LOOP GRASSMANNIAN CONSTRUCTION OF $\dot{U}(\check{\mathfrak{n}})$

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Not for distribution

This is a sketch of a Loop Grassmannian construction of $U(\check{\mathfrak{n}})$. An element x of $U\check{\mathfrak{g}}(\eta)$ gives a map between weight functors $F_\phi \to F_{\phi+\eta}$. For x in $U\check{\mathfrak{n}}$ this can be realized as a map between objects that represent weight functors - the constant sheaves on semi-infinite strata.

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1. Grassmannians over a curve

We will extend the semi-infinite stratification of a Grassmannian $\mathcal{G}$ to the "global" Grassmannians $\mathcal{G}_{X^n}$.

1.1. Semi-infinite subschemas of the BD-Grassmannian. Data $\alpha_1, ..., \alpha_n \in X_*(T)$ define an ind-subscheme $S_{\alpha_1,...,\alpha_n} \subseteq \mathcal{G}_{X^n}$. In order to reduce this to the ordinary Grassmannian one first defines $\overline{S_{\alpha_1,...,\alpha_n}} \subseteq \mathcal{G}_{X^n}$ - over a point with distinct coordinates $(x_1, ...x_n) \in X^n$ the fiber is the product of closures of semi-infinite orbits $\overline{S_{\alpha_i}} \subseteq \mathcal{G}_{x_i}$, and then $\overline{S_{\alpha_1,...,\alpha_n}}$ is the closure of its restriction to the regular part of X^n. Then $S_{\alpha_1,...,\alpha_n}$ is obtained by removing from $\overline{S_{\alpha_1,...,\alpha_n}}$ all $\overline{S_{\beta_1,...,\beta_n}}$ with $\beta_i \leq \alpha_i$ and $(\beta_1, ..., \beta_n) \neq (\alpha_1, ..., \alpha_n)$. The fiber of $S_{\alpha_1,...,\alpha_n}$ at $(x_1, ..., x_n)$ is

$$S_{\alpha_1,...,\alpha_n})_{(x_1,...,x_n)} = \prod_{y \in \{x_1,...,x_n\}} S_{\sum_{x_i=y} \alpha_i, y}.$$

Clearly, various $S_{\alpha_1,...,\alpha_n}$ are not disjoint off the generic part of X^n. To refine this to a stratification of $\mathcal{G}_{X^n}$ one should also take into account the stratification of X^n.

One can also get a stratification which is more crude. A less refined extension of the semi-infinite stratification to $\mathcal{G}_{X^n}$ is the partition $\mathcal{G}_{X^n} = \sqcup_\nu S_{\nu,X^n}$ where $S_{\nu,X^n} \overset{\text{def}}{=} \cup_{\sum \alpha_i = \nu} S_{\alpha_1,...,\alpha_n}$.

If $X = \mathbb{A}^1$ the G_m-action contracts X^n to the point $\mathbf{0} = (0, ..., 0)$ and $S_{\alpha_1,...,\alpha_n}$ to the intersection $S_{\alpha_1,...,\alpha_n} \cap \mathcal{G}_\mathbf{0}$ which is the same as $S_{\sum \alpha_i}$ under the identification $\mathcal{G}_\mathbf{0} \cong \mathcal{G}$.

1.2. Pro-objects $\Bbbk_S$ and ind-objects ω_S. Consider the triangulated category $D(\mathcal{G}, \Bbbk) \overset{\text{def}}{=} \lim\limits_{\substack{\rightarrow \\ C}} D^b_c(C, \Bbbk)$, the limit is over compact subschemas C of our ind-scheme $\mathcal{G}$.

For $C \overset{i}{\hookrightarrow} D$, $i_! = i_* : D^b_c(C) \to D^b_c(D)$ is an inclusion that commutes with Verdier duality, hence the category $D(\mathcal{G}, \Bbbk)$ is a union of all $D^b_c(C)$, and it has Verdier duality.

