check{\alpha}_I + \check{\alpha}_j)$. These are the 7 weights of $\check{\mathfrak{g}} = L(\check{\rho})$.

8.5. **The "determinant" map** $\det : (\overline{\mathcal{V}}, \mathcal{V}) \to (\overline{H}_{ad}, H_{\mathrm{ad}})$.

Lemma. (a) The largest commutative quotients of $\mathcal{V}$ and $\overline{\mathcal{V}}$ are H_{ad} and $\overline{H}_{ad}$. So, there is a canonical map of pairs[(58)]

$$(\overline{\mathcal{V}}, \mathcal{V}) \xrightarrow{\det} (\overline{H}_{ad}, H_{\mathrm{ad}}).$$

(b) The following square is Cartesian:

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\det} & H_{\mathrm{ad}} \\ {\scriptstyle\subseteq}\big\downarrow & & {\scriptstyle\subseteq}\big\downarrow \\ \overline{\mathcal{V}} & \xrightarrow{\det} & \overline{H}_{ad}. \end{array}$$

Proof. (a) The claim for $\mathcal{V}$ is obvious since $H_{\mathrm{ad}} = \mathcal{V}/G_{\mathrm{sc}}$ and G_{sc} is semisimple. The map $\overline{\mathcal{V}} \to \overline{H}_{ad}$ is $\square$[59] ■ still to be constructed.

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[58] When $G = SL(U) \cong SL_2$ then $(\overline{\mathcal{V}}, \mathcal{V}) \xrightarrow{\det} (\overline{H}_{ad}, H_{\mathrm{ad}})$ is really the determinnat map $(\mathrm{End}(U), GL(U)) \to (\mathbb{A}^1, G_m)$ (see 3.12).

[59] $\boxed{!}$?

(b) The claim is that if $v \in \overline{\mathcal{V}}$ has $\det(b) \in \overline{H}$ invertible then v itself is invertible. $\square$[60] $\blacksquare$
$\square$

8.6. **Appendix: History of Vinberg semigroups.** Definitions

8.6.1. *The current data.* To a parabolic $P = U \ltimes L$ and its normal subgroup V (between U and $[P,P]$) we associate

- $(G,(P/V)^o)$-spaces $G/V \subseteq (G/V)^{\mathrm{aff}}$;
- groupoid $\mathcal{S} = \mathcal{S}_{P,V}$ over $\mathcal{P} = G/P$ and its semigroupoid closure $\overline{\mathcal{S}}_{P,V}$;
- the Vinberg group $\mathcal{V} = \mathcal{V}_{P,V} \stackrel{\text{def}}{=} (G \times_{Z(G)} Z(P/V)$ and its semigroup closure $\overline{\mathcal{V}}_{P,V}$.

8.6.2. *The definitions and realizations of* $\overline{\mathcal{V}}_{P,V}$.

- The original definition of a Vinberg semigroup was in tems of its ring of functions. It was only valid in characteristic zero (and the data were the standard ones (G,B,N) with G simply connected).
- Alvaro Rittatore defined Vinberg semigroups over an arbitrary field. He considers the category $FM(G_0)$ of *"very flat" reductive monoids* M (irreducible, normal and with a nice abelianization map), such that the derived subgroup $(M^*)'$ of its ivertible part M^* is a given semisimple group G_0.

  Rittatore's abstract machinery is his classification of objects in $FM(G_0)$ from the point of view of the classiphication of spherical varities.

  Then certain data produce the Vinberg semigroup $\mathcal{U}_{G_0}$, However he also
  (1) proves the universal property of $\mathcal{U}_{G_0}$ (so he calls it the *envelope* of the semisimple group G_0);
  (2) and he constructs it geometrically from a torsor over the wonderful compactification.[61]
- My definition/construction should be

$$\overline{\mathcal{V}}_{G,P,V} \stackrel{\text{def}}{=} \mathrm{End}_{\mathcal{Z}_{G,P,V}}[(G/V)^{\mathrm{aff}}].$$

- A conjectural realization/definition via sections of semigroupoids: $\overline{\mathcal{V}}$ is the (affinization of?) stable sections of a certain semigroupoid over $\mathcal{B}$.
- My original notes used a complicated and conjectural definition (conjecture 21.1.2.d). via semigroupoids.
- Pluecker definition/realization?

  In $\mathbb{V} = \oplus\ V_i\ (G/N)^{\mathrm{aff}}$ is given by Pluecker equations. The semigroup $\mathrm{End}(\mathbb{V})$ contains $G = G_{\mathrm{sc}}$ and $\overline{H}$. Then it should also contain $\overline{\mathcal{V}}$ as the subsemigroup of $\mathrm{End}(\mathbb{V})$ generated by the two.[62]

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60 $\boxed{!}$?
61 I noticed this at some point. Did I claim or conjecture this before 2005?
62 This may be in Kapranov's paper on hypergeometric function on reductive groups (at least in some sense)?

Remarks. (0) The universal property of $\overline{\mathcal{V}}$ was formulated and proved by Vinberg (in caracteristic zero) and by Rittatore (over a field). This should be the definition of the Vinberg semigroup of a semisimple (or reductive?) group. Then one can choose his own favorite existence proof or construction.

These so far involve $\mathrm{End}_{\mathbb{H}}((G/N)^{\mathrm{aff}})$, wonderful compatification, Pluecker, classification of nice reductive monods.

(1) Definition over integers. Rittatore does not seem to mention that his construction is over integers (he does not in this paper). (The universal property has not been considered over integers, however this should follow from the result over fields, i.e., I think that flatness statements are checked over geometric points?.)

(2) The endomorphism construction seems philosophically intriguing. (One may think of it as some kind of induction?) That's the interesting part of my approaches.

It ihas been written so far for the standard Vinberg semigroup $\overline{\mathcal{V}}(G) = \overline{\mathcal{V}}_{G,B,N}\mathcal{B}(G)$ (theorem 8.2).

This seems closer to a reasonable definition of Vinberg semigroups.

8.6.3. *Sections below.* In the section 20 We calculate the invertible sections of the groupoid $\mathcal{S} = \mathcal{S}_{\mathcal{P},V}$ as the Vinberg group $\mathcal{V}_{\mathcal{P},G}$ and show that the sections of $\overline{\mathcal{S}}$ are controlled by the T-fixed points.

In the next section 21 we introduce the Vinberg semigroups $\overline{\mathcal{V}}_{\mathcal{P},V} \supseteq \mathcal{V}_{\mathcal{P},V}$ and give a conjectural comparison with the *stable* sections of $\overline{\Gamma^*}(\mathcal{P},\mathcal{S}) \subseteq \Gamma(\mathcal{P},\overline{\mathcal{S}})$ of the semigroupoid $\overline{\mathcal{S}}$.

8.7. **Appendix. The "correspondence" idea.** Let C be a central quotient $(A\times B)/Z$ of $A\times B$. A semigroup closure S of C is a kind of correspondence between A and B. The effect is that

$$\mathcal{G}(C,S) \subseteq \mathcal{G}(C) \ \to\ \mathcal{G}(A/Z)\times\mathcal{G}(B/Z)$$

is indeed a correspondence beetween $\mathcal{G}(A/Z)$ abd $\mathcal{G}(B/Z)$.

Remarks. (0) If A is semisimple and B is a torus then the disconnectedness of $\mathcal{G}(B)$ may pass onto $\mathcal{G}(C,S)$ where it has the effect of separating certain pieces of $\mathcal{G}(A)$.

(1) The price for this separation is that the newely separated pieces of $\mathcal{G}(A)$ acquire in $\mathcal{G}(C,S)$ some nonreduced directions from $\mathcal{G}(B)$.

So, what wet is a description of the said pieces of A as being the reduced part of the connected components of $\mathcal{G}(C,S)$.

8.7.1. *Step 0. The central torus in G.* The semigroup closures of a reductive group G are related to the maximal central torus Z. Group G is an extension $0 \to Z \to G \to\!\!\to G_{\mathrm{ss}} \to 0$ with G_{ss} semisimple. Then

- $\mathcal{G}(S)$ acts freely on $\mathcal{G}(G)$ and the quotient is $\mathcal{G}(\overline{G})$.
- This splits in the sense that

$$\mathcal{G}(S)_0 \times \mathcal{G}(G)_{red} \xrightarrow{\cong} \mathcal{G}(G)$$

and $\mathcal{G}(G)_{red} \to \mathcal{G}(\overline{G})$ is etale and an isomorphism ofconnected components

$$[\mathcal{G}(G)_x]_{red} \xrightarrow{\cong} / \mathcal{G}(\overline{G})_{\overline{x}}.$$

8.8. **Dennis: The Deconcini-Procesi Vinberg semi-group.** 4.2.1. Consider the group

$$G_{enh} \stackrel{\text{def}}{=} \mathcal{V} = \ G \times_{Z(G)} T.$$

Let us recall Vinberg's construction of the semi- group $G_{enh}^{+} = \overline{\mathcal{V}}$, whose locus of invertibility coincides with G_{enh}.

Defining such a semi-group is equivalent to specifying which representations of extend to it. Any representation of $\mathcal{V}$ is a direct sum of ones of the form $V \otimes \ \lambda_H$ the weights of representation V of G are in $\lambda + \check{Q}$. For $\overline{\mathcal{V}}$ we require that the weights of V are $\leq \lambda$.

xxx

Lemma. By construction, we have a canonical map $s : \overline{\mathcal{V}} \to \overline{H}$ with $\overline{\mathcal{V}} \times_{\overline{H}} H = \mathcal{V}$. □

8.8.1. *4.2.3.* For a parabolic P with Levi quotient M let $c_P \in \overline{H}$ be the point defined by the condition that

$$\langle \alpha, c_P \rangle = \delta_{\alpha \in M}$$

for simple roots α. Consider the preimage

$$s^{-1}(c_P) \subseteq \overline{\mathcal{V}}.$$

It contains an open subset isomorphic to

$$G/U \times_M G/U_-.$$

Lemma. 4.2.4. There exists a unique $\overline{\mathcal{V}} \times \overline{\mathcal{V}}$-invariant open subscheme $\overline{\mathcal{V}}^o \subseteq \overline{\mathcal{V}}$ such that for every parabolic P, the intersection

$$s^{-1}(c_P) \ \cap \ \overline{\mathcal{V}}^o$$

equals $G/U \times_M G/U_-$.

9. The wonderful compactification W of $G/Z(G)$

9.1. The wonderful compactification $\overline{G}$ and the Vinberg semigroup $\overline{\mathcal{V}}$.

9.1.1. *The stratification of the wonderful compactification.* The wonderful compactification $\overline{G}$ of G_{ad} has a stratification parameterized by subsets $J{\subseteq}I$. The open stratum $W_\emptyset$ is the subgroup $G/Z(G)$. In general one has $W_J{\cdot}W_K{\subseteq}\ W_{J\cup K}$. So the closures $\overline{W_J}\ = \cup_{K\supseteq J}\ W_K$, are ideals in the semigroup W, i.e., $W{\cdot}\overline{W_J}{\subseteq}\overline{W_J}{\supseteq}\overline{W_J}{\cdot}W$. In particular, $W_I\cong\mathcal{B}{\times}\mathcal{B}$ is a monoid for $(a,b){\cdot}(c,d)=(a,d)$.

For SL_2 the wonderful compactification is the same as the quasimap compactification of automorphisms of $\mathbb{P}^1$. The standard point of view on the wonderful compactification interprets $G/Z(G)$ as maps from $\mathfrak{g}$ to $\mathfrak{g}$. In general, there should be a point of view on the wunderbar compactification that interprets $G/Z(G)$ as stable maps from $\mathcal{B}$ to $\mathcal{B}$. [This has been done by Brion!]

9.1.2. Remember that there should be a resolution $\widetilde{W}$ of the Vinberg semigroup that is an extension of W by the semigroup closure of the Cartan.

I proposed to describe $\widetilde{W}$ as sections of a semigroupoid, in order to make the semigroup structure manifest.

9.2. **Wonderful compactification as a Hilbert scheme.**

9.2.1. *Lemma.* (a) The closed stratum $\mathcal{B}{\times}\mathcal{B}{\subseteq}\ W$ is interpreted in terms of the Hilbert scheme of subvarieties of $\mathcal{B}{\times}\mathcal{B}$ by: point $(p,q)\in\ \mathcal{B}^2$ corresponds to the cross $p{\times}\mathcal{B}\cup\mathcal{B}{\times}q$.

(b) Let $e\in\ \ \mathfrak{b}\in\ \ \mathcal{B}$ be a regular nilpotent and consider the line $\boldsymbol{e}^{\mathbb{C}e}$ in $G{\subseteq}\ W$. Its boundary is the unique B-fixed point in the closed stratum W.

(c) All subschemes of $\mathbb{B}^2$ that lie in W are reduced.

Proof. (b) The boundary point is the subscheme $Y=\ \lim\limits_{s\to\infty}\ (1,\boldsymbol{e}^{se}){\cdot}\ \Delta_\mathcal{B}$ of $\mathcal{B}^2$. Since for any $\mathfrak{b}'\in\mathcal{B}$ one has $\lim\limits_{s\to\infty}\ (1,\boldsymbol{e}^{se}){\cdot}\ \mathfrak{b}'=\ \ \mathfrak{b}$ (the only e-fixed point in $\mathcal{B}$), Y contains all $\lim\limits_{s\to\infty}\ (1,\boldsymbol{e}^{se}){\cdot}\ (\mathfrak{b}',\mathfrak{b}')=\ \ (\mathfrak{b}',\mathfrak{b})$, i.e., $\mathcal{B}{\times}\mathfrak{b}$. Since $\boldsymbol{e}^{\mathbb{C}e}{\cdot}\ \Delta_\mathcal{B}$ is invariant under the switch of coordinates, so is Y. So Y contains the cross $\mathfrak{b}{\times}\mathcal{B}\ \cup\ \mathcal{B}{\times}\mathfrak{b}$. Since both Y and the cross are in the same Hilbert scheme they are equal. So Y is the unique B-fixed point $(\mathfrak{b},\mathfrak{b})$ in the closed stratum $\mathbb{B}{\times}\mathcal{B}$ of W.

(c) follows from (a) since the most degenerate schemes in W are reduced.

9.2.2. *Example: SL_2.* Here W is $\mathbb{P}^3=\mathbb{P}(M_2)$ and $W_\emptyset=G/Z(G)=PGL_2$ is the projectivization rank two operators, while $W_I=\mathbb{P}^1{\times}\mathbb{P}^1$ is the projectivization of rank one operators. Its extension $\widetilde{W}\overset{\text{def}}{=}\overline{\Gamma(\mathcal{G})}{\rightarrow}$ is the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. The zero-section is an ideal with the above semigroup structure. The remaining torsor for $H\cong C^*$ consists of the Vinberg group $\widetilde{W_\emptyset}=GL_2$, and over W_I one has $\widetilde{W_I}=$ rank one operators. The semigroup structure is just the multiplication of matrices except in the case of two rank one matrices A,B with the composition of rank zero. Arguing by continuity, the product $A{\circ}B$ lies in the zero section and equals the pair of lines $(Im(A),Ker(B))$.

Affinization map from $\widetilde{W}$ to the Vinberg semigroup M_2 is the blow up.

Observe that $\widetilde{W}(=$ blow up of $M_2)$ acts on $G/N\underset{H}{\times}\overline{H}(=$ blow-up of $\mathbb{C}^2)$, this is the blow-up (continuous extension) of the action of matrices on vectors.

Makes you wonder about all other examples.

9.3. **Singularities of stable maps from $\mathcal{P}$ to $\mathcal{P}$.** W seems to be the closure of $Aut(\mathcal{P})$ in "stable maps" from $\mathcal{P}$ to itself.

9.3.1. *Example: projective space.* If $\mathcal{P}$ is the projective space $\mathbb{P}(V)$ for $G = SL(V)$, the semigroup is $\mathrm{End}(V)$ and $0 \neq A \in End(V)$ defines a rational map from $\mathbb{P}(V)$ to itself, which is not defined on $Ker(A)$. I guess that it is defined on the blow-up along $Ker(A)$.

There is no singularity if the kernel is a hyperplane, i.e., for the rank 1 operators, since the blow up does not change the space. However, one still remembers the kernel, i.e., while the associated map for $\mathcal{P}$ to itself is the constant map $Im(A)$, the stable map associated to A is in this case the pair $(Im(A), Ker(A)) \in \mathbb{P}(V)\times\mathbb{P}(V^*) = \mathbb{P}(\text{rank 1 operators})$.

In general, one should hopefully have singularities on the embedded partial flag varieties of Levi factors. It may be interesting to see what reductive generalization of the blow up will be needed to make the maps defined.

9.4. **The strata of W are groupoids?**

9.4.1. *General "wonderful compactifications" $W_{P,V}$?* For a parabolic P there isa Vinberg semigroup $\overline{\mathcal{V}}_P$ with the invertible part $\mathcal{V}_P \stackrel{\text{def}}{=} G \hookrightarrow G \underset{Z(G)}{\times} P^{\text{ab}}$. The corresponding wonderful compactification is the invariant theory quotient

$$W_P \stackrel{\text{def}}{=} \mathcal{V}_P//P^{\text{ab}} \stackrel{\text{def}}{=} ,$$

9.4.2. *Orbits in $(G/N)^{\text{aff}}$ and in $\overline{\mathcal{V}}$.*

Question. The number of G-orbits in the affine closure of G/N is 2^{rank}, and their mutual position is "toric", the same as for the G-orbits in the wonderful compactification of G. (See remark 24.3.5.)

10. **Reductive semigroup closures**

10.1. **Semigroup closures and normality.** Let L be a reductive group. A "partial toric" compactification of L is an open dense inclusion $L\subseteq\overline{L}$ of $L\times L$-spaces.

10.1.1. *Lemma.* If $\overline{L}$ is affine and normal it is a semigroup closure of L.

Proof. $L{\times}L$-inclusion means that $L{\times}L \to L$ extends to both

$$L{\times}\overline{L} \to \overline{L} \quad \text{and} \quad \overline{L}{\times}L \to \overline{L},$$

hence to $\overline{L}{\times}\overline{L} - \partial L{\times}\partial L \to \overline{L}$. Now, the codimension of $\partial L{\times}\partial L$ is ≥ 2, and $\overline{L}{\times}\overline{L}$ is normal while the target $\overline{L}$ is affine.

10.1.2. *Sublemma.* If $X \overset{open\ dense}{\supseteq} U \xrightarrow{f} Y$ and

- $\mathrm{codim}_X(\partial U) \geq 2$,
- X is normal,
- Y is affine

then f extends to X.

Proof. Y is closed in some $\mathbb{A}^n$, so we can assume that $Y = \mathbb{A}^n$, and then that $n = 1$.

10.1.3. *Remark.* (1) One needs X to be normal. Say X is the crossing of two $\mathbb{A}^2$'s called X_i at a point p, then $U = X - \{p\} = \cup\ X_i - \{p\}$ and a function f on U consists of two functions f_i on $X_i - \{p\}$, which extend to X_i. Now f extends to X iff $f_1(p) = f_2(p)$.

(2) One needs Y to be affine: Wonderful compactification W of an adjoint group G is not a semigroup, so extension fails for $X = W{\times}W$ (normal) and $U = X - \partial G{\times}\partial G$ (of codimension 2).

10.2. The Grothendieck resolution of a semigroup closure $\overline{G}$ of G.

10.2.1. *Lemma.* (a) $\overline{T}$ is an affine toric variety.

(b) N is closed in $\overline{G}$.

Proof. (a) by definitions. (b) N is an orbit of N in an affine variety $\overline{G}$, hence closed!

10.2.2. The incidence subvariety

$$\mathcal{M} \overset{\text{def}}{=} \{(x, B) \in \overline{G}{\times}\mathcal{B};\ x \in \overline{B}\}$$

projects to $\mathcal{B}$ as $G \underset{B}{\times} \overline{B}$. It maps , to $\overline{H} \overset{\text{def}}{=} \overline{B}//N$, and the most interesting map is

$$\mathcal{M} \xrightarrow{\pi} \overline{G}{\times}_{\overline{G}//G=\overline{H}//W}\overline{H}.$$

10.2.3. *Remark.* $N{\times}\overline{T} \to \overline{B}$ is not injective if $\overline{G}$ has a zero. It need not be surjective because of 14.1.5.

10.2.4. *Lemma.* π is proper, surjective and generically an isomorphism.

Proof. Even the map $\mathcal{M} \xrightarrow{\pi} \overline{G}\times_{\overline{G}//G}\overline{H} \to \overline{G}$ is proper since it factors as $\mathcal{M}\subseteq\overline{G}\times\mathcal{B} \to \overline{G}$. Next, over $G\times_{H//W}H$ this is the usual Grothendieck resolution, so it is an isomorphism over $G_{rs}\times_{H//W}H_r$. Finally, map is surjective since it is proper and generically an isomorphism.

10.2.5. *Question.* Let T be a torus. In an irreducible affine T-variety X, is the fixed point set X^T connected?

10.2.6. *Question.* (No) Is the map of sets of orbits $\overline{T}/T \to G\backslash(G/V)^{\mathrm{aff}}$ surjective?

10.2.7. *Conjecture.* $G\times L$-orbits in $(G/U)^{\mathrm{aff}}$, and $G\times L$-orbits in $(G/U)^{\mathrm{aff}}$, are both parameterized by the W_L-orbits in $\Delta_T(\mathfrak{u}^{\mathrm{ab}})$, i.e., the minimal roots in $\mathfrak{u}$.

11. Langlands duality of reductive semigroups

11.1. Tori.

11.2. Reductive groups.

Remark. In order that for a torus extension $\mathcal{V}$ of G and $Y\supseteq Y^o = \mathcal{V}/A$, $\mathcal{G}(\mathcal{V}, Y)$ is $G_{\mathcal{O}}$-equivariant, we need $A = 1$ so that $\ddot{A} = \mathcal{V}_{\mathcal{O}}$. So, Y must be a partial compactification of $\mathcal{V}$. In order to have a convolution on $\mathcal{G}(\mathcal{V}, Y)$ we then need the group structure on $Y^o = \mathcal{V}$ to extend to a monoid structure on Y ???

11.3. **Semigroups with zero.** Let V be a normal subgroup of P that contains U, so that $M = M$ is reductive.

11.3.1. *Lemma.* For the irreducible $L^G(\lambda)$ and the coWeyl module $\check{W}(\lambda)$, one has

$$L^G(\lambda)^V = L^M(\lambda) \quad \text{and} \quad \check{W}^G(\lambda)^V = \check{W}^M(\lambda)$$

if λ is orthogonal to the T-roots in $\mathfrak{V}$, and otherwise it is zero.

Proof. Representation $L^(\lambda)^V$ of M is an extension of irreducible representations $L^M(\mu)$. Each μ is dominant for $B/V \cap B$, hence also for B. So there is at most one term in the socle and it is $L^M(\lambda)$. It appears iff the B-highest weight space λ is V-fixed, i.e., iff $\lambda \perp \Delta_T(\mathfrak{V})$.