For an ind-subscheme $S \subseteq \mathcal{G}$ one can define $\Bbbk_S$ as a pro-object in $D(\mathcal{G})$: $C \subseteq D$ gives $C \cap S \overset{\rho}{\hookrightarrow} D \cap S$, hence $\Bbbk_{D \cap S} \to \rho_* \rho^* \Bbbk_{D \cap S} = \Bbbk_{C \cap S}$. This pro-object represents the functor $H^*_S(\mathcal{G}, -) : D(\mathcal{G}) \to D(pt)$, defined by

$$\mathcal{A} \mapsto \lim_{\rightarrow} \operatorname{Hom}(\Bbbk_{C \cap S}, \mathcal{A}) = \lim_{\rightarrow} \operatorname{Hom}[(C \hookrightarrow \mathcal{G})_! \Bbbk_{C \cap S}, \mathcal{A}) = \lim_{\rightarrow} H^*_{C \cap S}(\mathcal{G}, \mathcal{A}).$$

For each $\mathcal{A} \in D(\mathcal{G})$ this limit stabilizes.

For any two ind-subschemas X and Y we will denote

$$\mathrm{Ext}^i(\Bbbk_X, \Bbbk_Y) \overset{\mathrm{def}}{=} \mathrm{Hom}(\Bbbk_X, \Bbbk_Y[i]) \overset{\mathrm{def}}{=} \lim_{\substack{\leftarrow \\ D}} \mathrm{Hom}(\Bbbk_X, \Bbbk_Y \cap D[i])$$

$$\overset{\mathrm{def}}{=} \lim_{\substack{\leftarrow \\ D}} \lim_{\substack{\rightarrow \\ C}} \mathrm{Hom}(\Bbbk_X[C], \Bbbk_Y \cap D[i]).$$

However we only consider the cases when everything stabilizes.

1.2.1. Dually, one can define the dualizing sheaf ω_S as an ind-object $\omega_S \overset{\mathrm{def}}{=} \mathbb{D}(\Bbbk_S) = \lim\limits_{\rightarrow} \mathbb{D}(\Bbbk_{C\cap S}) = \lim\limits_{\rightarrow} \omega_{C\cap S}$, and also a functor $H_c^*(S,-) : D(\mathcal{G}) \to D(pt)$, by $H_c^*(S,-) = {}^{\mathbb{D}}H_S^*(\mathcal{G},-)$, i.e.,

$$H_c^*(S,\mathcal{A}) = \mathbb{D}\ H_S^*(\mathcal{G},\mathbb{D}\mathcal{A}) = \mathbb{D} \lim_{\rightarrow} H_{C\cap S}^*(\mathcal{G},\mathbb{D}\mathcal{A}) = \lim_{\leftarrow} H_c^*(C\cap S,\mathcal{A}).$$

However, we do not (co)represent the functor $H_c^*(S,-)$.

1.3. **Coincidence of local and compactly supported cohomology.**

1.3.1. *Lemma. If $\mathcal{A}$ is a sheaf on $\mathcal{G}_{X^n}$ such that ..., then*

$$(S_{\alpha_1,\dots,\alpha_n} \to X^n)_!\ (S_{\alpha_1,\dots,\alpha_n} \hookrightarrow \mathcal{G}_{X^n})^*\ \mathcal{A}\ \cong\ (T_{\alpha_1,\dots,\alpha_n} \to X^n)_*\ (T_{\alpha_1,\dots,\alpha_n} \hookrightarrow \mathcal{G}_{X^n})^!\ \mathcal{A}.$$

When $X = G_a$ a sufficient condition on the sheaf $\mathcal{A}$ is that it be monodromic for $T\times\ Aut(X)$. In general, one should replace automorphism group with a more local object, the groupoid of local isomorphisms.

1.3.2. *Lemma. For $X = \mathbb{A}^1$, the restriction of sheaves from $\mathcal{G}_{X^n}$ to $\mathcal{G}_{\mathbf{0}}$ gives isomorphisms*

$$\mathrm{Ext}^*_{\mathcal{G}_{X^n}}(k_{S_{\alpha_1,\dots,\alpha_n}}, \mathcal{A}) \xrightarrow{\cong} \mathrm{Ext}^*_{\mathcal{G}}(k_{S_{\alpha_1+\dots+\alpha_n}}, \mathcal{A}|\mathcal{G}_{\mathbf{0}}), \qquad \textit{and}$$

$$\mathrm{Ext}^*_{\mathcal{G}_{X^n}}(k_{S_{\nu,X^n}}, \mathcal{A}) \xrightarrow{\cong} \mathrm{Ext}^*_{\mathcal{G}}(k_{S_\nu}, \mathcal{A}|\mathcal{G}_{\mathbf{0}}).$$