11.3.2. *Lemma.* G/V is quasiaffine.

Proof. The image of the G-map $G/V \xrightarrow{\tau} (G/V)^{\mathrm{aff}}$ is a homogeneous space G/K for some $V\subseteq K\subseteq G$.

τ is an embedding on $M \overset{\mathrm{def}}{=} P/V$ since

Part 2. Pieces

12. Technical pieces

12.1. **Fibered products of quotient stacks $G_1\backslash Y_1 \times_{G_0\backslash Y_0} G_2\backslash Y_2$.** We consider a system of groups $G_1 \to G_0 \leftarrow G_2$ and a compatible system $Y_1 \xrightarrow{a_1} Y_0 \xleftarrow{a_2} Y_2$ of G_i-spaces Y_i.

Lemma. The fibered products of quotient stacks simplifies to

$$G_1\backslash Y_1 \times_{G_0\backslash Y_0} G_2\backslash Y_2 \;\cong\; G_0\backslash[(G_0{\times}_{G_1}Y_1) \times_{Y_0} (G_0{\times}_{G_2}Y_2)]$$

$$\cong\; G_1\backslash[Y_1{\times}_{Y_0} (G_0{\times}_{G_2}Y_2)] \;\cong\; G_2\backslash[(G_0{\times}_{G_1}Y_1){\times}_{Y_0} Y_2].$$

Proof. $G_1\backslash Y_1 \times_{G_0\backslash Y_0} G_2\backslash Y_2$ can be written as

$$G_0\backslash(G_0{\times}_{G_1}Y_1) \times_{Y_0} G_0\backslash(G_0{\times}_{G_2}Y_2)$$

where a_i extends tp $G_0{\times}_{G_i}Y_i \xrightarrow{\alpha_i} Y_0$ by $\alpha_i[\overline{(g_0,u_i)}] = \; g_0{\cdot}\; a_i(y_i)$. This is isomorphic to

$$G_0\backslash[(G_0{\times}_{G_1}Y_1) \times_{Y_0} (G_0{\times}_{G_2}Y_2)]$$

where G_0 acts diagonally. Then the last two nonsymmetric formulas follow. $\square$

$f \in\; Map[S, G_1\backslash Y_1 \times_{G_1\backslash Y_1} G_2\backslash Y_2]$ consists of maps $f_i : S \to G_i\backslash Y_1$ with compatibility of $\alpha_i{\circ}f_i$ for $\alpha_i : G_i\backslash Y_i \to G\backslash Y$. These are the diagrams $S \xleftarrow{\phi_i} P_i \xrightarrow{\psi_i} Y_i$ with an identification of the diagrams

$$S \xleftarrow{\phi_i} G{\times}_{G_i}P_i \xrightarrow{\psi_i} Y_i \xrightarrow{a_i} Y$$

, i.e.,

$$\begin{array}{ccccc}
S & \xleftarrow{\phi_1} & G{\times}_{G_1}P_1 & \xrightarrow{a_1\psi_1} & Y \\
{=}\downarrow & & \zeta\downarrow\cong & & {=}\downarrow \\
S & \xleftarrow{\phi_2} & G{\times}_{G_2}P_2 & \xrightarrow{a_2\psi_2} & Y
\end{array}$$

$\square$

12.2. **Describing the nonreduced directions.** It seems more difficult to me since the traditional tools like Bialnicky-Bitula are now unclear to me!

12.3. **Colored divisors of meromorphic sections of H-torsors.** For a simply connected G we have $\prod_{i\in I}\; \omega_i \;:\;\; H\breve{\omega}{G_m}^I$. So, an H-torsor S is a system of G_m-torsors $\mathsf{S}_i,\; i\in I,\;$ i.e., of the corresponding line bundles $L_i = \mathbb{A}^1{\times}_{G_m}\mathsf{S}_i$. A meromorphic section σ of S is locally a system of I meromorphic functions s_i. So, we can define its I-colored divisor $\mathrm{div}(\sigma) \in\; \mathbb{Z}[I]$.

For a Cartan subgroup $T\subseteq G$, a choice of a Borel $B\supseteq T$ defines $T\xrightarrow{\cong}H$ as $T\subseteq B\twoheadrightarrow H$. This moves T-torsors to H-torsors $S\mapsto S_B$ and similarly for sections $s\mapsto s_B$. So, a section s of a T-torsor S acquires an I-colored divisor $\mathrm{Div}_B(s) \stackrel{\text{def}}{=} \mathrm{Div}(s_B)$.

12.4. **The closure of a subindscheme.** The closure of a subindscheme $Y = \underset{\rightarrow}{\mathbf{lim}}\ Y_i$ in the indscheme $X = \underset{\rightarrow}{\mathbf{lim}}\ X_i$ can be described as the ind system of closures of subschemes: $\overline{Y} \stackrel{\text{def}}{=} \underset{\rightarrow}{\mathbf{lim}}\ \overline{Y_i} \subseteq X$.

Question. Does the passage from indschemes to schemes, i.e., commute with closures?

12.5. **Stratifications of schemes.** The interest here is in a definition.

Remark. However the goal of the present text seems to be to construct for $X \to S$ a stratification of X and maube S whose strata are flat (?) or the map is stratified flat? ("Flattening stratifications"). This stratification is preferably canonical.

12.5.1. *Functor.* Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module. For any scheme T over S we will denote $\mathcal{F}_{\mathcal{T}}$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_{\mathcal{T}}$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Since the base change of a flat module is flat we obtain a functor

$$F_{flat} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_{\mathcal{T}} \text{ is flat over } \mathcal{T}, \\ \emptyset & \text{else.} \end{cases} \tag{1}$$

12.5.2. *Flattening stratifications.* Just the definitions and an important baby case.

Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module. We say that the *universal flattening of $\mathcal{F}$ exists* if the functor F_{flat} defined in Situation ?is representable by a scheme S' over S. We say that the *universal flattening of X exists* if the universal flattening of $\mathcal{O}_{\mathcal{X}}$ exists.

Note that if the universal flattening S' [63] of $\mathcal{F}$ exists, then the morphism $S' \to S$ is a monomorphism of schemes such that $\mathcal{F}_{\mathcal{S}'}$ is flat over S' and such that a morphism $T \to S$ factors through S' if and only if $\mathcal{F}_{\mathcal{T}}$ is flat over T.

We define (compare with Topology, Remark ?a (locally finite, scheme theoretic) *stratification* of a scheme S to be given by closed subschemes $Z_i \subset S$ indexed by a partially

[63] The scheme S' is sometimes called the *universal flatificator*. In [**?**] it is called the *platificateur universel*. Existence of the universal flattening should not be confused with the type of results discussed in More on Algebra, Section ?

ordered set I such that $S = \bigcup Z_i$ (set theoretically), such that every point of S has a neighbourhood meeting only a finite number of Z_i, and such that

$$Z_i \cap Z_j = \bigcup\nolimits_{k \leq i,j} Z_k.$$

Setting $S_i = Z_i \setminus \bigcup_{j<i} Z_j$ the actual stratification is the decomposition $S = \coprod S_i$ into locally closed subschemes. We often only indicate the strata S_i and leave the construction of the closed subschemes Z_i to the reader. Given a stratification we obtain a monomorphism

$$S' = \coprod\nolimits_{i \in I} S_i \longrightarrow S.$$

We will call this the *monomorphism associated to the stratification*. With this terminology we can define what it means to have a flattening stratification.

12.5.3. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module. We say that $\mathcal{F}$ has a *flattening stratification* if the functor F_{flat} defined in Situation ?is representable by a monomorphism $S' \to S$ associated to a stratification of S by locally closed subschemes. We say that X has a *flattening stratification* if $\mathcal{O}_{\mathcal{X}}$ has a flattening stratification.

When a flattening stratification exists, it is often important to understand the index set labeling the strata and its partial ordering. This often has to do with ranks of modules, as in the baby case below.

Lemma. Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_{\mathcal{S}}$-module. The closed subschemes

$$S = Z_{-1} \supset Z_0 \supset Z_1 \supset Z_2 \ldots$$

defined by the fitting ideals of $\mathcal{F}$ have the following properties

(1) The intersection $\bigcap Z_r$ is empty.
(2) The functor $(\mathrm{Sch}/S)^{opp} \to Sets$ defined by the rule

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_{\mathcal{T}} \text{ is locally generated by } \leq \nabla \text{ sections} \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the open subscheme $S \setminus Z_r$.
(3) The functor $F_r : (\mathrm{Sch}/S)^{opp} \to Sets$ defined by the rule

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_{\mathcal{T}} \text{ locally free rank } \nabla \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the locally closed subscheme $Z_{r-1} \setminus Z_r$ of S.

If $\mathcal{F}$ is of finite presentation, then $Z_r \to S$, $S \setminus Z_r \to S$, and $Z_{r-1} \setminus Z_r \to S$ are of finite presentation.

Proof. We refer to More on Algebra, Section ?for the construction of the *fitting ideals* in the algebraic setting. Here we will construct the sequence

$$0 = \mathcal{I}_{-\infty} \subset \mathcal{I}_{\prime} \subset \mathcal{I}_{\infty} \subset \ldots \subset \mathcal{O}_{\mathcal{S}}$$

of fitting ideals of $\mathcal{F}$ as an $\mathcal{O}_{\mathcal{S}}$-module. Namely, if $U \subset X$ is open, and

$$\bigoplus\nolimits_{i\in I} \mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}^{\oplus\backslash} \to \mathcal{F}|_{\mathcal{U}} \to \prime$$

is a presentation of $\mathcal{F}$ over U, then $\mathcal{I}_{\nabla}|_{\mathcal{U}}$ is generated by the $(n-r)\times(n-r)$-minors of the matrix defining the first arrow of the presentation. In particular, $\mathcal{I}_{\nabla}$ is locally generated by sections, whence quasi-coherent. If $U = \mathrm{Spec}(A)$ and $\mathcal{F}|_{\mathcal{U}} = \widetilde{\mathcal{M}}$, then $\mathcal{I}_{\nabla}|_{\mathcal{U}}$ is the ideal sheaf associated to the fitting ideal $\mathrm{Fit}_r(M)$ as in More on Algebra, Definition ?Let $Z_r \subset S$ be the closed subscheme corresponding to $\mathcal{I}_{\nabla}$.

For any morphism $g : T \to S$ we see from More on Algebra, Lemma ?that $\mathcal{F}_{\mathcal{T}}$ is locally generated by $\leq r$ sections if and only if $\mathcal{I}_{\nabla} \cdot \mathcal{O}_{\mathcal{T}} = \mathcal{O}_{\mathcal{T}}$. This proves (2).

For any morphism $g : T \to S$ we see from More on Algebra, Lemma ?that $\mathcal{F}_{\mathcal{T}}$ is free of rank r if and only if $\mathcal{I}_{\nabla} \cdot \mathcal{O}_{\mathcal{T}} = \mathcal{O}_{\mathcal{T}}$ and $\mathcal{I}_{\nabla-\infty} \cdot \mathcal{O}_{\mathcal{T}} = \prime$. This proves (3).

The final statement of the lemma follows from the fact that if $\mathcal{F}$ is of finite presentation, then each of the morphisms $Z_r \to S$ is of finite presentation as $\mathcal{I}_{\nabla}$ is locally generated by finitely many minors. This implies that $Z_{r-1} \setminus Z_r$ is a retrocompact open in Z_r and hence the morphism $Z_{r-1} \setminus Z_r \to Z_r$ is of finite presentation as well. $\square$

Lemma ?notwithstanding the following lemma does not hold if $\mathcal{F}$ is a finite type quasi-coherent module. Namely, the stratification still exists but it isn't true that it represents the functor F_{flat} in general.

Lemma. Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{S}}$-module of finite presentation. There exists a flattening stratification $S' = \coprod_{r\geq 0} S_r$ for $\mathcal{F}$ (relative to $\mathrm{id}_S : S \to S$) such that $\mathcal{F}|_{\mathcal{S}_{\nabla}}$ is locally free of rank r. Moreover, each $S_r \to S$ is of finite presentation.

Proof. Suppose that $g : T \to S$ is a morphism of schemes such that the pullback $\mathcal{F}_{\mathcal{T}} = \}^*\mathcal{F}$ is flat. Then $\mathcal{F}_{\mathcal{T}}$ is a flat $\mathcal{O}_{\mathcal{T}}$-module of finite presentation. Hence $\mathcal{F}_{\mathcal{T}}$ is finite locally free, see Properties, Lemma ?Thus $T = \coprod_{r\geq 0} T_r$, where $\mathcal{F}_{\mathcal{T}}|_{\mathcal{T}_{\nabla}}$ is locally free of rank r. This implies that

$$F_{flat} = \coprod\nolimits_{r\geq 0} F_r$$

in the category of Zariski sheaves on Sch/S where F_r is as in Lemma ?It follows that F_{flat} is represented by $\coprod_{r\geq 0}(Z_{r-1} \setminus Z_r)$ where Z_r is as in Lemma ? $\square$

We end this section showing that if we do not insist on a canonical stratification, then we can use generic flatness to construct some stratification such that our sheaf is flat over the strata.

Lemma. [Generic flatness stratification] Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Let $\mathcal{F}$ be an $\mathcal{O}_{\mathcal{X}}$-module of finite presentation. Then there exists a $t \geq 0$ and closed subschemes

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that $S_i \to S$ is defined by a finite type ideal sheaf, $S_0 \subset S$ is a thickening, and $\mathcal{F}$ pulled back to $X \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$.

Proof. We can find a cartesian diagram

$$X \xrightarrow{d}\xrightarrow{r} X_0 \xrightarrow{d} S \xrightarrow{r} S_0$$

and a finitely presented $\mathcal{O}_{\mathcal{X}_\prime}$-module $\mathcal{F}_\prime$ which pulls back to $\mathcal{F}$ such that X_0 and S_0 are of finite type over $\mathbf{Z}$. See Limits, Proposition ?and Lemmas ?and ?Thus we may assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module.

Assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. By generic flatness as stated in Morphisms, Proposition ?there is a dense open $U_0 \subset S_0$ such that $\mathcal{F}$ pulled back to $X \times_S U_0$ is flat over U_0. Let $S_1 \subset S_0$ be the reduced closed subscheme whose underlying closed subset is $S \setminus U_0$. We continue in this way, provided $S_1 \neq \emptyset$, to find $S_0 \supset S_1 \supset \ldots$. Because S is Noetherian any descending chain of closed subsets stabilizes hence we see that $S_t = \emptyset$ for some $t \geq 0$. $\square$

12.6. **Partial closures of the group.** Any partial compactification $\overline{L}$ of a group L defines a submoduli $\mathcal{G}(L)_{L^\bullet} \subseteq \mathcal{G}(L)$ consisting of all $(\mathsf{S}, \sigma, \mathcal{D})$ in $\mathcal{G}(L)$ such that σ extends to a section $\sigma_{L^\bullet}$ of the $L^\bullet$-bundle $\mathsf{S}_{L^\bullet} \stackrel{\text{def}}{=} \mathsf{S} \times_L L^\bullet$.

Example. For a parabolic P let $L \subseteq P$ be a Levi factor, i.e., a subgroup section of $P \to \underline{P}$. A partial compactification $L^\bullet$ of L defines $\mathcal{G}(P)_{L^\bullet} \stackrel{\text{def}}{=} \mathcal{G}(P) \times_{\mathcal{G}(L)} \mathcal{G}(L)_{L^\bullet} \subseteq \mathcal{G}(P)$. (It consists of all $(\mathsf{S}_P, \sigma_P) \in \mathcal{G}_(P)$ such that $(\mathsf{S}_P/U, \sigma_{\underline{P}}) \in \mathcal{G}_(\underline{P})$ lies in $\mathcal{G}(L)_{L^\bullet}$, i.e., such that $\sigma_{\underline{P}}$ extends to a section $\sigma_{\underline{P}^\bullet}$ of $\mathsf{S}/U \times_{\underline{P}} \underline{P}^\bullet$.)

12.7. **General parabolic zastava spaces.** More generally, one can associate a zastava space

$$\mathcal{G}[G, (G/V)^{\text{aff}} \times (G/V^-)^{\text{aff}}/H_-]$$

to any choice of opposite parabolic subgroups $P^\pm$ of G and their normal subgroups $V^\pm$, such that the spaces $G/V^\pm$ are quasiaffine. (Here $P = P^+$ etc.) $\square$[64] $\blacksquare$ [65]

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64 $\boxed{!}$

65 Instead of H_- may use $P^+ \cap P^-$?

13. **Example:** $G = SL_2$ **(Versions)**

There are three parts below from three different sources:

- A. Present text. This is above in 3.12.
- B. From the original writing on Drinfeld's classifying formulation
- C. From Vinberg text This is reapeated below in 20.4.

Part A. deals with $\mathcal{G}(\mathcal{V}, Y)$ for $\mathcal{V} = GL(U)$ and $Y = (G/N)^{\mathrm{aff}}$ or $\overline{\mathcal{V}} = \mathrm{End}(U)$.

Part B. deals with the zastava space for SL_2. One first describes $\mathrm{End}(U) \supseteq (U-0)^2 \supseteq GL(U)$. Then the standard description of fibers of zastava space (as certain z-Grassmannians) is recondtrcted from ? point of view.

Part C. deals with reconstructing the Vinberg semigroup $\overline{\mathcal{V}}$ as sections of the semigroupoid $\overline{\mathcal{S}} = G \times_B (G/N)^{\mathrm{aff}}$ over $\mathbb{P}^1 = \mathbb{P}(U)$. The sections are described as the realization $\mathrm{End}(U)$ of $\overline{\mathcal{V}}$.

13.1. **B. From the original writing on Drinfeld's classifying formulation.**

13.1.1. *Realization of End(U) as* $(G/N^+)^{\mathrm{aff}} \times (G/N^-)^{\mathrm{aff}}$. Here, $G = SL(U)$ acts on the vector space $U = \langle e_+, e_- \rangle$ (we also denote $e_1 = e_+$ and $e_2 = e_-$). Let $e^{\pm}$ be the dual basis of U^*.

We choose N^+, N^- as stabilizers of e_+, e_- or equivalently of vectors e^-, e^+ in U^*. (In the basis (e_+, e_-) we have $e_+ = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $e_- = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, hence $e_+ = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)$, $e_- = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)$ and $N = N^+ = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and $N^- = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$.)

This gives identifications

$$G/N^{\pm} \xrightarrow{\cong} U - 0, \quad gN^{\pm} \mapsto ge_{\pm}, \quad \text{hence} \quad (G/N^{\pm})^{\mathrm{aff}} \xrightarrow{\cong} U.$$

The sum of two copies $U^{\pm}$ of U identified with $(G/N^{\pm})^{\mathrm{aff}}$ can be identified with two columns of the matrix algebra $\mathrm{End}(U)$:

$$(G/N^+)^{\mathrm{aff}} \times (G/N^-)^{\mathrm{aff}} \xrightarrow{\cong} U^+ \oplus U^- \ \cong \ \mathrm{End}(U) \ = \ U \otimes U^*.$$

13.1.2. *The open cell N^+TN_- in G.* It consists of all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G = SL_2$ such that $d \neq 0$. [66] This cell can also be described as all $g \in G$ such that the e_--component $[ge_- : e_-]$ of ge_- is nonzero. Its boundary in G is therefore given by $d = 0$ or by $ge_- \in \ \Bbbk e_+$.

Lemma. There are canonical identifications

$$\begin{tikzcd}
(G/N^+)^{\mathrm{aff}} \times (G/N^-)^{\mathrm{aff}} \arrow[d, "\cong"'] & G/N^+ \times G/N^- \supseteq \arrow[l, "\supseteq"'] \arrow[d, "\cong"'] & (G/N^+ \times G/N^-)^o \arrow[l, "\supseteq"'] \arrow[d, "\cong"'] \\
\mathrm{End} U & (U-0)^2 \arrow[l, "\supseteq"'] & GL(U) \arrow[l, "\supseteq"']
\end{tikzcd}$$

.

Proof. We know the identifications in the first two columns. , i.e., the columns are not zero. The open subset $(G/N^+ \times G/N^-)^o \subseteq G/N^+ \times G/N^-$ is given by all $(v_+, v_-) = (g_+ e_+ . g_- e_-) \in U \oplus U$ for $g_{\pm} \in G$ such that ${g_+}^{-1} g_-$ lies in the open cell N^+TN_-. This means that $(g_+, g_-) \in \ G(1 \times N^+ B^-)$. Since, $(1 \times N^+ B^-)(e_+, e_-) = \left(\begin{smallmatrix} 1 & \Bbbk \\ 0 & \Bbbk^* \end{smallmatrix}\right)$ lies in $GL(U)$ and contains $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \Bbbk^* \end{smallmatrix}\right)$, its G-image is precisely $GL(U)$. $\square$

13.1.3. *The divisor in $\mathcal{Y}_G$ is given by* $\det = 0$.

---

[66] $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot \begin{pmatrix} s & 0 \\ y & 1/s^{-1} \end{pmatrix} = \begin{pmatrix} s+xy & xs^{-1} \\ y & s^{-1} \end{pmatrix}$.

Corollary. The divisor $\partial\mathcal{Y}_G^o\subseteq\mathcal{Y}_G$ is given by the equation $\det = 0$ in the $G\times H$-torsor $(G/N^+)^{\text{aff}}\times(G/N^-)^{\text{aff}}\cong\text{End}(U)$ over $\mathcal{Y}_G$. □

13.1.4. *Modular description of fibers of the zastava spaces as a certain Grassmannian.* A map $\phi = (\phi^+,\phi^-)$ from (C,a) to $(G/N^+)^{\text{aff}}\times(G/N^-)^{\text{aff}}\xrightarrow{\cong} U^+\oplus U^- = \text{End}(U)$ defines two subsheaves $\mathcal{L}^\pm = \mathcal{O}_C f^\pm$ of the trivial rank two bundle $\mathcal{V} = U\otimes\mathcal{O}_C$ over C.

Lemma. This identifies the fiber $Z(G)_D$ of $Z_C(G)$ at $D = na\in\mathcal{H}_C$ with the space of $\mathcal{O}(D)$-submodules of $U_D = U\otimes\mathcal{O}(D) = U_\mathcal{O}/z^nU_\mathcal{O}$ of rank n.[67]

Proof. The condition that f is generically in $(G/N^+\times G/N^-)^o = GL(U)$ means that f^+,f^- is generically a frame of $\mathcal{V}$, i.e., that $\mathcal{L}^+ + \mathcal{L}*1\subseteq\mathcal{V}$ is generically an equality. In particular, $\mathcal{L}^\pm$ are locally free sheaves subsheaves of $\mathcal{V}$ of rank one. Let $\mathcal{L}^\pm\subseteq L^\pm\subseteq\mathcal{V}$ be their extensions to line subbundles of $\mathcal{V}$.