Proof. We will use

1.4. **Lemma.** superlemma to pass to the compactly supported cohomology

$$\mathrm{Ext}^*_{\mathcal{G}_{X^n}}(k_{S_{\alpha_1,\dots,\alpha_n}}, \mathcal{A}) \cong H^*_{S_{\alpha_1,\dots,\alpha_n}}(\mathcal{G}_{X^n}, \mathcal{A}) = H^*[X^n, (S_{\alpha_1,\dots,\alpha_n} \to X^n)_* (S_{\alpha_1,\dots,\alpha_n} \hookrightarrow \mathcal{G}_{X^n})^! \mathcal{A}]$$

as

$$\cong H^*[X^n, (T_{\alpha_1,\dots,\alpha_n} \to X^n)_! (T_{\alpha_1,\dots,\alpha_n} \hookrightarrow \mathcal{G}_{X^n})^* \mathcal{A}].$$

For $X = \mathbb{A}^1$, sheaves are G_m-monodromic. so the restriction from X^n to $\mathbf{0}$ identifies this cohomology with the stalk

$$[(T_{\alpha_1,\dots,\alpha_n} \to X^n)_! (T_{\alpha_1,\dots,\alpha_n} \hookrightarrow \mathcal{G}_{X^n})^* \mathcal{A}]_{\mathbf{0}}.$$

For $\alpha = \sum \alpha_i$, the intersection of $T_{\alpha_1,\dots,\alpha_n} \cap \mathcal{G}_{\mathbf{0}}$, i.e., the fiber of $T_{\alpha_1,\dots,\alpha_n}$ at $\mathbf{0}$, is T_α. So the stalk is the same as

$$H^*_c(T_\alpha, \mathcal{A}|\mathcal{G}_{\mathbf{0}}) \cong H^*_{S_\alpha}(\mathcal{G}, \mathcal{A}|\mathcal{G}_{\mathbf{0}}) \cong \mathrm{Ext}^*(k_{S_\alpha}, \mathcal{A}|\mathcal{G}_{\mathbf{0}}).$$

The proof for the second isomorphism is the same. $\square$

1.5. **Weight functors.** We fix opposite Borel subalgebras $\mathfrak{b} = \mathfrak{b}_+$ and $\mathfrak{b}_-$, and let $\mathfrak{h} \stackrel{\text{def}}{=} \mathfrak{b}_+ \cap \mathfrak{b}_-$. This defines semi-infinite orbits $S_\nu, T_\nu \subseteq \mathcal{G}$ as ind-subschemas. On $D(\mathcal{G})$ define the "weight functors"

$$F_\nu \stackrel{\text{def}}{=} H^*_c(S_\nu, -)[2\mathrm{ht}(\nu)] = H^*_c(\mathcal{G}, \Bbbk_{S_\nu}[2\mathrm{ht}(\nu)] \otimes -) \quad \text{and} \quad F^\nu \stackrel{\text{def}}{=} H^*_c(T_\nu, -)[-2\mathrm{ht}(\nu)].$$

On the subcategory of T_a-monodromic sheaves functors

$$F_\nu \cong H^*_{T_\nu}(\mathcal{G}, -)[2\mathrm{ht}(\nu)] \cong \mathrm{Ext}^*(\Bbbk_{T_\nu}[-2\mathrm{ht}(\nu)], -), \quad \text{and} \quad F^\nu \cong H^*_{S_\nu}(\mathcal{G}, -)[-2\mathrm{ht}(\nu)],$$

are represented by pro-objects $\Bbbk_{T_\nu}[-2\mathrm{ht}(\nu)]$ and $\Bbbk_{S_\nu}[2\mathrm{ht}(\nu)]$. We will regard functors F_ν as basic, while $w_0 : F^\nu \xrightarrow{\cong} F_{w_0 \cdot \nu}$.

1.6. **Convolution [MV].** We recall the convolution construction of Drinfeld and show that it is compatible with the weight functors.

Two perverse sheaves $\mathcal{A}, \mathcal{B} \in \mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$ define a perverse sheaf $\mathcal{A} \boxtimes \mathcal{B}$ on $\mathcal{G}_{X^2}$ which is $\mathcal{A} \boxtimes \mathcal{B}[2]$ off the diagonal and a $!*$-extension across the diagonal. The restriction to the diagonal is non-characteristic and the perverse restriction $i^0 \stackrel{\text{def}}{=} i^*[-1] \cong i^![1]$ gives the convolution $\mathcal{A} * \mathcal{B}$ spread over $\mathcal{G}_{X^2}|\Delta_X = \mathcal{G}_X$ as a perverse sheaf (so $\mathcal{A} \boxtimes \mathcal{B} \, |\mathcal{G}_{(a,a)} \cong \mathcal{A} * \mathcal{B}$ [2]).

One knows that the direct image to X^2 is a constant sheaf. We will state this more precisely.

1.6.1. *Lemma.* (a) $(S_{\nu,X^2}\to X^2)_!(S_{\nu,X^2}\hookrightarrow\mathcal{G}_{X^2})^*\mathcal{A}\boxast\mathcal{B}$ is a constant sheaf in the degree $2\mathrm{ht}(\nu)-2$.

(b) There is a canonical decomposition

$$(\mathcal{G}_{X^2}\to X^2)_!\mathcal{A}\boxast\mathcal{B} \;\cong\; \oplus_\nu\; (S_{\nu,X^2}\to X^2)_!(S_{\nu,X^2}\hookrightarrow\mathcal{G}_{X^2})^*\mathcal{A}\boxast\mathcal{B}.$$

Proof. (a) Since we know that each cohomology of $(\mathcal{G}_{X^2}\to X^2)_!\mathcal{A}\boxast\mathcal{B}$ is a constant sheaf, its summands are also constant sheaves and (a) follows from (b).

(b) To see that $(S_{\nu,X^2}\to X^2)_!(S_{\nu,X^2}\hookrightarrow\mathcal{G}_{X^2})^*\mathcal{A}\boxast\mathcal{B}$ is a sheaf shifted to the degree $2\mathrm{ht}(\nu)-2$, we observe that the stalk $[(S_{\nu,X^2}\to X^2)_!(S_{\nu,X^2}\hookrightarrow\mathcal{G}_{X^2})^*\mathcal{A}_1\boxast\mathcal{A}_2]_{(x_1,x_2)}$ is isomorphic to

$$\bigoplus_{\nu=\sum_{y\in\{x_1,x_2\}}\nu_y}\quad \bigotimes_{y\in\{x_1,x_2\}} H^*_c(S_{\nu_y},\ \mathop{*}_{x_i=y}\ \mathcal{A}_i\ )[2],$$

and the summands are concentrated in the degree $\sum_y\ 2\mathrm{ht}(\nu_y)-2=\ 2\mathrm{ht}(\nu)-2$.

If we partition a finite locally closed union $\mathcal{X}$ of S_{ν,X^2}'s into open and closed sub-unions $\mathcal{U}$ and $\mathcal{Y}$, the exact gluing triangle

$$(\mathcal{U}\to X^2)_!(\mathcal{U}\hookrightarrow\mathcal{G}_{X^2})^*\mathcal{A}\boxast\mathcal{B}\to\ (\mathcal{X}\to X^2)_!(\mathcal{X}\hookrightarrow\mathcal{G}_{X^2})^*\mathcal{A}\boxast\mathcal{B}\to(\mathcal{Y}\to X^2)_!(\mathcal{Y}\hookrightarrow\mathcal{G}_{X^2})^*\mathcal{A}\boxast\mathcal{B}$$

splits canonically. This is proved by induction in the numbers of S_{ν,X^2}'s involved. The splitting comes from the dual gluing triangle using

1.7. **Lemma.** superlemma. □

1.7.1. *Corollary.* There are canonical isomorphisms

$$F_\nu(\mathcal{A}*\mathcal{B})\;\cong\;\oplus_{\alpha+\beta=\nu}\ F_\alpha\mathcal{A}\otimes F_\beta\mathcal{B}.$$