We can assume that L^- is $\mathcal{O}_Ce_-$ or more precisely that $f_- = z^de_-$ where d is the order of vanishing of f_- at $a = 0$ in C. This reduces the symmetry from $(G\times H)(C)$ to the stabilizer (???) $(N^-\times H)(C)$ of z^de_-.

... □

13.2. **C. From Vinberg text.** Let $U = \langle e_+,e_-\rangle$ and $G = SL(U)$.

It deals with reconstructing the Vinberg semigroup $\overline{\mathcal{V}}$ as sections of the semigroupoid $\overline{\mathcal{S}} = G\times_B(G/N)^{\text{aff}}$ over $\mathbb{P}^1 = \mathbb{P}(U)$. The sections are described as the realization $\text{End}(U)$ of $\overline{\mathcal{V}}$.

13.2.1. For $G = SL_2$, the semigroupoid $\overline{\mathcal{S}} \overset{\text{def}}{=} (G/N)^o\to\mathcal{B}$ can be identified with the vector bundle $\mathcal{O}^2_{\mathbb{P}^1}(1)$ over $\mathbb{P}^1$ (which appears in various settings), and the sections $\Gamma(\mathcal{B},\overline{\mathcal{S}})$ with 2×2 matrices M_2.

13.2.2. *Corollary to the lemma 3.12.3.*

Corollary. (a) $G\times H$-equivariant bundle $\overline{\mathcal{S}} \overset{\text{def}}{=} (\overline{G/N})^0\to\mathcal{B}$ is isomorphic to the $G\times G_m$-equivariant vector bundle $(\mathfrak{g}/\mathfrak{n})^0\to\mathcal{B}$. Here G_m acts on the vector bundle in the standard way and we use identification $\rho: H\xrightarrow{\cong}G_m$.

(b) $(\mathfrak{g}/\mathfrak{n})^0 \cong \mathcal{O}(1)\oplus\mathcal{O}(1) \cong \mathcal{T}_\mathbb{H}$ (the twistor space of the hyperkähler manifold $\mathbb{H}$). In particular, the sections of $(\mathfrak{g}/\mathfrak{n})^0 = \widetilde{T}^*\mathcal{B}$ (the universal twisted cotangent bundle), can be identified with the set M_2 of 2×2 matrices.

13.2.3. *Identification of* $\Gamma(\mathbb{P}^1,\mathcal{S})$ *with* GL_2. In this case it is simpler to think of $\overline{\mathcal{S}}$ first. With the conjugation action of B, $\overline{G/N}$ has been identified with $U(1)$, and this induces $G/N \cong U(1)-\{0\}$. Therefore, sections of $\overline{\mathcal{S}}\to\mathcal{B}$ can be identified with M_2. A bases e_1,e_2 of U gives $\Gamma[\mathbb{P}^1,U(1)] = \Gamma[\mathbb{P}^1,\mathcal{O}(1)]e_1\oplus\Gamma[\mathbb{P}^1,\mathcal{O}(1)]e_2$, with each summand of dimension two and giving one row of M_2 (or a column?).

Sections of $\mathcal{S}\to\mathcal{B}$ are the non-vanishing sections of $U(1) = \mathcal{O}(1)\oplus\mathcal{O}(1)$. A non-zero section s of $\mathcal{O}(1)$ vanishes precisely once, its divisor is a point x in $\mathbb{P}^1$ and s is determined by x up to a scalar. A section $s = (s_1,s_2)$ of $U(1)$, vanishes iff one of s_i's is zero, or if they have the same divisor; but this is the same as saying that one is a multiple of the other, i.e., that the matrix with rows s_i is not invertible.

[67] In terms of the a local parameter z at a we consider the nilpotent operator z on the vector space $U_D = U\oplus Uz^{-1}\oplus\cdots Uz^{n-1}$. and we consider of its Springer fiber $Gr_n(_D)^z$ in the Grassmannian $Gr_n(U_D)^z$.

13.3. **(Intersection) Homology of moduli.** The question is that $IH(\mathcal{M})$ of a moduli $\mathcal{M}$ should itself be some kind of a moduli. The same for H_* and H^*

13.3.1. *$A_{\mathcal{M}}$ for a moduli of maps $\mathcal{M} = Map(\Sigma, X)$.*

Question. As functors we seems to have

$$A_{Map(\Sigma,X)} \ \cong \ Map(\Sigma, A_X).$$

13.4. **Standard MC questions.**

13.4.1. *Relation of moduli interpretation of $H^*(X)$ (Kontsevich) and the geometric interpretation A_X of relative motivic homology.*

13.4.2. *Question.* How does the moduli interpretation of $H^*(X)$ as the moduli of deformations of $\mathcal{C}oh(X)$ relate to the $H_*^{rel}(X, \mathbb{Z}) = A_X$?

13.4.3. *Absolute and relative motivic homology.* Seemingly there should be a map from H_*^{abs} to H_*^{rel} since any traditional (absolute) finite correspondence gives a relative one, i.e., there is a map

13.4.4. *The fundamental class in motivic cohomolgy?* It is of the diagoanal Hodge type?

Check the sources.

Part 3. Kapranov: Reductive semigroups

14. Reductive semigroups

14.0.1. *Setting.* G is a reductive group with a central torus Z. Fix $T{\subseteq}B{\subseteq}G$. Let $\overline{G}$ be a semigroup closure of G *with zero*. The closures $\overline{T},\overline{B}$ in $\overline{G}$ are affine semigroups.

14.1. **$\overline{G}$ in Tannakian terms.** $Rep(\overline{G}){\subseteq}Rep(G)$ is a full-subcategory of representations that extend to $\overline{G}$, in particular $Irr(\overline{G}){\subseteq}Irr(G)$. The Grothendieck semi-rings lead to an inclusion of based semirings $\mathbb{Z}_+[Irr(\overline{G})]{\subseteq}\mathbb{Z}_+[Irr(G)]$. In particular, $\mathcal{O}(\overline{T}) = \Bbbk[X^*(\overline{T})]$ lies in $\mathcal{O}(T) = \Bbbk[X^*(T)]$.

If we think of $Irr(G)$ as the dominant cone $X^*(T)_+$, then $Irr(\overline{G})$ is a subsemigroup we denote $X^*(\overline{T})_+$.

14.1.1. *Lemma.* [Kapranov, Vinberg] $\overline{G} = \ G{\cdot}\overline{T}{\cdot}G$.

14.1.2. *Corollary.* A representation of G extends to $\overline{G}$ iff its T-weights extend to $\overline{T}$.

14.1.3. *Corollary.* (a) For $\chi \in X^*(T)$, if $n\chi \in X^*(\overline{T})$ for some $n > 0$ then $\chi \in X^*(\overline{T})$.

(b) $X^*(\overline{T})$ is convex inside $X^*(T)$:

$$conv[X^*(\overline{T})] \cap X^*(T) = \ X^*(\overline{T}).$$

Proof. (a) For U open and dense in X, if some power of $f\mathcal{O}(U)$ extends to X then f also extends to X.

(b) follows since for $\mathcal{A}{\subseteq}\mathbb{Z}^n$, $conv(\mathcal{A}) \cap \mathbb{Z}^n$ lies in $\mathbb{Q}{\cdot}\mathcal{A}$.

14.1.4. *Questions.* (a) When is $conv[Irr(\overline{G})] \cap Irr(G){\subseteq}\ Irr(G)$ an equality?

(b) Does any G-orbit in the semisimple part of G (a constructible subset), meet $\overline{T}$?

(c) Is $B{\cdot}W{\cdot}\overline{T}{\cdot}B{\subseteq}G$ an equality? (Yes for $GL_2{\subseteq}M_2$.)

14.1.5. *Remark.* (1) $B{\cdot}\overline{T}{\subseteq}\overline{B}$ is a dense constructible subset. (2) $B{\cdot}\overline{T}$ and $\overline{T}{\cdot}B$ need not be the same. (In SL_2, products $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}a & b\\ 0 & c\end{smallmatrix}\right)$ with $a,b \neq 0$ give all matrices A with $A_{11} = 0{\Rightarrow}A_{12} = 0$, while $B\overline{T}$ is described by $A_{22} = 0{\Rightarrow}A_{12} = 0$.)

14.1.6. *Conjecture.* The following are the same

1. Semigroup closures $\overline{G}$ of G,
2. W-invariant cones C in $X^*(T)_{\mathbb{R}}$, generated by finitely many integral weights,
3. Full abelian subcategories $\mathcal{R}$ of $Rep(G)$ stable under $\otimes$ and subqouotients and extensions (?).

Proof. $\overline{(}G)\mapsto Rep(\overline{G})$ is a bijection by Tannakian formalism. They give $C = X^*(\overline{T})$ and conversely $\mathcal{R}$ consists of representations of G with weights in C.

14.2. **Stratifications of $\overline{G}$.**

14.2.1. *Questions.* (a) Adjoint quotients of semigroups:

$$\begin{array}{ccc} G//G & \longrightarrow & \overline{G}//G \\ \cong\Big\uparrow & & \cong\Big\uparrow ? \\ T//W & \longrightarrow & \overline{T}//W \end{array}.$$

(b) Do all G^2-orbits meet $\overline{T}$, i.e., is the map of finite sets $\overline{T}/W_{aff} \to G\backslash\overline{G}/G$ surjective?

14.2.2. *Conjecture.* The stratification of $\overline{G}$ by G^2-orbits should be encoded in a stratification of $\overline{T}//W$ by T-orbits.

True for $G = GL_n \subseteq M_n = \overline{G}$.

14.2.3. Define $G_{\text{ss}} \overset{\text{def}}{=} {}^G\overline{T}$, $G_r \overset{\text{def}}{=} \{x \in \overline{G},\ Z_G(x) = \dim(T)\}$. Then $G_{rs} \overset{\text{def}}{=} G_r \cap G_{\text{ss}}$ consists of elements of $\overline{G}$ that lie in the closure of one Cartan $\overline{T}$ of $\overline{G}$.

14.2.4. *Questions.*

14.2.5. *Lemma.* A representation of G extends to $\overline{G}$ iff its T-weights extend to $\overline{T}$.

Proof. for $p > 0$.

14.3. **Functions on $\overline{G}$.**

14.3.1. *Lemma.* $Gr\mathcal{O}(G) = \oplus_{L\in Irr(G)} L\otimes L^*$ contains $Gr\mathcal{O}(\overline{G}) = \oplus_{L\in Irr(\overline{G})} L\otimes L^*$.

Proof.

14.3.2. Observe that

14.3.3. *Lemma.* (?) $\overline{G}//Z$ is a projective variety and it is a compactification of the reductive group $\overline{G} \overset{\text{def}}{=} G/Z$.

15. **Semigroup closures**

15.0.1. *Data.* We start with a reductive group G and a finite subset $A \subseteq Irr(G)$. It defines a semigroup $G_c \overset{\text{def}}{=}$ the closure of the image of G in the semigroup $\text{End}(L_c)$.

If the representation $(L_c \overset{\text{def}}{=} \oplus_{L\in A} L$ is faithful then G_c is a semigroup closure of G.

15.1. **Projective variety X_c.** Under the

- *Homogeneity assumption*: the image of G in $GL(L_c)$ contains scalars

the semigroup G_c has a zero, and it is a cone. Then

$$X_c \stackrel{\text{def}}{=} \mathcal{P}(G_c) \hookrightarrow \mathbb{P}[\mathrm{End}(L_c)]$$

is a projective variety.

15.2. **Classification of reductive semigroups.**

15.2.1. *Lemma.* Isomorphism classes of semigroup closures $G \hookrightarrow \overline{G}$ are the same as ...

16. G-spherical varieties

A G-space X is said to be spherical if there is a dense B-orbit $\mathcal{O}$. Then X contains a (unique) spherical G-orbit $G{\cdot}\mathcal{O}$.

One also says that a subgroup H of G is spherical if the G-space G/H is spherical. Such pairs (G, H) are called Gelfand pairs.

The rank of a spherical space X is defined as $rank(G) - rank(H)$ if the open spherical orbit in X is of the form G/H.

16.0.1. *Lemma.* Subgroup H is spherical iff G/H has no multiplicities (i.e. $\mathcal{O}(G/H)$ has none).

16.0.2. *Examples.* (a) G/N_- is spherical – the open B-orbit is the big cell BB_-/N. (b) Symmetric subgroups are spherical. (c) $\Delta_G \subseteq G^2$ is spherical.

17. Newton polytope Q_c of a finite subset $A \subseteq Irr(G)$

The "Newton polytope" Q_c of a finite subset $A \subseteq Irr(G)$ is defined as the convex closure of the union of weights in A.

17.0.1. *Examples.* (a) In SL_n let A be the basic representation, then Q_c is the standard $(n-1)$-simplex: $W{\cdot}\omega_1 = S_n{\cdot}\varepsilon_1 = \{\varepsilon_1, ..., \varepsilon_n\}$.

The facets of Q_c are the subsets of $\{1, ..., n\}$, and the W-orbits in the set of facets of Q_c are indexed by the size $0 \le k \le n$.

(b) In SL_n let A be the fundamental representation $L(\omega_p) = \overset{p}{\wedge} \mathbb{k}^n$. Since $\omega_p = \varepsilon_1 + \cdots + \varepsilon_p$, the vertices of Q_c are the $(p-1)$-dimensional facets of $Q(L_{\omega_1})$.

17.0.2. *Theorem.* (a) $G{\times}G$-orbits in the semigroup G_c associated to a finite subset $A{\subseteq}Irr(G)$, are the same as the W-orbits in facets of the polytope Q_c.

(b) Let us associate to each face Γ of Q_c an idempotent operator e_Γ on L_c – the projector to the sum of all weight spaces in Γ. Then e_Γ is in the image of the G-action on L_c, i.e., in G_c, and it defines a G^2-orbit

$$G{\cdot}e_G a{\cdot}G \overset{\text{def}}{=} G_c(\Gamma){\subseteq}\, G_c.$$

17.0.3. *Lemma.* G_c is a spherical G-variety.

18. Tannakian approach ("monoidal set approach")

19. Appendix. Vinberg semigroup $\overline{\mathcal{V}}(G)=\overline{\mathcal{V}}_{\mathcal{B}}(G)$ via algebras of functions

Here we recall Vinberg's original definition of the (absolute) Vinberg semigroup (from "On reductive Algebraic Semigroups"). There Vinberg has defined $\overline{\mathcal{V}}=\overline{\mathcal{V}}(G)$ by describing its algebra of functions $\mathcal{O}(\overline{\mathcal{V}})$ as a certain subalgebra of functions on $\mathcal{V}\overset{\text{def}}{=}G\underset{Z(G)}{\times}H$. These formulas are only valid in characteristic zero.

Vinberg's formula for functions on $\overline{\mathcal{V}}(G)$ is

$$\mathcal{O}(\overline{\mathcal{V}}) \overset{\text{def}}{=} \underset{\lambda\in X_+,\ \mu\in\lambda+Q_+}{\oplus} L(\lambda){\otimes}L(\lambda)^*{\otimes}\mathbb{C}{\cdot}e^\mu \subseteq \mathcal{O}[G\underset{Z(G)}{\times}H] \overset{\text{def}}{=} \underset{\lambda\in X_+,\ \mu\in\lambda+Q}{\oplus} L(\lambda){\otimes}L(\lambda)^*{\otimes}\mathbb{C}{\cdot}e^\mu.$$

The only thing we do in this section is fix a choice of the matrix coefficient map.

19.1. Groups $G,\ \mathcal{V}(G)$ and the semigroup $\overline{H}$.

19.1.1. *Groups $G,\widetilde{G}$ and $\overline{H}$.* Let G be a semi-simple simply connected algebraic group. Denote by $\Delta{\subseteq}\ X \overset{\text{def}}{=}\ X^*(H)$ the abstract roots and let Δ^+ correspond to $\mathfrak{g}/\mathfrak{b}$. Then X contains the the Δ^+-dominant weights X_+, and also $X\supseteq Q\overset{\text{def}}{=}\ \mathbb{Z}{\cdot}\Delta\supseteq Q_+\overset{\text{def}}{=}\ \mathbb{Z}_+{\cdot}\Delta_+$.

19.1.2. *The Vinberg group $\widetilde{G}$ of G.* This is the group $\widetilde{G}\overset{\text{def}}{=}\ G\underset{Z(G)}{\times}H$.

19.1.3. *Semigroup closure $\overline{H}$ of the Cartan group.* Abstract Cartan group $H\overset{\text{def}}{=}\ \mathrm{Spec}(\mathbb{C}[X])$ lies in a semigroup $\overline{H}\overset{\text{def}}{=}\ \mathrm{Spec}(\mathbb{C}[X_+])$, corresponding to a cone X_+ in the lattice X. Observe that the lattice $\check{Q}$ dual to X has a basis of simple coroots $\check{\Pi}$, such that the cone it generates $\check{Q}_+\overset{\text{def}}{=}\ \mathbb{Z}_+{\cdot}\check{\Pi}$ is dual to X_+. We decompose H according to $\check{\Pi}$: $H=X_*(H){\otimes}G_m=(\underset{\alpha\in\Pi}{\oplus}\mathbb{Z}\check{\alpha})\otimes G_m=\prod_{\alpha\in\Pi}\check{\alpha}(G_m)\cong(G_m)^{\check{\Pi}}$. Cocharacters λ of $\check{Q}_+{\subseteq}\ \check{Q}=X_*(H)$ extend to semigroup maps $\lambda:\ \mathbb{C}{\to}\ \overline{H}$ and $\overline{H}=\prod_{\alpha\in\Pi}\check{\alpha}(\mathbb{C})\cong\mathbb{C}^{\check{\Pi}}$.

19.2. **Functions on $G, \widetilde{G}$ and $\overline{H}$.**

19.2.1. *Functions on G.* For a representation V of G matrix coefficient map $c = c^V : V\otimes V^* \to \mathcal{O}[G]$ defined by $c_{v,u}(x) \stackrel{\text{def}}{=} \langle v, xu\rangle$ for $x \in G$, $v \in V$ and $u \in V^*$, is $G\times G$-equivariant:

$$[(g,h)\ c_{v,u}](x) = c_{v,u}(g^{-1}\cdot x\cdot h) = \langle v, g^{-1}xh\cdot u\rangle = \langle gv, x\cdot hu\rangle = c_{gv,hu}(x).$$

For a dominant weight $\lambda \in X_+$, we denote by $L(\lambda)$ the irreducible representation of G with the highest weight λ. This means that for any Borel subgroup B the action of $H \cong B/N$ on $H_0[N, L(\lambda)]$ is by a character $\lambda_B \in X^*(B/N)$, dominant for $(\Delta^+)_B = \Delta(\mathfrak{g}/\mathfrak{b})$; while the action on $L(\lambda)^N$ is by $(w_0\lambda)_B$ dominant for $\Delta(\mathfrak{n})$.

We will use the identification $\underset{\lambda\in X_+}{\oplus} L(\lambda)\otimes L(\lambda)^* \stackrel{\cong}{\to} \mathcal{O}[G]$ given over $\mathbb{Q}$ by the above matrix coefficient maps.

To describe the algebra structure on $\mathcal{O}[G]$ in this realization, we use decompositions $L(\lambda')\otimes L(\lambda'') \cong \underset{i}{\oplus} L(\nu_i)$. For $v' \in L(\lambda')$, $v'' \in L(\lambda'')$, $u' \in L(\lambda')^*$, $u'' \in L(\lambda'')^*$, the decomposition above gives $v'\otimes v'' = \underset{i}{\oplus} v_i$ and $u'\otimes u'' = \underset{i}{\oplus} u_i$. Therefore,

$$c^{L(\lambda')}_{v',u'} \cdot c^{L(\lambda'')}_{v'',u''} = c^{L(\lambda')\otimes L(\lambda')}_{v'\otimes v'',u'\otimes u''} = \sum_i c^{L(\nu_i)}_{v_i,u_i}.$$

19.2.2. *Functions on H.* Characters $\mu \in X = X^*(H)$, give a basis of functions on H: $\mathcal{O}[H] = \mathbb{C}[X]$. Moreover, $\mathbb{C}\cdot\mu$ is an $H\times H$-submodule of $\mathbb{C}[H]$ isomorphic to $L(-\mu)\otimes L(\mu)$, since for $a, b, h \in H$ one has

$$[(a,b)\cdot\mu](h) = \mu(a^{-1}\cdot h\cdot s) = \mu(a)^{-1}\mu(b)\cdot\mu(h).$$

19.2.3. *Functions on $\widetilde{G}$.* We start with

$$\mathcal{O}[G\times H] = \mathcal{O}[G]\otimes\mathcal{O}[H] \cong \underset{\lambda\in X_+,\mu\in X}{\oplus} L(\lambda)\otimes L(\lambda)^*\otimes \mathbb{C}\mu.$$

Next, $\widetilde{G}$ is the quotient of $G\times H$ by the action of $Z(G)$ embedded into $G\times G\times H\times H$ by $z \mapsto (1, z, z, 1)$. The summand corresponding to (λ, μ), factors to $\widetilde{G}$ iff the actions of $Z(G)$ on $L(\lambda)^*$ (by $-\lambda$) and on $\mathbb{C}\cdot\mu$ (by $-\mu$) cancel, i.e., iff $Z(G)$ acts the same via λ and μ. This means that $\lambda - \mu \in Z(G)^\perp = Q \subseteq X$, hence

$$\mathcal{O}(\widetilde{G}) = \mathcal{O}(G)\otimes\mathcal{O}(H) \cong \underset{\lambda\in X_+,\ \mu\in\lambda+Q}{\oplus} L(\lambda)\otimes L(\lambda)^*\otimes \mathbb{C}e^\mu.$$

19.3. **The Vinberg semigroup $\overline{\mathcal{V}} = \overline{\mathcal{V}}(G)$ of G.**

The space $\overline{\mathcal{V}}$ is defined in terms of the subalgebra $\mathcal{O}(\overline{\mathcal{V}}) \subseteq \mathcal{O}(\widetilde{G})$:

Lemma. The following subspace $\mathcal{O}(\overline{\mathcal{V}})\subseteq\mathcal{O}(\widetilde{G})$ is a subalgebra:

$$\mathcal{O}[\overline{\mathcal{V}}] \overset{\text{def}}{=} \bigoplus_{\lambda\in X_+,\ \mu\in X,\ \lambda\le\mu} L(\lambda)\otimes L(\lambda)^*\otimes\mathbb{C}\cdot e^\mu,$$

(Here, $\lambda \le \mu$ is defined using the cone $Q^+\subseteq X$.)

Proof. To check that this is a subalgebra we notice that for $\lambda', \lambda'' \in X_+$ and $\mu', \mu'' \in X$ with $\lambda' \le \mu'$ and $\lambda'' \le \mu''$, one has $[L(\lambda')\otimes\ e^{\mu'}]\otimes[L(\lambda'')\ e^{\mu''}] \cong \bigoplus_i L(\nu_i)\otimes\ e^{\mu'+\mu''}$, and $\nu_i \le \lambda' + \lambda'' \le \mu' + \mu''$. $\square$

19.4. **The canonical map $\overline{\mathcal{V}}\to\ (G/N)^{\mathrm{aff}}$.** We will describe the map from the Vinberg semigroup into the affinization of G/N.

19.4.1. *The quotients by free actions.* We will denote by $A\backslash X$ the quotient under a free action of an affine group A. Here "free" means that there is a G-bundle $X \xrightarrow{p} B$ which is locally isomorphic to $A\times B\to B$, i.e., $X\to B$ is an A-torsor in the Zariski topology. Then $A\backslash X =\ B$ is also an invariant theory quotient, i.e., $\mathcal{O}_B\to\ (p_*\mathcal{O}_X)^A$ is an isomorphism.

19.4.2. *Functions on G/N.*

Lemma. (a) The functions on G/N are

$$\mathcal{O}[G/N] = \mathcal{O}[G]^{1\times N} = [\bigoplus_{\lambda\in X_+} L(\lambda)\otimes L(\lambda)^*]^{1\times N} = \bigoplus_{\lambda\in X_+} L(\lambda)\otimes[L(\lambda)^*]^N.$$

(b) A description of G/N as $G\underset{B}{\times}B/N$ gives

$$\mathcal{O}[G\underset{B}{\times}B/N] = \bigoplus_{\lambda\in X_+} L(\lambda)\otimes [L(\lambda)^*]^N\otimes \mathbb{C}\cdot e^\lambda \subseteq \bigoplus_{\lambda\in X_+,\ \mu\in X} L(\lambda)\otimes [L(\lambda)^*]^N\otimes \mathbb{C}\cdot e^\mu =\ \mathcal{O}[G\times H].$$

Proof. $\mathcal{O}[G\underset{B}{\times}B/N] =\ (\mathcal{O}[G]\otimes\mathcal{O}[B/N])^B =\ (\mathcal{O}[G]^{1\times N}\otimes\mathcal{O}[B/N])^{B/N}$ equals

$$= \left(\bigoplus_{\lambda\in X_+,\ \mu\in X} L(\lambda)\otimes[L(\lambda)^*]^N\otimes\ \mathbb{C}\cdot e^\mu\right)^{B/N} = \bigoplus_{\lambda\in X_+,\ \mu\in X} L(\lambda)\otimes([L(\lambda)^*]^N\otimes\ \mathbb{C}\cdot e^\mu)^{B/N}.$$

The (λ, μ)-summand is zero unless $-\mu$ is the N-highest weight of $L(\lambda)^*$, i.e., the lowest weight λ of $L(\lambda)$.

19.4.3. *The canonical map $\overline{\mathcal{V}}\to(G/N)^{\mathrm{aff}}$.* By our algebraic definition of $\overline{\mathcal{V}}$, $\mathcal{O}(\overline{\mathcal{V}})$ lies in $\mathcal{O}(G\times H)$. Moreover, by 19.4.2(a), $\mathcal{O}(\overline{\mathcal{V}})$ contains the subalgebra $\mathcal{O}[G/N]$ of $\mathcal{O}[G\times H]$. This gives a map of affine varieties $\overline{\mathcal{V}}\to(G/N)^{\mathrm{aff}}$. Actually, one has

$$\begin{tikzcd}
G\times H \arrow[r, "open"] \arrow[d, "="'] & \overline{\mathcal{V}} \arrow[r, "="] & \overline{\mathcal{V}} \arrow[d, "surj?"] \\
G\times H \arrow[r, "surj"] & G\underset{B}{\times}H = G/N \arrow[r] & (G/N)^{\mathrm{aff}}
\end{tikzcd}$$

Remark. Is $\overline{\mathcal{V}}(G)$ hyperkähler? (Yes in SL_2 and the dimensions are always even, but this may be all of the relation there is).

Part 4. Schubert polynomials via semigroups [Knutson–Miller]

Here the double Schubert polynomials are exaplined in terms of the Vinberg semigroup. (Hence no triple Schubert polynomials.)

19.4.4. Let $\mathcal{B}$ be the flag variety of a simply connected semisimple group A. Let G be a reductive group containing A and $\overline{G}$ its semigroup closure (a semigroup with open dense part G). Consider the map

$$H^*(\mathcal{B}) = H^*_{1\times B}(G) \leftarrow H^*_{1\times B}(\overline{G}) \cong H^*_T$$

for a Cartan T of a Borel B of G. The last step requires that $\overline{G}$ have a zero, hence be equivariantly contractible.

19.4.5. For a cycle C in $\mathcal{B}$ let $\mathcal{C}$ be its inverse in G. The fundamental class of $[\overline{\mathcal{C}}] \in H^*_{1\times B}(\overline{G}) \cong H^*_T$ maps to $[C] \in H^*(\mathcal{B})$, so it is a "refinement" of C.

19.5. **Positivity.** Suppose that G is $A \underset{Z(G)}{\times} \mathcal{T}$ for a torus $\mathcal{T} \subseteq \mathrm{End}_G(\check{W})$ for a "faithful" representation $\check{W}$ of G, and $\overline{G}$ is the closure of G in linear operators on $\check{W}$.

Choose a basis $\mathbb{B}$ of $\check{W}$ in which the Cartan $T = T_c \cdot \mathcal{T} \subseteq G$ diagonalizes, and use a cocharacter ζ of the torus $(G_m)^{\mathbb{B}}$ to degenerate in the Hilbert scheme the subvariety $\overline{\mathcal{C}} \subseteq \overline{G} \subseteq \mathrm{End}(\check{W})$ (it is G_m-invariant, so the degeneration happens in the projective space) to a $(G_m)^{\mathbb{B}}$-invariant subscheme $\overline{\mathcal{C}}_\zeta$, i.e., a union of coordinate planes.

This should show that $[\overline{\mathcal{C}}]$ is a $\mathbb{Z}_+$-combination of monomials in the polynomial ring H^*_T.

19.6. **The case of the semigroup of matrices $\overline{G} = M_n$ [Knutson-Miller].** This was done by Knutson and Miller for $A = SL_n$ and matrices $\overline{G} = M_n$ to get a construction of the Schubert polynomial $\mathfrak{S}_w[x_1, ..., x_n]$, $w \in S_n$ with manifest positivity and stability (under $N \geq n$). They also prove nice properties of $\overline{\mathcal{C}}$ and $\overline{\mathcal{C}}_\zeta$, Cohen-Macaulay and reduced, which may be interesting in a larger generality.

19.7. **Double Schubert polynomials and the Vinberg semigroup.** If $\overline{G}$ is the Vinberg semigroup ($\check{W}$ is the sum of fundamental representations), and $\mathcal{C}$ is a Schubert cycle, one should get the double Schubert polynomials.

B. Vinberg semigroups, semigroupoids and affinization

These are the elements of an old study if these topics. In particular this is not up to date.

19.8. **Definitions of Vinberg semigroups.**

19.8.1. *The data.* To a parabolic $P = U \ltimes L$ and its normal subgroup V (between U and $[P, P]$) we associate

- $(G, (P/V)^o)$-spaces $G/V \subseteq (G/V)^{\mathrm{aff}}$;
- groupoid $\mathcal{S} = \mathcal{S}_{P,V}$ over $\mathcal{P} = G/P$ and its semigroupoid closure $\overline{\mathcal{S}}_{P,V}$;
- the Vinberg group $\mathcal{V} = \mathcal{V}_{P,V} \stackrel{\mathrm{def}}{=} (G \times_{Z(G)} Z(P/V)$ and its semigroup closure $\overline{\mathcal{V}}_{P,V}$.

19.8.2. *The definitions and realizations of* $\overline{\mathcal{V}}_{P,V}$.

- The original definition of a Vinberg semigroup was in tems of its ring of functions. It was only valid in characteristic zero (and the data were the standard ones (G, B, N)).
- The "up to date" definition/construction should be

$$\overline{\mathcal{V}}_{G,P,V} \stackrel{\mathrm{def}}{=} \mathrm{End}_{\mathcal{Z}_{G,P,V}}(G/V).$$

- More conjectural realizations/definitions:
  (1) affinization of a certain torsor over the wonderful compactification;
  (2) (affinization of?) stable sections of a certain semigroupoid over $\mathcal{B}$.
  The text below, uses a complicated and conjectural definition (conjecture 21.1.2.d). along the lines of (2).

The endomorphism construction is known for the standard Vinberg semigroup $\overline{\mathcal{V}}(G) = \overline{\mathcal{V}}_{G,B,N}\mathcal{B}(G)$, i.e., it has a realization as endomorphisms $\mathrm{End}_H((G/N)^{\mathrm{aff}})$ (theorem 8.2).

This seems closer to a reasonable definition of Vinberg semigroups.

19.8.3. *Sections below.* In the section 20 We calculate the invertible sections of the groupoid $\mathcal{S} = \mathcal{S}_{\mathcal{P},V}$ as the Vinberg group $\mathcal{V}_{\mathcal{P},G}$ and show that the sections of $\overline{\mathcal{S}}$ are controlled by the T-fixed points.

In the next section 21 we introduce the Vinberg semigroups $\overline{\mathcal{V}}_{\mathcal{P},V} \supseteq \mathcal{V}_{\mathcal{P},V}$ and give a conjectural comparison with the *stable* sections of $\overline{\Gamma^*}(\mathcal{P}, \mathcal{S}) \subseteq \Gamma(\mathcal{P}, \overline{\mathcal{S}})$ of the semigroupoid $\overline{\mathcal{S}}$.

19.9. **Summary: Vinberg semigroup and wonderful compactification.** $\square$[68] $\blacksquare$ Let G be semisimple and $\overline{G}$ be the wonderful compactification of G_{ad}.

[68] $\boxed{!}$ his material appaears in 8.3

19.9.1. $\overline{\mathcal{V}}$ *via* $(G/N)^{\text{aff}}$. The following property seems to be a reasonable definition of $\overline{\mathcal{V}}$.

Lemma. $\mathcal{V}\subseteq\overline{\mathcal{V}}$ act on $G/N\subseteq(G/N)^{\text{aff}}$ and

$$\mathcal{V}\xrightarrow{\cong}\mathrm{Aut}_H(G/N) = \mathrm{Aut}_H((G/N)^{\text{aff}}) \quad \text{and} \quad \overline{\mathcal{V}} = \mathrm{End}_H((G/N)^{\text{aff}}).$$

19.9.2. $\overline{\mathcal{V}}$ *and the wonderful compactification* $\overline{G_{\text{ad}}}$.

Lemma.

- (a) The H-free part $\overline{\mathcal{V}}^o$ of the Vinberg semigroup $\overline{\mathcal{V}}$ is an H_{sc}-torsor over the wonderful compactification $\overline{G_{\text{ad}}}$ of G_{ad}.
- (b) As an H_{sc}-torsor over $\overline{G_{\text{ad}}}$, $\overline{\mathcal{V}}^o$ is the product of G_m-torsors that correspond to the decomposition of $\partial G_{\text{ad}} = \overline{G} - G_{\text{ad}}$ into irreducible G-invariant divisors $D_i,\ i \in I$.
- (c) $\overline{\mathcal{V}}$ is the affinization of $\overline{\mathcal{V}}^o$.

Proof. (a) The free part $\overline{\mathcal{V}}^o$ of $\overline{\mathcal{V}}$ contains $\mathcal{V}$, so $\overline{\mathcal{V}}^o/H_{\text{sc}}$ contains $\mathcal{V}/H_{\text{sc}} = G_{\text{ad}}$. .... □

Remark. $\overline{G_{\text{ad}}}$ is the geometric invariant theory quotient $\overline{\mathcal{V}}//H_{\text{sc}}$ of $\overline{\mathcal{V}}$ by H_{sc} in the sense that it is the quotient of the free part of the space. So, G_{ad} is open in the G_{ad}-stack $\overline{\mathcal{V}}/H$.

19.9.3. $\overline{\mathcal{V}}$ *via a semigrouoid* $\overline{\mathcal{S}}$ *over* $\mathcal{B}$. We consider a certain groupoid $\mathcal{S}$ over $\mathcal{B}$ with a semigroupoid closure $\overline{\mathcal{S}}$ (see 19.9.4). The notion of *sections* $\Gamma(\mathcal{B},\mathcal{S})$ *of a groupoid* means the sections of $\mathcal{S} \to \mathcal{B}^2 \xrightarrow{pr_2} \mathcal{B}$. The sections form a semigroup $\Gamma(\mathcal{B},\mathcal{S})$ and its invertible part $\Gamma^*(\mathcal{B},\mathcal{S})$ is a group.

Define the *stable sections* of $\overline{\mathcal{S}}$ as the Hilbert scheme closure of $\Gamma^*(\mathcal{B},\mathcal{S})$ in all sections $\Gamma(\mathcal{B},\overline{\mathcal{S}})$ of $\overline{\mathcal{S}}$.

Lemma. $\mathcal{V}$ is the group of invertible sections $\Gamma^*(\mathcal{B},\mathcal{S})$ of $\mathcal{S}$.

Conjecture. The stable sections of $\overline{\mathcal{S}}$ form precisely the Vinberg semigroup $\overline{\mathcal{V}}$.

19.9.4. *Groupoid* $\mathcal{S}$ *over* $\mathcal{B}$ *and its semigroupoid closure* $\overline{\mathcal{S}}$. The action groupoid for the G-action on $\mathcal{B}$ is $\mathcal{G}_0 = \mathcal{B}\times G \to \mathcal{B}^2$ by $(\mathfrak{b}', g)\mapsto({}^g\mathfrak{b}', \mathfrak{b}')$. (We will also think of it as $G\times_B G$ for the conjugation action of B on G.)

Its vertical part $\mathcal{G}_0^{\uparrow} = \mathcal{G}|_{\Delta_{\mathcal{B}}}$ is the group bundle $G\times_B B$ (for the conjugation action of B on B). It contains a normal group subbundle $\mathcal{G}_- \overset{\text{def}}{=} G\times_B N$. Our groupoid over $\mathcal{B}$ will be

$$\mathcal{S} \overset{\text{def}}{=} \mathcal{G}_0/\mathcal{G}_-.$$

So, the fiber at $(B'', B') \in \mathcal{B}^2$ is

$$\mathcal{S}_{B'',B'} = \{gN' \in G/N';\ {}^gB' = B''\} = \{N''g \in G/N';\ {}^gB' = B''\}.$$

Also, the total space of $\mathcal{S}$ is $G\times_B G/N$ and the second projection map $pr_2 : \mathcal{S} \to \mathcal{B}$ is $G\times_B G/N \to G/B = \mathcal{B}$ by $(g, xN) \mapsto (g, xN) \mapsto g$.

We define $\overline{\mathcal{S}}$ as the relative affinization of $\mathcal{S}$ over $\mathcal{B}$

$$\overline{\mathcal{S}} \stackrel{\text{def}}{=} \mathcal{S}^{\text{aff}}_{\mathcal{B}} = G\times_B (G/N)^{\text{aff}}.$$

Lemma. $\overline{\mathcal{S}}$ is a semigroupoid closure of the groupoid $\mathcal{S}$.

19.9.5. *The canonical resolution* $\widetilde{\widetilde{\mathcal{V}}}.$ *of* $\overline{\mathcal{V}}$.

- (a) $\overline{\mathcal{V}}$ has a canonical resolution $\widetilde{\widetilde{\mathcal{V}}}.$
- (b)$\overline{\mathcal{V}}$ and $\widetilde{\widetilde{\mathcal{V}}}.$ are stable sections of certain groupoids over $\mathcal{B}$.

Part 5. Vinberg semigroups and sections of semigroupoids

Example. For $G = G_{\mathrm{ad}} = PGL_2$ we have $\mathcal{V} = G{\times}H$. If $G = PGL_2$ then $(G/N)^{\mathrm{aff}}$ is the nilpotent cone $\mathcal{N}$ and $G/N = \mathcal{N}_{reg}$.

In particular, one of its orbits provides a canonical map $\overline{\mathcal{V}}(G) \to (G/N)^{\mathrm{aff}}$.

20. Sections of the semigroupoid $\overline{\mathcal{S}}$ associated to $[P,P]{\supseteq}V{\supseteq}U$

To a parabolic $P = U \ltimes L$ and its normal subgroup V (between U and $[P,P]$) we associate a groupoid $\mathcal{S} = \mathcal{S}_{\mathcal{P},V}$, its semigroupoid closure $\overline{\mathcal{S}}$ and the Vinberg group

$$\mathcal{V} = \mathcal{V}_{\mathcal{P},V} \stackrel{\mathrm{def}}{=} (G{\times}_{Z(G)}\ Z(M)$$

for $M = P/V$.

We calculate the invertible sections of the groupoid $\mathcal{S} = \mathcal{S}_{\mathcal{P},V}$ as the Vinberg group

$$\overline{\Gamma}^*(\mathcal{P},\mathcal{S})\ \cong\ \mathcal{V}_{\mathcal{P},G}$$

(under some technical assumptions on $\mathcal{P}$, see 20.2.1 which hold for the full flag variety).

Next, we are interested in the closure of $\overline{\Gamma^*}(\mathcal{P},\mathcal{S})$ in all sections $\Gamma(\mathcal{P},\overline{\mathcal{S}})$ of the semigroupoid $\overline{\mathcal{S}}$. Here we only notice that for $\mathcal{P} = \mathcal{B}$ the sections of $\overline{\mathcal{S}}$ are controlled by the T-fixed points.

In the next section 21 we introduce the where Vinberg semigroups $\overline{\mathcal{V}}_{\mathcal{P},V}{\supseteq}\mathcal{V}_{\mathcal{P},V}$ and we attempt to compare them with $\overline{\Gamma^*}(\mathcal{P},\mathcal{S})$.

20.0. **Data** (P,V)**.** Here, P is a parabolic subgroup with the unipotent radical U. V is a normal subgroup of P that lies between U and $[P,P]$. Then $M \stackrel{\mathrm{def}}{=} M$ is reductive as a quotient of the Levi group $\underline{P} \stackrel{\mathrm{def}}{=} P/U$ by V/U.

On the level of Lie algebras, $\mathfrak{l} = \mathbb{Z}(\mathfrak{l}){\oplus}\ \oplus_{j\in J}\mathfrak{l}_j$ for simple factors $\mathfrak{l}_j$. Then $\mathfrak{v} = \mathfrak{v}_0{\oplus}{\oplus}_{j\in J'}\ \mathfrak{l}_j$ for $\mathfrak{v}_0 = \mathfrak{v}{\cap}Z(\mathfrak{l})$ and some $J'{\subseteq}J$ while $\mathfrak{m} = \mathfrak{m}_0{\oplus}{\oplus}_{j\in J''}\ \mathfrak{l}_j$ for $\mathfrak{m}_0 = Z(\mathfrak{l})/\mathfrak{v}_0$ and $J'' = J{-}J'$.

Example. The interesting choices are $V = U$ and $V = [P,P]$. Then M is the Levi group $\overline{P} = P/U$ of P or the commutative quotient P^{ab} of P.

20.0.1. *Example $P = B$: sections of semigroupoids $\mathcal{S}{\subseteq}\overline{\mathcal{S}}$ over $\mathcal{B}$.* Here the only choice is $V = N = [B,B]$. We will only formulate the key objects.

For a semi-simple algebraic group G we consider a G-equivariant bundle $\mathcal{S} \stackrel{\mathrm{def}}{=}\ G\underset{B}{\times}\ G/N{\to}\ G/B\ =\ \mathcal{B}$. Here, the B-action on G/N is by conjugation.(69)

69 The translation action extends to a G-action and then the G-bundle is trivial.

We will see that $\mathcal{S}$ is a groupoid over $\mathcal{B}$ and the above map $\mathcal{S} \to \mathcal{B}$ is its first projection. We are interested in global sections S over $\mathcal{B}$. This gives the space

$$\overline{\mathcal{S}} \times \overline{G/N}$$

over $\mathcal{B}$ which is the first projection map for a semigroupoid $\overline{\mathcal{S}}$ over $\mathcal{B}$. Here we denote the global sections by $\overline{S} = \Gamma(\mathcal{B}, \overline{\mathcal{S}}$.

20.1. Groupoid $\mathcal{S} = \mathcal{S}_{P,V}$ over G/P.

20.1.1. *The group bundles* $0 \to V_{\mathcal{P}} \to P_{\mathcal{P}} \to M_{\mathcal{P}} \to 0$ *over* $\mathcal{P}$. For a reductive group G we consider the data $(\mathcal{P}, V)$ as above.

Let $\mathcal{P}$ be the partial flag variety that contains P. It carries several G-equivariant bundles, the tautological group bundle $TPP = G \times_P$ with a normal subgroup $V_{\mathcal{P}} \stackrel{\text{def}}{=} G \times_P V$ and the the quotient group $M_{\mathcal{P}} = G \times_P M \stackrel{\text{def}}{=} M_{\mathcal{P}}$ (here we use the conjugation actions of P).

20.1.2. *The groupoid* $\mathcal{S}_{P,V}$ *over* $\mathcal{P}$.

- Let $\mathcal{G} = \mathcal{G}_{P,V} \cong G \times \mathcal{P}$ be the action groupoid for the G-action on $\mathcal{P}$.
- Its vertical part $\mathcal{G}|_{\Delta_{\mathcal{P}}}$ is the stabilizer group bundle $P_{\mathcal{P}}$. We define the groupoid $\mathcal{S} = \mathcal{S}(\mathcal{P}, V) = \mathcal{S}(\mathcal{P}, V)$ over $\mathcal{P}$ as the quotient $\mathcal{G}/V_{\mathcal{P}}$.

The fibers of $\mathcal{G}$ are

$$\mathcal{G}_{P'', P'} = \{g \in G;\ {}^g P' = P''\} \subseteq Isom(P, P') \ = \ Isom(\mathcal{T}_P, \mathcal{T}_{P'}).$$

So, one can think of $\mathcal{G}$ as the "groupoid $G - \mathrm{Aut}(P_{\mathcal{P}})$ of G-automorphisms" of the group bundle $P_{\mathcal{P}}$. Then $\mathcal{S}$ would be the groupoid $G - \mathrm{Aut}(M_{\mathcal{P}})$ of "G-automorphisms" of the group bundle $M_{\mathcal{P}}$ over $\mathcal{P}$.

Lemma. (a) The restriction $\mathcal{S}|_{\mathcal{P} \times \{P\}}$ is a bitorsor for $(M_{\mathcal{P}}, M \times \mathcal{P})$. It consists of all trivializations of the group bundle $M_{\mathcal{P}}$.

(b) The restriction $\mathcal{S}|_{\mathcal{P} \times \{P\}}$ is G/V. So, $\mathcal{S}$ can be written as $G \times_P G/V$ (for the conjugation action on the second factor). $\square$

Example. When V is the derived group $[P, P]$, the group bundle $\mathcal{S}|\Delta_{\mathcal{P}} \cong \ P^{\mathrm{ab}} \times \ \mathcal{P}$ is trivial.

20.2. Sections of $\mathcal{S}$.

For the notion of sections of (semi)groupoids see the appendix 23.

20.2.1. *Assumptions.* We will assume that

(1) $\mathfrak{p}$ does not contain any simple summands of $\mathfrak{g}$. Otherwise, we could consider the same $\mathcal{P}$ from the point of view of a smaller group G.
(2) $G \twoheadrightarrow \mathrm{Aut}(G/P)$.

Here, (1) implies that the map $G/Z(G) \hookrightarrow \mathrm{Aut}(\mathcal{P})$ is injective.

Requirement (2) can probably be avoided by replacing in formulations G sometimes with the possibly larger group $\widetilde{\mathrm{Aut}}(\mathcal{P}) \overset{\text{def}}{=} G \times_{G_{\text{ad}}} \mathrm{Aut}(\mathcal{P})$ (an extension of $\mathrm{Aut}(\mathcal{P})$ by $Z(G)$). See remark 20.2.4.

20.2.2. *Sections of $M_{\mathcal{P}}$.*

Lemma. If $V \in \{U, P'\}$ then the global sections of $M_{\mathcal{P}}$ are given by the center of M :

$$\Gamma(\mathcal{P}, M_{\mathcal{P}}) \cong Z(M) \quad \text{and} \quad \Gamma(\mathcal{P}, \mathfrak{m}_{\mathcal{P}}) \cong Z(\mathfrak{m})].$$

Proof. **(1)** *Lie algebras.* For the Lie algebra claim we have $\mathfrak{m} = \mathfrak{m}' \oplus Z(\mathfrak{m})$. Here, $Z(\mathfrak{m})$ gives a summand of $\mathfrak{m}_{\mathcal{P}}$ which is a trivial G-bundle. We will see that $\Gamma(\mathcal{P}, \mathfrak{m}_{\mathcal{P}}')$ is the sum of all simple factors of $\mathfrak{g}$ contained in $\mathfrak{p}$, i.e., $\Gamma(\mathcal{P}, G \times_P \mathfrak{m}') = 0$.

If $\mathfrak{m}'$ is non-trivial then $V = U$ and M is the Levi factor $\underline{P}$. Its Lie algebra is a sum of simple Lie algebras $\mathfrak{m}_i$ corresponding to connected subsets J_i of the set I of simple coroots. Let ϕ_i be the highest root of $\mathfrak{m}_i$, the vector bundle $G \underset{P}{\times} \mathfrak{m}'$ is the direct image from the full flag variety of the sum of line bundles $\oplus\ \mathcal{O}_{\mathcal{B}}(\phi_i)$.

It remains to see that if ϕ_i is dominant then $\mathfrak{m}_i$ is a simple summand of $\mathfrak{g}$. Since $\phi_i = \sum_{\alpha \in J_i} n_\alpha \cdot \alpha$ with all $n_\alpha > 0$, and for simple roots $\alpha \in J_i$ and $\beta \notin J_i$ one has $\langle \phi_i, \beta \rangle \leq 0$, the dominance implies that any simple roots $\alpha \in J_i$ and $\beta \notin J_i$ have to be orthogonal.

(2) *Groups.* The group $\Gamma(\mathcal{P}, \frac{M_{\mathcal{P}}}{Z[M_{\mathcal{P}}]})$ is finite since its Lie algebra $\Gamma(\mathcal{P}, \mathfrak{m}_{\mathcal{P}}')$ is zero.

A section of $M_{\mathcal{P}}$ (or $\frac{M_{\mathcal{P}}}{Z[M_{\mathcal{P}}]}$)) stays in the closure of one "conjugacy class". For that observe that $M_{\mathcal{P}}$ maps to the invariant theory quotient $M//M$ for the conjugation action, hence any section of $M_{\mathcal{P}}$ gives a map from $\mathcal{P}$ to the affine space $M//M$, which has to be a constant.

Now we see that the evaluation of sections at $P \in \mathcal{P}$ is injective. If a section σ of $\frac{M_{\mathcal{P}}}{Z[M_{\mathcal{P}}]}$ has value 1 at P then σ stays in the unipotent cone, but it also has finite order.

Finally, group G acts on the bundle $\frac{M_{\mathcal{P}}}{Z[M_{\mathcal{P}}]}$, and on its global sections. The orbit of a Levi factor $L \subseteq P$ through a section σ is isomorphic to the conjugacy class of $\sigma(P)$. Since it is finite $\sigma(P)$ is central and so is σ. $\square$

20.2.3. *The direct image of (A_X, τ^*A_Y)-bitorsors for a map $X \xrightarrow{\tau} Y$.* We consider a group A_X on X and a group A_Y over Y, where X and Y are related by a map $X \xrightarrow{\tau} Y$.

We consider a bitorsor $\mathcal{T}$ for the pair of groups (A_X, τ^*A_Y) on X. We define its "direct image in spaces" $\tau_*^{\mathrm{Sp}}(\mathcal{T}) \to Y$ as the total space of $\mathcal{T}$, considered as a variety over Y.

Lemma. The group A_Y acts on $\tau_*^{\mathrm{Sp}}\mathcal{T}$ on the right and

$$Aut_{A_Y}(\tau_*^{\mathrm{Sp}}\mathcal{T}) \ = \ \tau_*^{\mathrm{Sh}}A_X.$$

(Here, τ_*^{Sh} is the direct image in sheaves, i.e., a fiber $[\tau_*^{\mathrm{Sh}}A_X]_y$ is the space $\Gamma(X_y, A_X)$ of sections of A_X on the fiber at y.)

Example. Consider the case of the map $\mathcal{P} \xrightarrow{a} pt$, the group $M_{\mathcal{P}}$ on $\mathcal{P}$ and its fiber $[M_{\mathcal{P}}]_P = M$ as a group on the point pt. We know that the total space of the restriction $\mathcal{S}|_{\mathcal{P}\times\{P\}}$ of the groupoid $\mathcal{S} = \mathcal{S}_{\mathcal{P},V}$ to a copy of $\mathcal{P}$ is the map $G/V = \to G/P = \mathcal{P}$ and that this is a bitorsor on $\mathcal{P}$ for $(M_{\mathcal{P}}, M\times\mathcal{P})$ (lemma 20.1.2).

The direct a_*^{Sp} image of this bitorsor $\mathcal{S}|_{\mathcal{P}\times\{P\}} \to \mathcal{P}$ is its total space G/V. The lemma now says that

Corollary. For the action of M on G/V on the right

$$\mathrm{Aut}_M(G/V) \ = \ Z(M).$$

Proof. $Aut_M(G/V)$ is $\mathrm{Aut}_M\big(a_*^{Sp}(G/V \to \mathcal{P})\big)$ and by the lemma this is $(\mathcal{P} \to \mathrm{pt})_*^{Sh} M_{\mathcal{P}} = \Gamma(\mathcal{P}, M_{\mathcal{P}})$. However, this was calculated in lemma 20.2.2 as $Z(M)$ $\square$

20.2.4. *M-automorphisms of G/V.*

Corollary. If $G \to \mathrm{Aut}(\mathcal{P})$ is surjective then

$$\mathrm{Aut}_M\big(\mathcal{S}|_{\mathcal{P}\times\{P\}}\big) = \ \mathrm{Aut}_M(G/V) = \ G \underset{Z(G)}{\times} Z(M).$$

Proof. Recall that the fiber $(G/V)_x$ of $G/V \to G/P$ at a point $x = gP \in \mathcal{P}$ is gP/V, and this is a bitorsor for $({}^gP/g^V, M)$ where the first group is the fiber of $M_{\mathcal{P}}$ at $x = gP$.

(1) Let $\pi : G/V \to G/P = \mathcal{P}$. The maps i and q in the sequence

$$0 \to \Gamma(\mathcal{P}, M_{\mathcal{P}}) \ \xrightarrow{i} \ \mathrm{Aut}_M(G/V) \ \xrightarrow{q} \ \mathrm{Aut}(G/P) \to 0$$

are defined by

$$[i(\gamma)](y) \overset{\mathrm{def}}{=} \ \gamma\big(\pi(y)\big)\cdot y \quad \text{and} \quad [q(\alpha)](x) \overset{\mathrm{def}}{=} \ \pi[\alpha(y)] \ \text{ for } \ y \in G/V, \ x = \pi(y) \in G/P.$$

This sequence is exact. First, the map q is surjective by the canonical map

$$G \underset{Z(G)}{\times} Z(M) \xrightarrow{\iota} \mathrm{Aut}_M(G/V)$$

and by our assumption that $G\twoheadrightarrow \mathrm{Aut}(G/P)$.

The map i is injective since for $\gamma\in \mathrm{Ker}(i)\subseteq\Gamma(\mathcal{P},M_{\mathcal{P}})$ and any x,y as above we have $y=[\gamma(x)](y)=\gamma(x){\cdot}y$, i.e., $\gamma(x)\in (M_{\mathcal{P}})_x$ acts trivially on the torsor $(G/V)_x$. So, $\gamma(x)=1$ and $\gamma=1$.

Now, elements α of $\mathrm{Ker}[\mathrm{Aut}_M(G/V)\xrightarrow{q}\mathrm{Aut}_M(G/P)])$ preserve each fiber $(G/V)_x$ and act on it by automorphisms of the right M-torsor structure. So, such α acts on $(G/V)_x$ as a unique element $\widetilde{\alpha}(x)\in (M_{\mathcal{P}})_x$, hence α acts on G/V as a section $\widetilde{\alpha}\in\Gamma(\mathcal{P},M_{\mathcal{P}})$

(2) By the lemma 20.2.2 $\Gamma(\mathcal{P},M_{\mathcal{P}})\cong\ Z(M)$. □

Remark. In order to extend the lemma to the general situation in the formulation $\mathrm{Aut}_M(G/V)\ =\ \widetilde{\mathrm{Aut}}(\mathcal{P})\underset{Z(G)}{\times}Z(M)$, we would need surjectivity of $\mathrm{Aut}_M(G/V)\to\mathrm{Aut}(G/P)$.

20.2.5. *Invertible sections of the groupoid $\mathcal{S}_{\mathcal{P},V}$.*

Lemma. Under the assumptions 20.2.1

$$\Gamma^*(\mathcal{P},\mathcal{S}_{\mathcal{P},V})\ \cong\ G\times_{Z(G)} Z(M).$$

Proof. A section $s\in\ \Gamma(\mathcal{P},\mathcal{S})\to\mathcal{P})$ of the groupoid $\mathcal{S}\xrightarrow{(q,p)}\mathcal{P}^2$, is a pair $s=(f,\sigma)$ of a map $f:\ \mathcal{P}\to\mathcal{P}$ and a section σ of $\mathcal{G}\to X^2$ over the graph of f, i.e., $\sigma:\ X\to\mathcal{G}$ and $\sigma(x)\in\ \mathcal{S}_{f(x),x}$. Subgroup $\Gamma^*(\mathcal{P},\mathcal{S})$ is the invertible part of the semigroup $\Gamma(\mathcal{P},\mathcal{S})$, it consists of all $f\in\ \mathrm{Aut}(\mathcal{P})$.

Group $G\underset{Z(G)}{\times}Z(M)$ acts on $\mathcal{S}\xrightarrow{p}\mathcal{P}$ and on sections $\Gamma^*(\mathcal{P},\mathcal{S})$. The translations of the canonical section $\mathbf{1}$ give an embedding $G\underset{Z(G)}{\times}Z(M)\xrightarrow{\iota}\Gamma^*(\mathcal{P},\mathcal{S})$. There is an exact sequence $0\to\Gamma(\mathcal{P},\mathcal{S}|\Delta_{\mathcal{P}})\to\Gamma^*(\mathcal{P},\mathcal{S})\xrightarrow{\tau}\mathrm{Aut}(\mathcal{P})\to 0$, since (1) by assumption, G surjects onto $\mathrm{Aut}(\mathcal{P})$, and (2) the kernel of τ consists of sections $s=\ (1,\sigma)$, with σ a section of $\mathcal{S}|\Delta_{\mathcal{P}}=\ M_{\mathcal{P}}$. Now ι is also surjective since by lemma 20.2.2, $\Gamma(\mathcal{P},\mathcal{S}|\Delta_{\mathcal{P}})=\ \Gamma(\mathcal{P},M_{\mathcal{P}})=Z(M)$.

20.2.6. *The case of the flag variety.* For the flag variety $\mathcal{B}$ the only choice of $V\subseteq B$ is the unipotent radical N. Now, $\mathcal{S}_{B'',B'}=\ \{g\in G,\ {}^gB'=\ B''\}$ is a bitorsor for $(B''/N'',B'/N')$. Group bundle $\boldsymbol{B}_{\mathcal{B}}/\boldsymbol{N}_{\mathcal{B}}=\ G\underset{B}{\times}B/N$ is the trivial bundle $\mathcal{B}\times\ H$ for the abstract Cartan $\mathcal{B}$ and the group of sections is H (by its definition). Finally,

$$\mathrm{Aut}_H(G/N)\cong\ G\underset{Z(G)}{\times}H\cong\ \Gamma^*(\mathcal{B},\mathcal{S}).$$

20.3. **Semigroupoid $\overline{\mathcal{S}}_{P,V}=\ G\times_P\ (G/V)^{\mathrm{aff}}$ and its sections.** We define $\overline{\mathcal{S}}$ as the affinization of $\mathcal{S}$ relative to a projection to a single copy of $\mathcal{P}$.

Conjecture. $\overline{\mathcal{S}}$ is a semigroupoid over $\mathcal{P}$

The only things established here are certain injectivity results for the restriction of sections to $\mathcal{B}^T = W$ and the case of SL_2.

20.3.1. *Evaluation of sections at points of $\mathcal{B}$.* We will see in 20.4 that for $G = SL_2$ the evaluation at opposite Borels $\mathfrak{b}_\pm$ yields an isomorphism

$$\Gamma(\mathcal{B}, \overline{\mathcal{S}}) \xrightarrow{\cong} \overline{G/N_+} \times \overline{G/N_-}.$$

In general we have

Lemma. Restriction of sections to $\mathcal{B}^H$, yields an embedding $\Gamma(\mathcal{B}, \overline{\mathcal{S}}) \hookrightarrow \prod_{w \in W} G/{}^wN$.

Proof. Image of $1 \in G \times H$ is the collection $\mathbf{1}$ of unit cosets. Since the map is equivariant under $G \times H$ it suffices to find the fiber at $\mathbf{1}$. It consists of all $(g, h) \in G \times H$ such that g lies in all ${}^wB$, $w \in W$,, i.e., $g \in H$; and for each $w \in W$, h_B equals $g_{{}^wB} = {}^w(g_B)$. So the conditions are that $g \in H^W = Z(G)$ and then also $h = g_B$ lies in $Z(G) \subseteq H$.

Remarks. (a) In general, the evaluation at $\mathfrak{b}_\pm$ is not injective. [Let ρ be the composition $G \times H \hookrightarrow S \to \overline{G/N_+} \times \overline{G/N_-}$, then $\rho(g, s) = (1, 1)$ implies that $g \in B_- \cap B_+ = H$ and then it is equivalent to $h = g_{B_\pm}$. So the conditions are that $h = g_{B_-} = {}^{w_0}g_{B_+} = {}^{w_0}h$, i.e., $g \in H^{\{1, w_0\}}$ and $h = g_{B_\pm}$.]

(b) The fact that a section of $\mathcal{S} \to \mathcal{B}$ is determined by the values at W points ${}^w\mathfrak{b}$, $w \in W$ seems to be a generalization of the fact that in the case of SL_2, for $\overline{\mathcal{S}} = \mathbb{C}^2(1)$, any section is determined by the values at two points. [However, considerably fewer points may suffice?]

20.4. **The case $G = SL_2$.** In this case, $\overline{\mathcal{S}} \stackrel{\text{def}}{=} (G/N)^0 \to \mathcal{B}$ can be identified with the vector bundle
$OO^2_{\mathbb{P}^1}(1)$ over $\mathbb{P}^1$ (which appears in various settings), and the sections $\Gamma(\mathcal{B}, \overline{\mathcal{S}})$ with 2 by 2 matrices M_2.

20.4.1. *Lemma.* (a) The affine closure of G/N is $\mathsf{W} \stackrel{\text{def}}{=} \mathbb{C}^2$

(b) The action of G on G/N becomes the standard G-action on W and the H-action becomes via $\rho : H \xrightarrow{\cong} G_m$ the standard action of G_m on a vector space.

(c) The conjugation action of B on G/N gives a new structure of a B-module on W, isomorphic to the B-module $\mathfrak{g}/\mathfrak{n}$.

Proof. We fix the notation. Let $G = SL_2 \supseteq B = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) = N \cdot H$ for $H = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. Let $\Delta^+ = \{\alpha\}$ and $\rho = \alpha/2$, so that $\Delta_H(\mathfrak{g}/\mathfrak{b}) = \{\alpha_B\}$. Denote $\mathsf{W} = \mathbb{C}^2$ and fix the basis $(e_1, e_2) = (e, f)$.

(a) The stabilizer of $e = \binom{1}{0}$ is $G_e = N$, so $\iota : G/N \xrightarrow{\cong} \mathsf{W} - \{0\}$, $gN \mapsto g{\cdot}e$ is an identification of homogeneous spaces of G.

For (b) observe that $s = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in H$ acts on e by $a = (-\rho_B)(s)$. So for $h \in H$ and $g \in G$,

$$\iota(h\ \ gN) = \iota(gN{\cdot}(h_B)^{-1}) = \iota(g\ (h_B)^{-1}\ N) = g\ (h_B)^{-1}{\cdot}e = \rho_B(h_B){\cdot}ge = \rho(h){\cdot}\iota(g).$$

(c) For $b \in B$ and $v \in \mathsf{W}$ one has $b \underset{new}{\cdot} v = \rho_B(b){\cdot}bv$, since for $g \in G$ and $v = ge$

$$b \underset{new}{\cdot} ge \overset{\text{def}}{=} ({}^b g){\cdot}e = b{\cdot}g{\cdot}b^{-1}{\cdot}e = (\rho_B(b){\cdot}b){\cdot}ge.$$

So with the new action this B-module is $\mathsf{W}(1) \overset{\text{def}}{=} \mathsf{W}{\otimes}\rho_B$, and this is non-canonically isomorphic to $\mathfrak{g}/\mathfrak{n}$ (both are indecomposable B-modules with the same weights $2, 0$).

20.4.2. *Corollary.* (a) $G{\times}H$-equivariant bundle $\overline{\mathcal{S}} \overset{\text{def}}{=} (\overline{G/N})^0 \to \mathcal{B}$ is isomorphic to the $G{\times}G_m$-equivariant vector bundle $(\mathfrak{g}/\mathfrak{n})^0 \to \mathcal{B}$. Here G_m acts on the vector bundle in the standard way and we use identification $\rho : H \xrightarrow{\cong} G_m$.

(b) $(\mathfrak{g}/\mathfrak{n})^0 \cong \mathcal{O}(1){\oplus}\mathcal{O}(1) \cong \mathcal{T}_{\mathbb{H}}$ (the twistor space of the hyperkähler manifold $\mathbb{H}$). In particular, the sections of $(\mathfrak{g}/\mathfrak{n})^0 = \widetilde{T}^*\mathcal{B}$ (the universal twisted cotangent bundle), can be identified with the set M_2 of $2{\times}2$ matrices.

20.4.3. *Identification of* $\Gamma(\mathbb{P}^1, \mathcal{S})$ *with* GL_2. In this case it is simpler to think of $\overline{\mathcal{S}}$ first. With the conjugation action of B, $\overline{G/N}$ has been identified with $\mathsf{W}(1)$, and this induces $G/N \cong \mathsf{W}(1) - \{0\}$. Therefore, sections of $\overline{\mathcal{S}} \to \mathcal{B}$ can be identified with M_2. A bases e_1, e_2 of W gives $\Gamma[\mathbb{P}^1, \mathsf{W}(1)] = \Gamma[\mathbb{P}^1, \mathcal{O}(1)]e_1 \oplus \Gamma[\mathbb{P}^1, \mathcal{O}(1)]e_2$, with each summand of dimension two and giving one row of M_2 (or a column?).

Sections of $\mathcal{S} \to \mathcal{B}$ are the non-vanishing sections of $\mathsf{W}(1) = \mathcal{O}(1){\oplus}\mathcal{O}(1)$. A non-zero section s of $\mathcal{O}(1)$ vanishes precisely once, its divisor is a point x in $\mathbb{P}^1$ and s is determined by x up to a scalar. A section $s = (s_1, s_2)$ of $\mathsf{W}(1)$, vanishes iff one of s_i's is zero, or if they have the same divisor; but this is the same as saying that one is a multiple of the other, i.e., that the matrix with rows s_i is not invertible.

Odds

20.5. **Action groupoid $\mathcal{G}$.** The action of G on a partial flag variety $\mathcal{P}$ defines the action groupoid $\mathcal{G} \xrightarrow{p'', p'} \mathcal{P}^2$ with fibers $\mathcal{G}_{P'', P'} = \{g \in G,\ {}^g P' = P''\}$. Its restriction to the diagonal $\mathcal{G}^{\uparrow} = \mathcal{G}|_{\Delta_{\mathcal{P}}}$ is the stabilizer group bundle $\boldsymbol{P}_{\mathcal{P}}$ with the fiber at $P \in \mathcal{P}$ equal P.

20.5.1. *A base point* $P \in \mathcal{P}$. For any choice of $P \in \mathcal{P}$, the restriction $\mathcal{G}|_{\mathcal{P}\times\{P\}}$ is a bitorsor for $(P_{\mathcal{P}}, P\times\mathcal{P})$ since the fiber $\mathcal{G}_{P,P}$ is a (P,P)-bitorsor. Its total space is G (for any action groupoid, $\mathcal{G}\xrightarrow{p'}\mathcal{P}$ is identified with $G\times\mathcal{P}$). Also, $P_{\mathcal{P}}$ can be written as $G\underset{P}{\times} P$ for the conjugation action on the second factor.

20.5.2. *Normal subgroup* V *of* P. Fix $P \in \mathcal{P}$ and its normal subgroup V which will be either the derived subgroup P' or the unipotent radical U of P. It defines a normal subgroup $V_{\mathcal{P}}$ of $P_{\mathcal{P}}$, hence for each $P \in \mathcal{P}$ a normal subgroup $V = V_P \subseteq P$.

If $\mathcal{P} = \mathcal{B}$ then $V = B' = U$.

20.6. **Groupoid** $\mathcal{S} = \mathcal{S}_{\mathcal{P},V}$. Since $V_{\mathcal{P}}$ is also a normal subgroupoid of $\mathcal{G}$, we also get a groupoid

$$\mathcal{S} \overset{\text{def}}{=} \mathcal{G}/V_{\mathcal{P}} \xrightarrow{p'',p'} \mathbb{P}^2$$

with fibers $\mathcal{S}_{P'',P'} = V_{P''}\backslash\mathcal{G}_{P'',P'} = \mathcal{G}_{P'',P'}/V_{P'}$.

The restriction $\mathcal{S}|_{\mathcal{P}\times\{P\}}$ is a bitorsor for $(M_{\mathcal{P}}, M\times\mathcal{P})$, and its total space is G/V. So, $\mathcal{S}$ can be written as $G\underset{P}{\times} G/V$ for the conjugation action on the second factor.

20.7. **Use of quasimaps for q-cohomology of flag varieties?** 19.9.3

Givental's used quasimaps for the q-cohomology of flag varieties. In our paper [FFKM], there is only a map in one direction and this is also what is expected here: $\overline{\mathcal{V}}(G)$ mapping to "Hilbert maps".

Here, $\overline{\mathcal{V}}$ is conjecturally constructed in terms of "stable maps" whic we defined as "Hilbert maps", i.e., the Hilbert space closure of the space of graphs of maps. Prsumably the stable curves of Kontsevich have the same AG interpretation.

Is the present construction related to Giventhal's mystery?

21. **Vinberg semigroups as sections of semigroupoids (Conjectures)**

My old proposal to construct these as "stable sections" (a la Kontsevich's stable curves), has been partially accomplished by Brion – he does wunderbar compactifications as the stable automorphisms of the (partial) flag variety. My problem was that I did not have a definition of stable maps in general, but of course these are just closures of maps in the Hilbert scheme. With this definition everything should be clear.

We will describe the Vinberg semigroup as

- (i) stable sections of a bundle over the flag variety and
- (ii) affinization of a torsor over the very wonderful compactification.

Let $\pi_1(G) = 0$ and let the subset $J \subseteq I$ of simple coroots correspond to Levi factors of parabolic subgroups in a partial flag variety $\mathcal{P}$. For a Cartan subgroup T of $P = UL \in \mathcal{P}$ one has $X_*(T) = \underline{\mathbb{Z}[I]}$ and $X_*(P^a b) = \mathbb{Z}[I-J]$. The cone $\mathbb{Z}_+[I-J]$ corresponds to a semigroup closure $\overline{P^{\mathrm{ab}}}$.

21.1. **Conjectures on Vinberg semigroups.** For a parabolic flag variety $\mathcal{P}$ consider the torsor $\widetilde{\mathcal{P}} = G/P' \to G/P = \mathcal{P}$ for $H_{\mathcal{P}} \overset{\text{def}}{=} P^{\mathrm{ab}}$. Its automorphism groupoid $\mathcal{G} \overset{\text{def}}{=} \mathcal{A}ut(\widetilde{\mathcal{P}}/\mathcal{P}) \xrightarrow{(q,p)} \mathcal{P}^2$, has fibers $\mathcal{G}_{\mathfrak{p}'',\mathfrak{p}'} = Isom_{H_{\mathcal{P}}}(\widetilde{\mathcal{P}}_{\mathfrak{p}'}, \widetilde{\mathcal{P}}_{\mathfrak{p}''})$.

It lies in a semigroupoid $\overline{\mathcal{G}} \overset{\text{def}}{=} \mathcal{G} \underset{H_{\mathcal{P}}}{\times} \overline{H_{\mathcal{P}}}$, for the canonical semigroup closure $\overline{H_{\mathcal{P}}}$ of $H_{\mathcal{P}}$.

Let us also consider the relative affinization $(\mathcal{G}_{\mathcal{P}})^{\mathrm{aff}} \overset{\text{def}}{=} [\mathcal{G}_{\mathcal{P}} \xrightarrow{p} \mathcal{P}]^{\mathrm{aff}}$. It is a non-symmetric object, i.e., it maps to only to the second copy of $\mathcal{P}$.

21.1.1. *Lemma.* $(\mathcal{G}_{\mathcal{P}})^{\mathrm{aff}}$ is the affinization of $\widetilde{\mathcal{G}_{\mathcal{P}}}$ and $\widetilde{\mathcal{G}_{\mathcal{P}}}$ is a resolution of $\overline{\mathcal{G}_{\mathcal{P}}}$.

XXX $\square$[70]$\blacksquare$ The following is also in 8.3.6 at least in part.

21.1.2. *Conjectures.* (a) $\Gamma(\mathcal{G}) = \Gamma^*(\mathcal{G})$ is the Vinberg group $G_{\mathcal{P}} \overset{\text{def}}{=} G \underset{Z(G)}{\times} H_{\mathcal{P}}$.

(b) The closure $\overline{\Gamma(\mathcal{G})}$ of $\Gamma^*(\mathcal{G})$ in $\Gamma(\overline{\mathcal{G}})$ is a semigroup.

(b') If $\mathcal{P} = \mathcal{B}$ this is an extension of the wonderful compactification $\overline{G/Z(G)}$ by $\overline{H_{\mathcal{P}}}$.

(b") An open part $\overline{\Gamma(\mathcal{G})}^0$ is an extension by $H_{\mathcal{P}}$ and $\overline{\Gamma(\mathcal{G})} = \overline{\Gamma(\mathcal{G})}^0 \underset{H_{\mathcal{P}}}{\times} \overline{H_{\mathcal{P}}}$. As an $H_{\mathcal{P}}$-torsor over $\overline{G/Z(G)}$, this open part $\overline{\Gamma(\mathcal{G})}^0$ corresponds to the I-colored divisor which is minus the boundary of bolshaya yacheyka in $G/Z(G)$.

[70] $\boxed{!}$

(c) The affinization of $\overline{\Gamma(\mathcal{G})}$ is a semigroup which we **call the Vinberg semigroup** $\overline{G_{\mathcal{P}}}$ associated to the partial flag variety $\mathcal{P}$. In turn, $\overline{\Gamma(\mathcal{G})}$ is a resolution of $\overline{G_{\mathcal{P}}}$. In particular when $\mathcal{P}$ is the flag variety $\mathcal{B}$ and G is simply connected, we get the usual Vinberg semigroup.

Add the formulation for general $\mathcal{P}$.

(d) There should be another statement concerning the action of $\widetilde{\mathcal{G}_{\mathcal{P}}}$ on $G/P' \underset{H_{\mathcal{P}}}{\times} \overline{H_{\mathcal{P}}}$.

Proof. (a) is known for $\mathcal{P} = \mathcal{B}$, and all is known for SL_2.

YYY

Example. For $G = G_{\mathrm{ad}} = PGL_2$ we have $\mathcal{V} = G \times H$. If $G = PGL_2$ then $(G/N)^{\mathrm{aff}}$ is the nilpotent cone $\mathcal{N}$ and $G/N = \mathcal{N}_{reg}$.

21.1.3. *The absolute Vinberg semigroup $\overline{\mathcal{V}}(G) = \overline{\mathcal{V}}_{\mathcal{B}}(G)$ as endomorphisms of $(G/N)^{\mathrm{aff}}$.* To a simply connected semisimple group G one attaches the *Vinberg group* $\mathcal{V} \stackrel{\mathrm{def}}{=} G \underset{Z(G)}{\times} H$ and its semigroup closure, the *Vinberg semigroup* $\overline{\mathcal{V}} = \overline{\mathcal{V}}(G)$. Some of its features:

(1) Vinberg semigroup $\overline{\mathcal{V}}(G)$ acts on $(G/N)^{\mathrm{aff}}$ by H-endomorphisms and actually it is precisely the semigroup of H-endomorphisms of $(G/N)^{\mathrm{aff}}$ (8.2).
(2) In particular, one of its orbits provides a canonical map $\overline{\mathcal{V}}(G) \to (G/N)^{\mathrm{aff}}$.
(3) In characteristic zero, $\overline{\mathcal{V}}(G)$ has been introduced by describing its algebra of functions (a subalgebra of $\mathcal{O}(\mathcal{V})$).

22. **Semigroupoid $\mathcal{Z}$ over $\mathcal{P}$**

22.1. **Version one: $\mathcal{Z}$.** A groupoid $\mathcal{S} \xrightarrow{(q,p)} \mathcal{P}^2$ defines a space over $\mathcal{P}$ $\mathcal{S}' \stackrel{\text{def}}{=} (\mathcal{S} \xrightarrow{p} \mathcal{P})$ and its affinization

$$(\mathcal{S}')^{\text{aff}} \stackrel{\text{def}}{=} (\mathcal{S} \xrightarrow{p} \mathcal{P})^{\text{aff}}.$$

One can extend the structure map (q,p) to a correspondence

$$\mathcal{Z} \subseteq \mathcal{S}^{aff}_{/2} \times \mathcal{P}^2$$

which is the closure of the graph of (q,p). So, $(\mathcal{S}^{aff}_{/2})_c = (\mathcal{S}_{\mathcal{P}\times a})^{\text{aff}}$ need not map to $\mathcal{P}$, and $\mathcal{Z}_{\mathcal{P}\times a} =$

22.1.1. *Lemma.* $\mathcal{Z}$ is a semigroupoid.

Proof. We need to extend the multiplication $\mathcal{S} \underset{\mathcal{P}}{\times} \mathcal{S} \to \mathcal{S}$ i.e., $\overline{\mathcal{S}}_{c,b} \times \overline{\mathcal{S}}_{b,a} \to \overline{\mathcal{S}}_{c,a}$.

$\mathcal{S}_{\mathcal{P}\times b} \times \mathcal{S}_{(b,a)} \to \mathcal{S}_{\mathcal{P}\times a}$ gives

$$\overline{\mathcal{S}}_{\mathcal{P}\times b} \times \mathcal{S}_{(b,a)} = \left(\mathcal{S}_{\mathcal{P}\times b} \times \mathcal{S}_{(b,a)}\right)^{\text{aff}} \to (\mathcal{S}_{\mathcal{P}\times a})^{\text{aff}} = \overline{\mathcal{S}}_{\mathcal{P}\times a}.$$

Because of the G-equivariance one has $Z = G \underset{P}{\times} Z_P$ for the fiber Z_P at $P \in \mathcal{P}$, and the fiber is a partial compactification of the group $(G/V)_P = M$.

22.1.2. *Conjecture.* Affinization $q^{\text{aff}} : Z^{\text{aff}} \to (G/V)^{\text{aff}}$ is an isomorphism.

" *Proof.* " Map q is proper and generically it is the isomorphism $\Gamma_\pi \xrightarrow{\cong} G/N$. In particular, q is surjective. Since G/N is irreducible, so is Z.

22.1.3. *Semigroup $\overline{P^{\text{ab}}}$.* Let $\pi_1(G) = 0$ and let $\mathcal{P}$ correspond to a subset J of the set I of simple coroots. For a Cartan subgroup T of a Levi factor L of $P = U{\cdot}L$, one has $X_*(T) = \mathbb{Z}[I]$ and $X_*(P^ab) \xrightarrow{\cong} \mathbb{Z}[I-J]$. The cone $\mathbb{Z}_+[I-J]$ consists of cocharacters that extend to maps from $\overline{G_m} = (G_c, \cdot)$ to the semigroup closure $\overline{P^{\text{ab}}}$. We identify $\overline{P^{\text{ab}}}$ with G_c^{I-J} by cocharacters $I-J$.

To $J \subseteq I-J$ one can associate a subsemigroup $P^{\text{ab}}_J \subseteq P^{\text{ab}}$ with degeneracy J: $P^{\text{ab}}_J = \{(z_i)_{I-J} \in G_c^{I-J},\ z_i = 0 \text{ for } i \in J\}$.

22.2. **Version two: Z.** The extension of the map $G/V \xrightarrow{\pi} G/P$ to the affinization $(G/V)^{\text{aff}}$ is the correspondence $Z \overset{q,p}{\subseteq} (G/V)^{\text{aff}} \times \mathcal{P}$ – the closure of the graph of π.

Because of the G-equivariance one has $Z = G \underset{P}{\times} Z_P$ for the fiber Z_P at $P \in \mathcal{P}$, and the fiber is a partial compactification of the group $(G/V)_P = M$.

22.2.1. *Conjecture.* Affinization $q^{\text{aff}}: \ Z^{\text{aff}} \to (G/V)^{\text{aff}}$ is an isomorphism.

“ *Proof.* ” Map q is proper and generically it is the isomorphism $\Gamma_\pi \xrightarrow{\cong} G/N$. In particular, q is surjective. Since G/N is irreducible, so is Z.

22.2.2. *Semigroup* $\overline{P^{\text{ab}}}$. Let $\pi_1(G) = 0$ and let $\mathcal{P}$ correspond to a subset J of the set I of simple coroots. For a Cartan subgroup T of a Levi factor L of $P = U{\cdot}L$, one has $X_*(T) = \mathbb{Z}[I]$ and $X_*(P^a b) \xrightarrow{\cong} \mathbb{Z}[I-J]$. The cone $\mathbb{Z}_+[I-J]$ consists of cocharacters that extend to maps from $\overline{G_m} = (G_c, \cdot)$ to the semigroup closure $\overline{P^{\text{ab}}}$. We identify $\overline{P^{\text{ab}}}$ with G_c^{I-J} by cocharacters $I-J$.

To $J \subseteq I-J$ one can associate a subsemigroup $P^{\text{ab}}_J \subseteq P^{\text{ab}}$ with degeneracy J: $P^{\text{ab}}_J = \{(z_i)_{I-J} \in G_c^{I-J}, \ z_i = 0 \text{ for } i \in J\}$.

22.3. Y**.** Let $Y \stackrel{\text{def}}{=} G \times_P \overline{M}$ for a certain semigroup closure $\overline{M}$ of the reductive group M.

23. Appendix. Sections of (semi)groupoids

23.0.1. *Sections.* The space of sections of a semigroupoid $\mathcal{G} \xrightarrow{(q,p)} X^2$ is defined as

$$\Gamma(\mathcal{G}) = \Gamma(X, \mathcal{G}) \stackrel{\text{def}}{=} \Gamma(\mathcal{G} \xrightarrow{p} X).$$

Lemma. (a) $\Gamma(X, \mathcal{G})$ consists of pairs $s = (f, \sigma)$ of a map $f: X \to X$ and a section σ of $\mathcal{G} \to X^2$ over the graph Γ_f, i.e., $\sigma: X \to \mathcal{G}$ and $\sigma(x) \in \mathcal{G}_{f(x),x}$.[71]

(b) $\Gamma(X, \mathcal{S})$ is a semigroup for

$$(f'', \sigma'') \cdot (f', \sigma') \stackrel{\text{def}}{=} (f'' \circ f', \sigma), \quad \text{for} \quad \sigma(x) \stackrel{\text{def}}{=} \sigma''(f'(x)) \cdot \sigma'(x).$$

(c) A semigroupoid $\mathcal{G}$ over X also defines a semigroup $\mathcal{G}^\uparrow \stackrel{\text{def}}{=} \mathcal{G}|_{\Delta_X}$ over X, and the corresponding semigroup $\Gamma(X, \mathcal{G}^\uparrow)$. There is an exact sequence of pointed sets

$$0 \to \Gamma(X, \mathcal{G}^\uparrow) \to \Gamma(\mathcal{G}) \to End(X).$$

Proof. (b) Here, $\sigma'(x) \in \mathcal{G}_{f'(x),x}$ and $\sigma''(f'(x)) \in \mathcal{G}_{f''(f'(x)),f'(x)}$, hence $\sigma(x) \in \mathcal{G}_{f''(f'(x)),x}$. $\square$

Remark. If $\mathcal{G}$ is a groupoid, then $\mathcal{G}^\uparrow$ and its sections are groups and so is $\Gamma^*(X, \mathcal{G}) \subseteq \Gamma(X, \mathcal{G})$ defined as the inverse of $Aut(X)$ in $\Gamma(\mathcal{G})$. Then one has an exact sequence of groups

$$0 \to \Gamma(X, \mathcal{G}^\uparrow) \to \Gamma^*(\mathcal{G}) \to Aut(X).$$

[71] (One can say that $s(x) = (f(x), \sigma(x), x)$ for $x \in X$.)

23.0.2. *Semigroup closures from semigroupoid closures.* We see that any groupoid $\mathcal{G}$ generates a group $\Gamma^*(\mathcal{G})$. If a groupoid $\mathcal{G}$ lies in a semigroupoid $\overline{\mathcal{G}}$, then the group $\Gamma^*(\mathcal{G})$ naturally extends to a semigroup $\overline{\Gamma^*(\mathcal{G})}$ – the closure of $\Gamma^*(\mathcal{G})$ in $\Gamma(\overline{\mathcal{G}})$.

The "closure" here means the closure of the space of graphs of sections $\gamma_{f,\sigma} = \sigma(X) \subseteq \overline{\mathcal{G}}$ in the Hilbert scheme of the total space of $\overline{\mathcal{G}}$. I will informally call elements of $\overline{\Gamma^*(\mathcal{G})}$ the "stable" sections of $\mathcal{G}$.

23.0.3. *A framework for constructing extensions of groups: Extensions of automorphism groups of objects over a given space.* Suppose that our groupoid $\mathcal{G}$ is of the form $\mathcal{A}ut(\mathfrak{X})$, the groupoid of symmetries of an object $\mathfrak{X}$ over X. Then, $(f,\gamma) \in \Gamma^*[\mathcal{A}ut(\mathfrak{X})]$ that lies above some $f \in Aut(X)$ is a family of $\gamma_x \in Isom(\mathfrak{X}_x, \mathfrak{X}_{f(x)})$, i.e., an isomorphism of $\mathfrak{X}$ and $f^*\mathfrak{X}$.

The image of the projection $\Gamma^*[\mathcal{A}ut(\mathfrak{X})] \to Aut(X)$, $(f,\sigma) \mapsto f$, is the stabilizer subgroup $Aut(X)_{[\mathfrak{X}]}$ of the isomorphism class $[\mathfrak{X}]$ of $\mathfrak{X}$, i.e., all $f \in Aut(X)$ such that $f^*\mathfrak{X}$ is *isomorphic* to X. So we get an exact sequence

$$0 \to Aut(\mathfrak{X}) \to \Gamma^*[\mathcal{A}ut(\mathfrak{X})] \to Aut(X)_{[\mathfrak{X}]} \to 0.$$

Therefore, $\widetilde{Aut}(\mathfrak{X}) \stackrel{\text{def}}{=} \Gamma^*[\mathcal{A}ut(\mathfrak{X})]$ is an extension of $Aut(X)_{[\mathfrak{X}]}$ by $Aut(\mathfrak{X})$.

Remark. Such extensions come with semigroup closures $\widetilde{\mathrm{Aut}}(X, [\mathfrak{X}])$ from the semigroupoid closure $\mathrm{End}(\mathfrak{X}/X) \supseteq \mathrm{Aut}(X, [\mathfrak{X}]))$.

Example.

Part 6. Affinization of G/V for $V = U_P$ or $V = P'$

Here we use the affinization of quasiaffine schemes as a construction tool.

23.0.4. *Notation.* Here we assume that G is simply connected semisimple group over a closed field $\Bbbk$. Fix a Borel subgroup $B = N \ltimes T$ and let $H = B/N$. For a dominant weight λ let $\check{W}(\lambda)$ be the corresponding coWeyl module. Let $\check{W}_i = \check{W}(\omega_i)$, $i \in I$, and $\check{W} \overset{\text{def}}{=} \oplus_{i\in I} \check{W}_i$.

24. Affinization of $(G/P')^{\text{aff}}$

24.1. Subgroups $V\subseteq G$ with G/V quasiaffine.

Question. Is G/V is quasiaffine for any unipotent subgroup $V\subseteq G$?

Is G/V not quasiaffine precisely when V meets some subgroup S with $Lie(S) \cong sl_2$ in a Borel subgroup B_S? (In that case G/V contains $S/B_S \cong \mathbb{P}^1$.) If not then there should be a representation V and a vector v such that G_v is V?

Example. For an affine G-space X and any point $x \in X$, the stabilizer G_x has the property that G/G_x is quasiaffine (since $G/G_x \cong \ {}^G_x{\subseteq}X$). This in particular applies to centralizers in G of elements of $\mathfrak{g}$ or G.

24.2. **Summary.** Each $(G/P')^{\text{aff}}$ has a resolution which is obtained by extending the map $G/P' \to G/P$ (a P^{ab}-torsor), to the affinization $(G/P')^{\text{aff}}$. The resolution is a $\overline{P^{\text{ab}}}$-torsor over G/P. The appearance of $\overline{P^{\text{ab}}}$ in the resolution is just a fancy way of saying that $\overline{P^{\text{ab}}}$ is the closure of P^{ab} in $(G/P')^{\text{aff}}$.

The semigroup structure on $\overline{P}^{\text{ab}}$ is the same problem as the construction of the semigroupoid structure on $\overline{\mathcal{S}}$.

G-orbits in the resolution and in $(G/P')^{\text{aff}}$ are in a bijection. For instance, G-orbits in $(G/N)^{\text{aff}}$ are of the form G/P'_J for all $J\subseteq I$. In this way $(G/P'_J)^{\text{aff}}$ embeds into $(G/N)^{\text{aff}}$ as the closure of G/P'_J.

24.3. **Affinization $\mathcal{Y}$ of G/N.** For convenience we choose a frame y_i of $\check{W}_i^N$. This gives $y = (y_i)_{i\in I} \in \check{W}$ and a G-orbit $\mathcal{Y}_\phi \overset{\text{def}}{=} G{\cdot}y\subseteq\check{W}$. Moreover, for any $J\subseteq I$ there is a version with degeneracy J:

$$\mathcal{Y}_J \overset{\text{def}}{=} G{\cdot}y_J \quad for \quad (y_J)_i = \begin{cases} y_i & \text{if } i \in I - J, \\ 0 & \text{if } i \in J. \end{cases}$$

Let $P_J = L_J U_J$ be the standard parabolic obtained by adding J to B (and to T).

be the subvariety of $\check{W} = \prod_{i\in I}\ \check{W}_i$ given by all systems of vectors $v = \ (v_i)_{i\in I}$ that satisfy Plücker equations. It is stratified by degeneracy $J\subseteq I$:

$$\mathcal{Y} = \ \sqcup_{J\subseteq I}\ \mathcal{Y}_J, \quad \mathcal{Y}_J \stackrel{\text{def}}{=} \ \{v\in\mathcal{Y},\ v_i = 0 \text{ iff } i\in J\}.$$

24.3.1. *Lemma.* (a) For $J\subseteq I$, the stabilizer of y_J in G is P'_J, hence $G/P'_J \xrightarrow{\cong} \mathcal{Y}_J$.

(b) Vector bundle $G\times_{P_J}[\prod_{i\notin J}\ \check{W}_i^N]$ is a resolution of $\overline{\mathcal{Y}_J}$.

(c) The closure of $\mathcal{Y}_J$ in $\check{W}$ is the union of all $\mathcal{Y}_K$ with $J\subseteq K$ (the orbits more degenerate then $\mathcal{Y}_J$). Moreover, the resolution from (b) is a bijection on G-orbits.

Proof. (a) is standard.

(b) Since $\prod_{i\in I}\omega_i :\ T\xrightarrow{\cong}(G_M)^I$, one has $T{\cdot}y_J = \ \prod_{i\notin J}\ \check{W}_i^N - \{0\}$, hence $\overline{\mathcal{Y}_J}\supseteq\overline{T{\cdot}y_J} = \prod_{i\notin J}\ \check{W}_i^N$. Therefore the map $G\times_{P_J}\prod_{i\notin J}\ \check{W}_i^N \to \overline{\mathcal{Y}_J}$ is well defined. It is proper since its composition with $\overline{\mathcal{Y}_J}\subseteq\check{W}$ factors into $G\times_{P_J}\prod_{i\notin J}\ \check{W}_i^N \subseteq G\times_{P_J}\check{W} \cong G/P_J\times\check{W} \to \check{W}$. Above $\mathcal{Y}_J$ the map is an isomorphism $G\times_{P_J}[\prod_{i\notin J}\ \check{W}_i^N - \{0\}] \underset{\cong}{\leftarrow} G\times_{P_J} P_J/P'_J \cong G/P'_J$.

(c) G-orbits in $G\times_{P_J}[\prod_{i\notin J}\ \check{W}_i^N]$ are the same as the orbits in $\prod_{i\notin J}\ \check{W}_i^N$ of P_J, i.e., of P_J^{ab} which is identified by $\prod_{i\notin J}\omega_i$ with $(G_m)^{I-J}$. So they are given precisely by the degeneracy $J\subseteq K\subseteq I$.

Since $G\times_{P_J}[\prod_{i\notin J}\ \check{W}_i^N] \to \overline{\mathcal{Y}_J}$ is surjective, the same is true for the map of sets of orbits. However, orbits $\mathcal{Y}_K = G{\cdot}y_K$ are clearly distinct.

24.3.2. We will denote $\overline{G/P'_J} \stackrel{\text{def}}{=} \ \overline{\mathcal{Y}_J}$.

24.3.3. *Corollary.* Let P be a parabolic subgroup P_J.

(a) Space G/P' is quasi affine.

(b) The affine closure of G/P' is the normalization of the closure $\overline{G/P'_J}$ of $\mathcal{Y}_J$ in $\check{W}$.

(c) The lift of the generic stratum to the resolution $G/P'\hookrightarrow G\times_P(\prod_{J^o}\check{W}_i^N)$ is an isomorphism on affinizations.

Proof. $\overline{\mathcal{Y}_J}$ is affine and if for $J\subseteq K\subseteq I$ there is some $i\in K-J$,

$$\dim\mathcal{Y}_J - \ \dim\mathcal{Y}_K = \ \dim G/(P_J)' - \ \dim G/(P_K)' = \ \dim\mathfrak{g}/(\mathfrak{p}_J)' - \ \dim\mathfrak{g}/(\mathfrak{p}_K)'$$
$$= \ \dim[(\mathfrak{p}_K)'/(\mathfrak{p}_J)'] \ \geq \ \dim(\mathfrak{g}_i\oplus\Bbbk{\cdot}\check{i}) = \ 2.$$

So, we see that the affine closure of $\mathcal{Y}_J$ is the normalization of $\overline{\mathcal{Y}_J}$ by 27.1.1.

(c) $G/P' \cong G\times_P(\prod_{J^o}\check{W}_i^N - \{0\}) \subseteq G\times_P(\prod_{J^o}\check{W}_i^N)$ Here, $\mathcal{O}_{G\times_P(\prod_{J^o}\check{W}_i^N)\big/G/P)}$ is a sum of line bundles $\mathcal{O}(\sum_{J^o}\ r_i\omega_i$ on G/P over all $r\in\mathbb{Z}_+^{J^o}$. It lies in $\mathcal{O}_{G\times_P(\prod_{J^o}\check{W}_i^N)\big/G/P)}$ which is such sum over all $r\in\mathbb{Z}^{J^o}$. But a line bundle $\mathcal{O}(\sum_{J^o}\ r_i\omega_i$ has sections only for $r\in\mathbb{Z}_+^{J^o}$.

24.3.4. *Conjecture.* $\overline{G/P'_J}$ ($\stackrel{\text{def}}{=}$ $\overline{\mathcal{Y}_J}$) is normal. So it is the affine closure $(G/P'_J)^{\text{aff}}$ of G/P'_J.

24.3.5. *Remark.* The number of G-orbits in the affine closure of G/N is 2^{rank}, and their mutual position is "toric", the same as for the G-orbits in the wonderful compactification of G.

24.3.6. *Corollary.* The restriction of the the resolution of $\overline{G/N}$ to $\overline{G/P'_J}$ factors into a bundle over the resolution of $\overline{G/P'_J}$, with fibers isomorphic to the flag variety of the Levi factor L_J.

Proof. Let $J\subseteq K$, the fiber at $y_K \in \mathcal{Y}_K \cong G/P'_K$ of the the resolutions of $\overline{G/N}$ is the set of all Borels B such that B' lies in the stabilizer P'_K of v_K, i.e., $B\subseteq P_K$; while the fiber of the the resolutions of $\overline{G/P'_J}$ is the set of all parabolics $P \in \mathcal{P}_J$ such that $P'\subseteq P'_K$, i.e., $P\subseteq P_K$

24.3.7. *Conjecture.* The fibers of the above resolutions of all $\overline{G/P'_J}$ are reduced.

24.3.8. *Remarks.* Denote by $\widetilde{X}$ the resolutions of $X = \overline{G/P'_J}$. The resolution factors through the affinization as $\widetilde{X} \xrightarrow{q} \widetilde{X}^{\text{aff}} \xrightarrow{p} X$. Now p is the normalization and q is surjective since it is proper and generically an isomorphism. Maps p and q are bijections of sets of G-orbits (both maps are surjective hence surjective on G-orbits and the composition is a bijection of orbits).

Moreover, for any orbit α in $\widetilde{X}$, map $q(\alpha) \to p(\alpha)$ is an isomorphism. To start with, $\alpha \to p(q(\alpha))$ has reduced and connected fibers (it can be written as $G/P' \cap B \to G/P'$ and the fibers $P'/P' \cap B$ are partial flag varieties). Since $q|\alpha$ is surjective and flat, $p|q(\alpha)$ also has reduced and connected fibers. However the fibers are also finite since p is a normalization.

It seems that

(1) The scheme theoretic inverse of the K-stratum G/P'_K in the resolution is the K-stratum of the resolution: $G/P'_K \cap B$. (Above $y_K \in \mathcal{Y}_K$ one has a point (B, y_K) in the K-stratum of the resolution, and the map of the K-strata is $G/P'_K \cap B \to G/P'_K$).
(2) The scheme-theoretic fiber at the vertex 0 of the cone $\overline{G/P'_J}$ is the zero section of the resolutions as a vector bundle over G/P_J,

This would imply the conjecture: since q has connected reduced fibers and p is proper and surjective, then p also has connected reduced fibers. Since p is finite (affine proper and surjective), it is an isomorphism.

One can also argue that p is an isomorphism on strata and that the fibers can be calculated on the strata.

24.4. **Resolution of** $(G/P')^{\mathrm{aff}}$.

24.4.1. *Corollary.* Let $\mathcal{P}_J$ be the partial flag variety that contains $P = P_J$.

(a) A resolution of $\overline{G/P'}$ is given by the correspondence

$$\widetilde{\overline{G/P'}} \overset{\mathrm{def}}{=} \{(v,P) \in \overline{G/P'}\times\mathcal{P}_J,\ P' \text{ fixes } v\}$$

between $\overline{G/P'}$ and $\mathcal{P}_J$.

(b) Above a G-orbit $\mathcal{Y}_K \cong G/P_K{}'$, the fibers are isomorphic to the partial flag variety $\mathcal{P}(L_J)_J$ of all parabolics of type J in the Levi factor L_K of P_K.

(c) Over $\mathcal{P}_J$, the resolution is the sum $\oplus_{i\notin J}\ \mathcal{O}_{-\omega_i}$, of all duals of fundamental line bundles, that make sense on $\mathcal{P}_J$.

Proof. (a) is clear form the definition of the resolution.

(b) The fiber of the resolution at v_K consists of all parabolics $P \in \mathcal{P}_J$ such that P' lies in the stabilizer $G_{v_K} = P_K{}'$, i.e., $P{\subseteq}P_K$. These are the same as the parabolics of type J in the Levi factor of P_K.

(c) The fiber of the resolution at $P_J \in \mathcal{P}_J$ is $\oplus_{i\notin I}\ \check{W}_i^N = \ \oplus_{i\notin I}\ \Bbbk_{-\omega_i}$. $\square$

24.4.2. *Remark.* These resolution are usually not minimal models, even for $(G/N)^{\mathrm{aff}}$.

(1) For $G = SL_2$, $\widetilde{\mathcal{Y}}{\to}\mathcal{Y}$ is the blow-up of $\mathcal{Y} = \mathbb{C}^2$.

(2) Also, for each partial flag variety $\mathcal{P}$ we have an analogous space $\widetilde{\mathcal{Y}}_{\mathcal{P}}{\subseteq}\ \mathcal{Y}{\times}\mathcal{P}$, and some of these spaces are smooth.

(3) For $G = SL_3$ and $\check{W} = L_{\omega_1} = \mathbb{A}^3$, $\mathcal{Y} \cong \{(v,\lambda) \in \check{W}{\times}\check{W}^*,\ \langle\lambda,v\rangle = 0\ \}$, has a small resolution $\{(v,\lambda,L) \in \check{W}{\times}\check{W}^*{\times}\mathbb{P}(\check{W}),\ \lambda \perp L \ni v\ \}$. This is rank three vector bundle over $\mathbb{P}(\check{W})$ and the fiber at $(\lambda,v) \in\ \mathcal{Y}$ is: $\mathbb{P}^2$ at $(v,\lambda) = (0,0)$ (codimension 5), $\mathbb{P}^1$ if just one of v and λ vanishes (codimension 3). (In general I do not expect the existence of semi-small resolutions.)

24.5. **Semigroup** $\overline{P^{\mathrm{ab}}}$**.** In honor of the normality conjecture 24.3.4 above, we denote here $\overline{G/P_J'}$ by $(G/P')^{\mathrm{aff}}$.

24.5.1. *Lemma.* Let $P =\ LU$ be $P_J =\ L_JU_J$.
(a) Let $\overline{P^{\mathrm{ab}}}$ be the closure of $P^{\mathrm{ab}} = P/P'$ in $[G/P']^{\mathrm{aff}}$, the identification $(\omega_i)_{i\in I}$: $P^{\mathrm{ab}} \overset{\cong}{\to} G_m^{I-J}$, extends to $\overline{P^{\mathrm{ab}}} \overset{\cong}{\to} G_c^{I-J}$.

(b) $\overline{P^{\mathrm{ab}}}$ is canonically a semigroup with an open subgroup P^{ab}. The action of P^{ab} on G/P' extends to an action of $\overline{P^{\mathrm{ab}}}$ on $[G/P']^{\mathrm{aff}}$.

(c) For any right P^{ab}-orbit $\mathcal{O}$ in G/P', the closure in $[G/P']^{\mathrm{aff}}$ is described by $\mathcal{O} \underset{P^{\mathrm{ab}}}{\times} \overline{P^{\mathrm{ab}}} \xrightarrow{\cong} \overline{\mathcal{O}}$.

Proof. In (a), $[G/P']^{\mathrm{aff}}$ consists of all $v = (v_i)_{i\in I} \in \check{W} = \oplus_{i\in I} \check{W}_i$, that satisfy Plücker equations and $v_i = 0,\ i \in J$. Here G/P' is given by: $v_i = 0$ iff $i \in J$.

Observe that $P^{\mathrm{ab}} = P/P'$ embeds into G/P' as the fixed point set for the left multiplication by P'. So in the Plücker model

$$P/P' = \{v \in \oplus \check{W}_i^{P'},\ v_i \neq 0 \text{ iff } i \in I - J \ \&\ \text{Plücker condition}\}.$$

This is $\prod_{i\in I-J} (\check{W}_i^N - 0)$, since (i) $(\check{W}_i)^{P'}$ is $\check{W}_i^N$ for $i \in I - J$ and 0 otherwise, and (ii) Plücker condition is automatic in $\oplus \check{W}_i^N$. Therefore the closure of P/P' is $\oplus_{i\in I-J} \check{W}_i^N$. In terms of functions this description becomes

$$\mathcal{O}(P^{\mathrm{ab}}) = \mathbb{C}[\oplus_{i\in I-J}\mathbb{Z}\omega_i] \supseteq \mathbb{C}[\oplus_{i\in I-J}\mathbb{Z}_+\omega_i] = \mathcal{O}(\overline{P^{\mathrm{ab}}}),$$

and puts a semigroup structure on $\overline{P^{\mathrm{ab}}}$. This semigroup acts on $G \underset{P}{\times} \overline{P^{\mathrm{ab}}}$, hence also on its affinization $[G/P']^{\mathrm{aff}}$.

Now (c) follows from (a) by left translations.

24.6. **Plücker model $\underline{G/N}$ of G/N.** We denote by $\underline{G/N}$ the G-orbit $\mathcal{Y}_\phi = G{\cdot}y$ in $\check{W}$. "Pluecker equations" means any set of generators of the ideal of the closure of $\underline{G/N} = G{\cdot}y$ in $\check{W}$. We will recall the standard choice of Pluecker equations in characteristic zero. In general Pluecker equations seem only known in characteristic zero.

24.6.1. *I-data.* We say that a system of vectors $v_i \in L(\omega_i),\ i \in I$, satisfies Pluecker equations if for any multiplicities $\lambda_i \geq 0$, the projection of $\otimes_{i\in I} \check{W}(\omega_i)^{\otimes \lambda_i}$ to the unique G-invariant complement of $\check{W}(\sum_{i\in I} \lambda_i\omega_i)$, kills $\otimes_{i\in I} v_i^{\otimes \lambda_i}$.

24.6.2. *$X^*(T)_+$-data.* Recall that for dominant λ_i's, the space

$$\mathrm{Hom}[\check{W}(\lambda_1)\otimes\cdots\otimes\check{W}(\lambda_s), \check{W}(\lambda_1 + \cdots + \lambda_s)]$$

is one dimensional. Let us supply each $\check{W}(\lambda)$ with a frame y_λ of $\check{W}(\lambda)_\lambda$. Then the above space of homomorphisms has a canonical frame $m_{\lambda_1,\ldots,\lambda_s}$ characterized by $y_{\lambda_1}\otimes\cdots\otimes y_{\lambda_s} \mapsto y_{\lambda_1+\cdots+\lambda_s}$.

Now, equivalently, we say that a system of vectors $v_\lambda \in \check{W}(\lambda),\ \lambda$ dominant, satisfies Pluecker equations if for any dominant λ and μ,

- (i) $m_{\lambda,\mu}(y_\lambda\otimes y_\mu) = y_{\lambda+\mu}$,
- (ii) the projection of $\check{W}(\lambda)\otimes\check{W}(\mu)$ to the unique G-complement of $\check{W}(\lambda+\mu)$, kills $v_\lambda\otimes v_\mu$.

24.6.3. *Identification data.* A particular identification of G/N with this Pluecker model, requires fixing in each fundamental representation $\check{W}_i$ of G a frame y_i of $\check{W}_i^N$. Then $G/N \overset{\iota}{\hookrightarrow} \prod \check{W}_i$ by $gN \mapsto (gy_i)_{i\in I}$.

24.6.4. *The open cell and its boundary divisors.* The complement of the open cell: $B_- \xrightarrow{\cong} B_-\cdot N/N \subseteq G/N$, is a divisor in G/N. Its irreducible components $Y_i, i \in I$, are given by the conditions $v_i \perp (\check{W}_i^*)^N$.

25. **Affinization of** G/U

25.1. $(G/V)^{\mathrm{aff}}$**.** We consider a normal subgroups V of a parabolic $P = U \ltimes L$. We assume that $M = M$ is reductive, i.e., that V contains U. Then

25.1.1. *Conjecture.* G/V is quasiaffine.

We will usually only cover the cases $V = P'$ (24) and $V = U$. Then the above conjecture is in Corollary 25.3.2.

25.2. **Stratifications of** $(G/N)^{\mathrm{aff}}$ **and** $(G/N)^{\mathrm{aff}} \times (G/N)^{\mathrm{aff}}$**.** Let $G \times H^2$ act on $G/N \times G/N$ by $(g, b'N.b''N)\cdot(xN, yN) = (gxb'N, gyb''N)$.

25.2.1. *Lemma.* (a) G-orbits in $(G/N)^{\mathrm{aff}}$ are indexed by subsets $J \subseteq I$, $J \mapsto \mathcal{Y}_J$.

(a) B-orbits in $(G/N)^{\mathrm{aff}}$ are indexed by pairs (J, w) of a subset $J \subseteq I$ an a coset $w \in W/W_J$, $(J, w) \mapsto \mathcal{Y}_J^w$.

(c) The orbits of $G \times H^2$ in $(G/N)^{\mathrm{aff}} \times (G/N)^{\mathrm{aff}}$ are indexed by triples (J, K, w) of subsets $J, K \subseteq I$ and a coset $w \in W_K \backslash W/W_J$, $(J, K, w) \mapsto \mathcal{Y}_{J,K}^w$.

Proof. (a) is in the lemma 24.3.1. The orbit associated to $J \subseteq I$ is isomorphic to G/P'_J, hence the B-orbits in it are the same as for $B\backslash G/P_J$, and therefore indexed by $W/W_J \ni \mapsto B\cdot wy_J$.

(c) A $G \times G$-orbit in $(G/N)^{\mathrm{aff}} \times (G/N)^{\mathrm{aff}}$ is by (a) associated to two subsets $J, K \subseteq I$ and of the form $G/P'_J \times G/P'_K$. The orbits of $G \times H^2$ in $G/P'_J \times G/P'_K$. are the same as the orbits of G in $G/P_J^{\mathrm{aff}} \times G/P_K$, but $G\backslash[G/P_J^{\mathrm{aff}} \times G/P_K] \xrightarrow{\cong} P_J{}^{-1}\backslash G/P_K \cong W_J\backslash W/W_K$. Let $w \in W$ define a coset $\bar{w} \in W_J\backslash W/W_K$, the corresponding $G \times H^2$-orbit passes through (y_J, wy_K).

25.2.2. *Question.* Describe the closure relations in the stratifications above,

25.2.3. *Question.* (KL-exercise) Find the IC-stalks for these three stratifications.

25.2.4. *Remarks.* (1) Case (a) should be doable from the above resolutions of G-orbits in $(G/N)^{\mathrm{aff}}$.

(2) Cases (b) and (c) should be combinations of the standard KL-theory and (a).

(3) Hopefully, settings (b) and (c) give more symmetries (of the Fourier type), then the standard KL-theory. For instance the Kazhdan-Laumon extension of intertwining functors to $(G/N)^{\mathrm{aff}}$.

25.3. G-orbits in $(G/N)^{\mathrm{aff}}\times(G/N)^{\mathrm{aff}}$.

25.3.1. *Stratification of* $(G/N)^{\mathrm{aff}}\times(G/N)^{\mathrm{aff}}$ *according to the* G*-action.* The stratification of $(G/N)^{\mathrm{aff}}\times(G/N)^{\mathrm{aff}}$ given by the action of G is the stratification by the orbits of $G\times H^2$, so any G-orbit is of one of the types

$$Y_{J,K}^{\bar{w}} \stackrel{\mathrm{def}}{=} G\cdot(y_J, wy_K) \ \cong\ G/G_{(y_J,wy_K)} \ =\ G/P'_J \cap^w P'_K.$$

Conversely, for any two parabolic subgroups P,Q of G, $G/P'\cap Q'$ is isomorphic to one of $Y_{J,K}^w$.

For fixed J and K, the union of all $G\times H^2$-orbits $\mathcal{Y}_{JK}^w$ is a G^2-orbit $\mathcal{Y}_J\times\mathcal{Y}_K \cong G\times G/P'_J\times P'_K$. It is a $P_J^{\mathrm{ab}}\times P_K^{\mathrm{ab}}$-torsor over a partial flag variety $G/P_J\times G/P_K$. So, the mutual position of all $G\times H^2$-orbits $\mathcal{Y}_{JK}^{\bar{w}},\ \bar{w}\in W_J\backslash W/W_K$.

25.3.2. *Corollary.* (a) For any two parabolic subgroups P and Q, $G/P'\cap Q'$ is quasiaffine. In particular:

(b) $G/N\cap\ {}^wN$ is quasiaffine for $w\in W$.

(c) [Grosshans] For any parabolic P, G/U is quasiaffine.

Proof. Clearly (b) is a case of (a), also (c) is a case of (b) with $w=w_0^L$. Finally, (a) is seen by embedding the orbits $\mathcal{Y}_{J,K}^w$ into $(G/N)^{\mathrm{aff}}\times(G/N)^{\mathrm{aff}}$.

25.3.3. *Question.* Describe the G-orbits in the closure of a single G orbit $Y_{JK}^w = G/P'_J\cap^w P'_Q$ in $(G/N)^{\mathrm{aff}}\times(G/N)^{\mathrm{aff}}$, and their closure relations.

25.3.4. *Remarks.* (1) In the special case when the orbit $G/P'\cap Q'$ has $P'\cap Q' =^w N\cap N = U$ for $w=w_0^L$, G-orbits in the closure are parameterized by the W_L-orbits in the set of T-roots in $\mathfrak{u}^{\mathrm{ab}}$.

So there may be a generalization of the action of W_L on $\Delta(\mathfrak{u}^{\mathrm{ab}})$ involving $\Delta_T([\mathfrak{n}\cap^w\mathfrak{n}]^{\mathrm{ab}})$.

(2) In some sense the largest G-orbit is $Y_{\phi\phi}^{w_0} = G\cdot(y, w_0\cdot y)\cong G/N\cap^{w_0} N = G$. However, this orbit is closed.

The corresponding $G{\times}H^2$-orbit is $\mathcal{Y}_{\phi\phi}^{w_0} = \Delta_G{\cdot}(T{\cdot}y, Tw_0{\cdot}y) = \Delta_G{\cdot}(T{\cdot}y, w_0{\cdot}y) = \Delta_G{\cdot}(y, w_0{\cdot}y){\cdot}H{\times}1$ (so it is an orbit of $G{\times}(H{\times}1)$ but not of the Vinberg group since the center acts from the left in a diagonal fashion, but not from the right).

25.4. **Semigroup $\overline{M}$.** If $M \overset{\text{def}}{=} M$ embeds into $(G/V)^{\text{aff}}$, say, when G/V is quasiaffine, we define $\overline{M} = \overline{M}$ as the closure of M in $(G/V)^{\text{aff}}$.

25.4.1. *Question.* When is $\overline{M}$ (a) normal, (b) smooth?

25.4.2. *Lemma.* $\overline{M}$ is a semigroup closure of M and $G{\times}\overline{M}$ acts on $(G/V)^{\text{aff}}$.

Proof. The claim is that the action $(G/V)^{\text{aff}}{\times}M \to (G/V)^{\text{aff}}$ extends to $(G/V)^{\text{aff}}{\times}\overline{M} \to (G/V)^{\text{aff}}$. Then in particular the multiplication on M extends to $\overline{M}$ so $\overline{M}$ is a semigroup.

We start with the G-action on $(G/V)^{\text{aff}}$: $G{\times}(G/V)^{\text{aff}} \to (G/V)^{\text{aff}}$, and we restrict it to

$$G{\times}\overline{M} \to (G/V)^{\text{aff}}.$$

Since the left multiplication action of V on $\overline{M}$ is trivial, this factors to

$$G/V{\times}\overline{M} \to (G/V)^{\text{aff}},$$

and now we just take affinizations

$$(G/V)^{\text{aff}}{\times}\overline{M} = [G/V{\times}\overline{M}]^{\text{aff}} \to (G/V)^{\text{aff}}.$$

(M is affine since V is normal in P.) To see that this is compatible with $(G/V)^{\text{aff}}{\times}M \to (G/V)^{\text{aff}}$ repeat the above procedure with M instead of $\overline{M}$.

25.5. **$G{\times}L$-orbits in $(G/U)^{\text{aff}}$.**

25.5.1. *Theorem.* (Conjecture.) $G{\times}L$-orbits on $(G/U)^{\text{aff}}$ and $L{\times}L$-orbits on $\overline{L}$ (a generalization of the rank stratification of matrices), are both parameterized by the W_L-orbits in $\Delta_T(\mathfrak{u}^{\text{ab}})$.

25.6. **Examples of affinizations $(G/U)^{\text{aff}}$.**

25.6.1. *Vector spaces.* Let $G = SL(A)$ and $A = E{\oplus}F$ with $\dim E = n$ and $\dim F = 1$. Let P and P_- be the stabilizers of E and F, they have Levi decompositions $P_\pm = U_\pm \ltimes L$ where $L = P \cap P_-$ is the stabilizer of $\{E, F\}$.

Let $\mathcal{Y}_\phi$ be the set of all $v \in A^n$ which are independent.

Lemma. (a) A choice of a basis $e=(e_1,...,e_n)$ of E gives $G/U\xrightarrow{\cong}\mathcal{Y}_\phi$ by $gU\mapsto ge$.

(b) This induces (G,L)-identifications with G acting on A and P on E,

$$\begin{array}{ccccccc} (G/U)^{\mathrm{aff}} & \xrightarrow{\cong} & \mathrm{Hom}_{\Bbbk}(E,A) & & (G/P')^{\mathrm{aff}} & \xrightarrow{\cong} & A \\ {\scriptstyle\subseteq}\uparrow & & {\scriptstyle\subseteq}\uparrow & & {\scriptstyle\subseteq}\uparrow & & {\scriptstyle\subseteq}\uparrow \\ G/U & \xrightarrow{\cong} & Inj_{\Bbbk}(E,A) & & G/P' & \xrightarrow{\cong} & A-\{0\} \end{array}$$

25.6.2. *Corollary.* (a) $P_-\subseteq G/U$ is now described as all $\binom{\alpha}{a}\in\mathrm{Hom}(E,A)=\mathrm{End}(E)\oplus\mathrm{Hom}(E,F)$, such that α is invertible.

(b) $L\cong GL(E)$ and $\overline{L}\cong\mathrm{End}(E)$ is a matrix semigroup.

25.6.3. *Maximal parabolics in type A.* Let $G=SL(A)$ and $A=E\oplus F$ with $\dim E=e$ and $\dim F=f$. The stabilizers P and P_- of E and F have Levi decompositions $P_\pm=U_\pm\ltimes L$ with a common Levi subgroup $L\overset{\mathrm{def}}{=}P\cap P_-$ which is the stabilizer of $\{E,F\}$.

Let $\mathcal{A}_\phi$ be the set of all pairs $(v,u)\in A^e\times(A^*)^f$ such that $v_i\perp u_j$ and that v_i's and u_j's are independent.

These data are the same as a basis of a subspace $W\subseteq A$ of dimension $e=\dim(E)$ plus a basis of A/W, so $GL(A)$ acts transitively on such pairs. If v and u are bases of E and $E^\perp$, then the stabilizer $GL(A)_{u,v}$ is the unipotent radical U of the parabolic $P=G_E$ (for $G=GL(A)$ or $G=SL(A)$).

Therefore,

$$[GL(A)/U]^{\mathrm{aff}}=\ A^e\oplus(A^*)^f=\ \mathrm{Hom}(E,A)\oplus\mathrm{Hom}(A,F).$$

($GL(A)/U$ is open dense in the RHS which is affine and normal.) Also, one can describe $[SL(A)/U]^{\mathrm{aff}}$ by fixing a frame ϕ in the line $\mathrm{Hom}(\overset{top}{\wedge}F,\overset{top}{\wedge}E)$, then $[SL(A)/U]^{\mathrm{aff}}$ is the closure in $A^e\oplus(A^*)^f$ of all $(v,u)\in A^e\oplus(A^*)^f$ such that $v_1\wedge\cdots\wedge v_e\otimes u_1\wedge\cdots\wedge u_f=\ \phi$.

25.7. **Functions on $\overline{M}$: a mess.** The difficulties appear in positive characteristic. Here, the asymptotic cone is much simpler to understand then the group itself.

25.7.1. *Lemma.* (a) $\mathcal{O}(\overline{M})=\ Im[\mathcal{O}(G/V)\to\mathcal{O}(M)$.

(b) $Gr[\mathcal{O}(M)=\oplus_{M\text{-dominant }\mu}\ L^M(-\mu)\otimes L^M(\mu)$ contains

$$Gr[\mathcal{O}(\overline{M})]=\ \oplus_{G\text{-dominant }\lambda}\ L^M(-\lambda)\otimes L^M(\lambda).$$

Proof. (a) follows from the conjecture 25.1.1. (b) follows from (a).

Proof. Denote $\lambda^*\overset{\mathrm{def}}{=}-w_0\lambda$ and by $W(\mu)$ denote the coWeyl module with an extremal weight λ.

Recall that $\mathcal{O}(G)$ has an increasing filtration $F_\lambda,\ \lambda\in X_*(T)_+$; such that

$$Gr[\mathcal{O}(G)] = \ \oplus_{G\text{-dominant }\lambda}\ \check{W}(\lambda^*)\otimes\check{W}(\lambda).$$

Since the modules $\check{W}(\lambda)^V = L($???

25.7.2. *Two-sided quotients of* G.

25.7.3. *Lemma.* $\overline{L}\xrightarrow{\cong} U_-\backslash\backslash G//U$. and $\overline{L^{\mathrm{ab}}}\xrightarrow{\cong} P'_-\backslash\backslash G//P'$.

26. Partial affine closures of G/P' (A dull and aching pain)

A dull and empty section on nothing (instead of nothingness). The idea was to list systematically all partial affinizations of G/P'. *Should be skipped.*

26.1. **Notation.** We will denote by H' the derived subgroup of H and $H^{\mathrm{ab}} \overset{\text{def}}{=} H/H'$.

26.1.1. *Lemma.* (a) Let $\mathcal{P}$ be a partial flag variety, let $P\in\mathcal{P}$ be a parabolic subgroup with a Levi decomposition $U{\cdot}L$ and denote $\bar{P}\overset{\text{def}}{=} P/U$.

(i) L' is connected and semi-simple and $P'=L'{\cdot}U$.

(ii) $L=L'{\cdot}Z(L)_0$ and $P=P'{\cdot}Z(L)_0$.

(iii) Groups $Z_{\mathcal{P}}\overset{\text{def}}{=} Z(L){\cdot}U/U\xrightarrow{\cong}Z(\bar{P})$ and $H_{\mathcal{P}}\overset{\text{def}}{=} P^{ab}\xrightarrow{\cong}\bar{P}^{ab}$ are canonically independent of the choice of $P\in\mathcal{P}$. The canonical map $Z_{\mathcal{P}}\to H_{\mathcal{P}}$ is a finite cover and it factors into $Z_{\mathcal{P}}\hookrightarrow H\twoheadrightarrow H_{\mathcal{P}}$¿ In particular $Z_{\mathcal{B}}=H=H_{\mathcal{B}}$.

(b) Let $Q\subseteq P$ be another parabolic subgroup, then

(i) $L'\cap Q$ is a Levi subgroup in L', and

(ii) $(L'\cap Q)'=L'\cap Q'$.

26.2. **Partial affine closures of** G/N**.** Instead of G/N one can consider its affine closure $\overline{G/N}$ and then $\bar{\mathcal{S}}\overset{\text{def}}{=} G\underset{B}{\times}\overline{G/N}\to\mathcal{B}$ is the affine closure of the $\mathcal{B}$-variety $\mathcal{S}$.

More generally, for a pair of parabolic subgroups $P\supseteq Q$, consider the G-bundle $G/Q'\to G/P'$ with the fiber $P'/Q'\cong L'/L'\cap Q'=L'/(L'\cap Q)'$. Denote its relative affine closure by $\mathcal{Y}_G(Q\subseteq P)\overset{\text{def}}{=}\ (G/Q'\to G/P')^{\mathrm{aff}}$. This is a G-bundle over G/P' with the fiber $[L'/(L'\cap Q)']^{\mathrm{aff}} =\ \mathcal{Y}_{L'}(L'\cap Q\subseteq L')$.

Actually, in general, $\mathcal{Y}_G(P\subseteq P)\ =\ \ (G/P'\to G/P')^{\mathrm{aff}}\ =\ \ G/P'$ lies in $\mathcal{Y}_G(P\subseteq G)\ =\ (G/P'\to G/G')^{\mathrm{aff}}=(G/P')^{\mathrm{aff}}$. This is a special case of the following *functoriality*

(1) $Q_1\subseteq Q_2\subseteq P$ gives a map $\mathcal{Y}_G(Q_1\subseteq P)\to\mathcal{Y}_G(Q_2\subseteq P)$,

(2) $Q\subseteq P_1\subseteq P_2$ gives a map $\mathcal{Y}_G(Q\subseteq P_1)\to\mathcal{Y}_G(Q\subseteq P_2)$.

Here, if $Q_1 \subseteq Q_2 \subseteq P$ there is a G/P-map $G/Q'_1 \twoheadrightarrow G/Q'_2$, and it induces a map $(G/Q'_1 \twoheadrightarrow G/P')^{\mathrm{aff}} \to (G/Q'_2 \twoheadrightarrow G/P')^{\mathrm{aff}}$. Also, if $Q \subseteq P_1 \subseteq P_2$, there is a G/P'_2-map $G/Q' \twoheadrightarrow G/P'_1$, and it induces $(G/Q' \twoheadrightarrow G/P'_1)^{\mathrm{aff}} \to (G/Q' \twoheadrightarrow G/P'_2)^{\mathrm{aff}}$.

In the basic case $Q = B \in \mathcal{B}$, one has $\mathcal{Y}_G(B \subseteq B) = G/N \subseteq (G/N)^{\mathrm{aff}} = \mathcal{Y}_G(B \subseteq G)$. The G-bundle $G/N = G/B' \to G/P'$ has fibers $P'/N = L' \cdot U/N_{L'} \cdot U \cong L'/N_{L'}$ for $N_{L'} \stackrel{\mathrm{def}}{=} N \cap L'$. So the fibers of the relative affine closure $(G/N \to G/P')^{\mathrm{aff}} \to G/P'$ are isomorphic to $(L'/N_{L'})^{\mathrm{aff}}$.

26.3. **Partial closures of $\mathcal{S}$.** If $Q \subseteq P$ then Q acts by conjugation on $G/Q' \twoheadrightarrow G/P'$, so there is a G-bundle over G/Q,

$$\mathcal{S}_G(Q \subseteq P) \stackrel{\mathrm{def}}{=} G \underset{Q}{\times} (G/Q' \twoheadrightarrow G/P')^{\mathrm{aff}} = G \underset{Q}{\times} \mathcal{Y}_G(Q \subseteq P).$$

The relations of $\mathcal{S}$'s is the same as for $\mathcal{Y}$'s. In particular $\mathcal{S}(P, G, G) = G \underset{P}{\times} (G/P' \twoheadrightarrow G/G')^{\mathrm{aff}} = G \underset{P}{\times} (G/P')^{\mathrm{aff}}$ is the affine closure of the G/P-variety $\mathcal{S}_G(P \subseteq P) = G \underset{P}{\times} (G/P' \twoheadrightarrow G/P')^{\mathrm{aff}} = G \underset{P}{\times} G/P'$.

Over $G/B = \mathcal{B}$ we have $\mathcal{S} \stackrel{\mathrm{def}}{=} \mathcal{S}_G(B \subseteq B) \subseteq \overline{\mathcal{S}} \stackrel{\mathrm{def}}{=} \mathcal{S}_G(B \subseteq G)$.

26.4. **Global sections.** We are interested in the spaces $S_G(Q \subseteq P) \stackrel{\mathrm{def}}{=} \Gamma[Q/Q, \mathcal{S}(Q \subseteq P)]$ and in the inclusion $S_G(P \subseteq P) = \Gamma[G/P, G \underset{P}{\times} G/P'] \subseteq S_G(P \subseteq G) = \Gamma[G/P, G \underset{P}{\times} (G/P')^{\mathrm{aff}}]$.

26.4.1. *Symmetries.* For a parabolic subgroup $P \in \mathcal{P}$, group $G \times P^{\mathrm{ab}} = G \times H_{\mathcal{P}}$ acts on $G/P' \to G/P \stackrel{\mathrm{def}}{=} \mathcal{P}$. For a pair $Q \subseteq P$, via $H \twoheadrightarrow Q^{\mathrm{ab}} \twoheadrightarrow P^{\mathrm{ab}}$, we get an action of $G \times H$ on $G/Q' \to G/P'$, hence also on $\mathcal{S} \subseteq \mathcal{S}(Q, P, G) \subseteq \overline{\mathcal{S}}$ and the corresponding sections.

27. Appendix, Affinization

27.1. **Affinization functor.** Affinization of a scheme $X \to B$ with a base B is

$$X^{\mathrm{aff}} = (X \to B)^{\mathrm{aff}} \stackrel{\mathrm{def}}{=} \mathit{Spec}\ \pi_* \mathcal{O}_X.$$

Lemma. (a) Affinization with a base is a functor.

(b) Affinization is the right adjoint of the inclusion of affine B-varieties into all B-varieties. This includes a canonical map $(X/B) \to (X/B)^{\mathrm{aff}}$.

Proof. (a) For $X \xrightarrow{f} Y \xrightarrow{b} B = X \xrightarrow{a} B$ one has a map $a_* \mathcal{O}_X = b_* f_* \mathcal{O}_X \leftarrow b_* \mathcal{O}_Y$, i.e., simply the pull back of functions $f_* \mathcal{O}_X \leftarrow \mathcal{O}_Y$.

(b) For any B-variety $X \to B$ there is a canonical B-map $X \to X^{\mathrm{aff}}$, such that any B-map $X \to Y$ with Y a B-affine variety factors through $X \to X^{\mathrm{aff}}$: $\mathrm{Hom}[X, Y] = \mathrm{Hom}(X^{\mathrm{aff}}, Y)$.

(c) [Dependence on the base.] Composable maps $X \xrightarrow{f} B' \to B$ give a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & (X/B')^{\mathrm{aff}} \arrow[r, "\alpha"] \arrow[d] & (X/B)^{\mathrm{aff}} \arrow[d, "\beta"] \\ B' \arrow[r, "="'] & B' \arrow[r] & (B'/B)^{\mathrm{aff}}. \end{tikzcd}$$

(d) [Functoriality in pairs.] In general, any map of pairs $\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ B' \arrow[r, "g"] & B, \end{tikzcd}$ gives a map of afinizations of pairs which is a combination of the maps of the types α (changes base) and β (changes the scheme itself) above:

$$(X'/B')^{\mathrm{aff}} \to (X/B)^{\mathrm{aff}} \stackrel{\mathrm{def}}{=} [(X'/B')^{\mathrm{aff}} \xrightarrow{\alpha} (X'/B)^{\mathrm{aff}} \xrightarrow{\beta} (X/B)^{\mathrm{aff}}].$$

$\square$

27.1.1. *Quasiaffine spaces, resolutions and normality.* We say that X/B is *quasiaffine* if it is open in some affine map Y/B.

Recall that the normalization of an affine variety is affine ([Ha] Exc. 3.17), and if X is of finite type over a field then the normalization is a finite map ([Ha] Exc. 3.8).

Lemma. (a) If X is quasiaffine(?), $X \to X^{\mathrm{aff}}$ is an open dense embedding.

(b) For a normal quasiaffine X, the affinization X^{aff} is again normal.

(c) X^{aff} as a normalization.] If a normal variety X is open and dense in an affine variety Y and the *boundary is in codimension* 2, then X^{aff} is the normalization of Y.

(d) Let $\widetilde{Y} \xrightarrow{\pi} Y$ be a "resolution" of an affine variety Y (in the sense that it is proper and generically an isomorphism), If the fibers are connected then π is a Y-affinization.

Proof. (a) Let X be open in an affine scheme Y. Then $X \subseteq Y$ factors through $X \to X^{\mathrm{aff}}$, hence $X \to X^{\mathrm{aff}}$ is also an embedding. Since the closure $\overline{X}$ of X in X^{aff} is affine, it equals X^{aff}. (b) Since the normalization $\widetilde{X^{\mathrm{aff}}} \to X^{\mathrm{aff}}$ is affine and an isomorphism over X, it is an isomorphism.

(c) X is also open and dense in the normalization $\tilde{Y}$ of Y. Since $\tilde{Y} \to Y$ is finite, the boundary of X in $\tilde{Y}$ is again in codimension 2. Together with the normality of $\tilde{Y}$ it implies that the functions on X extend uniquely to $\tilde{Y}$. So $\tilde{Y}$ is affine and $\mathcal{O}(X) = \mathcal{O}(\tilde{Y})$.

(d) Under these conditions $\pi_* \mathcal{O}_{\widetilde{Y}} \xleftarrow[\cong]{} \mathcal{O}(Y)$.

27.1.2. *Examples of affinization of varieties.*

(1) For $G = SL(U)$ and P the maximal parabolic such that P' is the stabilizer of $0 \neq e \in U$, $(G/P')^{\mathrm{aff}} = U$ is smooth.
(2) For $G = SL(U) \cong SL_3$, $(G/N)^{\mathrm{aff}} = \{(v, u^*) \in U \oplus U^*,\ v \perp u^*\}$.
(3) [Nilpotent orbits.] The affinization of $\mathcal{O} \in G/\mathcal{N}$ is the normalization of $\overline{\mathcal{O}}$.
(4) $\widetilde{\mathfrak{g}}^{\mathrm{aff}} = = \mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{h}$

Proof. (2) One embeds G/N into $U \oplus U^*$ as the orbit of the B-highest vectors (v_0, u_0^*). The only singularity is the vertex (the differential $d_{v,\lambda}\langle -, - \rangle = \langle v, - \rangle + \langle -, \lambda \rangle$ vanishes only for $v = \lambda = 0$).

(3) is by the lemma 27.1.1.c.

(4) Use lemma 27.1.1.d. □

Remark. An example of "resolutions" that are not affinizations are normalizations. (Normalizations are affine so they can be affinizations.)

Question. If Y is normal, is any resolution an affinization?

27.1.3. *Affinization of the map $G/N \to G/P'$ for $G = SL_3$.* For a maximal parabolic subgroup $P = P_\beta \supseteq B$ and $U = \mathbb{C}^3$ one has

$$\begin{array}{ccccc}
G/N & \longrightarrow & (G/N \to G/P')^{\mathrm{aff}} & \longrightarrow & (G/N)^{\mathrm{aff}} = \{(v, u) \in U \oplus U^*,\ v \perp u\} \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle \phi}\big\downarrow & & {\scriptstyle pr_1}\big\downarrow \\
G/P' = U - \{0\} & \xrightarrow[=]{} & G/P' & \xrightarrow{\subseteq} & (G/P')^{\mathrm{aff}} = U.
\end{array}$$

The fiber of f is $P'/N \cong \mathbb{A}^2 - \{0\}$, so ϕ is an $\mathbb{A}^2$-bundle. The fiber of pr_1 at the single boundary point 0 in $(G/P')^{\mathrm{aff}} - G/P'$ jumps to $\mathbb{A}^3$.

27.2. **Resolutions from correspondence extensions of maps to affinizations.** We are looking for a resolution of X^{aff} for smooth X, that maps into a convenient complete variety Y. If X is open in some $\mathfrak{X}$, any B-map $f : X \to Y$ extends to a correspondence $Y \leftarrow F \to \mathfrak{X}$, with F the closure of the graph $\Gamma \subseteq X \times Y$ in $\mathfrak{X} \times Y$.

If Y is complete then $F \to \mathfrak{X}$ is (a) proper (as a composition $F \subseteq \mathfrak{X} \times Y \to \mathfrak{X}$), (b) generically an isomorphism (over X it is $\Gamma \xrightarrow{\cong} X$) and (c) surjective (by (a) and (b)). So if Y is complete and F is smooth then F is a resolution of $\mathfrak{X}$.

Finally, we consider a quasiaffine X open in $\mathfrak{X} = X^{\mathrm{aff}}$ and a map f from X to a proper Y. The affinization $F^{\mathrm{aff}} \to X^{\mathrm{aff}}$ is a finite map – it is proper and the fiber at $y \in X^{\mathrm{aff}}$ embeds into $\pi_0([F \to X^{\mathrm{aff}}]^{-1} y)$. So if X^{aff} is normal, the affinizations of X and F coincide.

27.2.1. *Examples.* (1) Let V be a vector space and $X = \ V - \{0\} \xrightarrow{f} \mathbb{P}(V) = \ Y$. The correspondence $X^{\mathrm{aff}} \leftarrow F \to Y$ is the blow up $V \leftarrow \widetilde{V} \to \mathbb{P}(V)$.

(2) P^{ab}-torsor $X = G/P' \to Y = G/P$ gives correspondence $(G/P')^{\mathrm{aff}} \leftarrow F \to G/P$. Here, F is a resolution of $(G/P')^{\mathrm{aff}}$ and it is a $\overline{P^{\mathrm{ab}}}$-torsor over G/P (see 24.3.1).

(These examples coincide for $G = SL_2$ in (2) and $V = \mathbb{C}^2$ in (1).)

References

[Ha] Hartshorne.
[Ra] Sam Raskin.