Proof. The left hand side is the stalk on the diagonal

$$F_\nu(\mathcal{A}*\mathcal{B})[-2\mathrm{ht}(\nu)]\;\cong\;H^*_c(\mathcal{G},k_{S_\nu}\otimes[\mathcal{A}*\mathcal{B}])\;\cong\;H^*_c(\mathcal{G}_{(0,0)},k_{S_{\nu,X^2}}\otimes[\mathcal{A}\boxast\mathcal{B}])[2]$$
$$\cong\;[(S_{\nu,X^2}\to X^2)_!(S_{\nu,X^2}\hookrightarrow\mathcal{G}_{X^2})^*\mathcal{A}\boxast\mathcal{B}]_{(x,x)}[2].$$

The RHS is the stalk at (a,b) for distinct $a,b\in X$. Under the identification $\mathcal{G}_{(a,b)}\cong\mathcal{G}\times\mathcal{G}$ one has $S_{\nu,X^2}\cap\mathcal{G}_{(a,b)}\cong\ \sqcup_{\alpha+\beta=\nu}\ S_\alpha\times S_\beta$, so

$$[(S_{\nu,X^2}\to X^2)_!(S_{\nu,X^2}\hookrightarrow\mathcal{G}_{X^2})^*\mathcal{A}\boxast\mathcal{B}]_{(a,b)}[2]\;\cong\;\oplus_{\alpha+\beta=\nu}\ H^*_c(S_\alpha,\mathcal{A})\otimes H^*_c(S_\beta,\mathcal{B})$$
$$\cong\;\oplus_{\alpha+\beta=\nu}\ F_\alpha\mathcal{A}[-2\mathrm{ht}(\alpha)]\otimes F_\beta\mathcal{B}[-2\mathrm{ht}(\beta)].$$

2. Algebra $\dot{U}_+$

2.1. **Maps between constant sheaves.** To start with we calculate the maps between weight functors that come from the maps between representing objects.

2.1.1. *Lemma.* (a) For two co-weights ν, μ

$$\mathrm{Ext}^*(\Bbbk_{S_\mu}, \Bbbk_{S_\nu}) \;\cong\; H_c^*(S_\nu \cap T_\mu, k) \;\cong\; \mathrm{Ext}^*(\Bbbk_{T_\nu}, \Bbbk_{T_\mu}).$$

(b) There is a canonical commutative diagram

$$\begin{array}{ccccc}
\Bbbk[Irr(S_\nu \cap T_\mu)] & \xrightarrow{\cong} & \mathrm{Hom}(\Bbbk_{S_\mu}[2\mathrm{ht}(\mu)], \Bbbk_{S_\nu}[2\mathrm{ht}(\nu)]) & \longrightarrow & \mathrm{Hom}[F_\mu, F_\nu] \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\
\Bbbk[Irr(S_\nu \cap T_\mu)] & \xrightarrow{\cong} & \mathrm{Hom}(\Bbbk_{T_\nu}[-2\mathrm{ht}(\nu)], \Bbbk_{T_\mu}[-2\mathrm{ht}(\mu)]) & \longrightarrow & \mathrm{Hom}[F_\mu, F_\nu]
\end{array}.$$

Proof. (a) One can choose $\mathcal{G}$-filling compact subvarieties of the form $D = \overline{\mathcal{G}_\lambda}$ and large enough to contain $T_\nu \cap S_\mu$. Since $T_\mu \cap D$ is T_a-invariant, $\mathrm{Ext}^*(\Bbbk_{T_\nu}, \Bbbk_{T_\mu \cap D}) \cong H^*_{T_\nu}(\mathcal{G}, \Bbbk_{T_\mu \cap D})$ can be identified with $H_c^*(S_\nu, \Bbbk_{T_\mu} \cap D) = H_c^*(S_\nu \cap T_\mu \cap D, k) = H_c^*(S_\nu \cap T_\mu, k)$. The other identification is obtained by switching S and T.

(b) Recall that $S_\nu \cap T_\mu \neq \emptyset \Leftrightarrow \mu \leq \nu$, and then the intersection is of pure dimension $\mathrm{ht}(\nu - \mu)$.

So, in this case,

$$\mathrm{Hom}(\Bbbk_{T_\nu}[-2\mathrm{ht}(\nu)], \Bbbk_{T_\mu}[-2\mathrm{ht}(\mu)]) \;=\; \mathrm{Ext}^{2\mathrm{ht}(\nu-\mu)}(\Bbbk_{T_\nu}, \Bbbk_{T_\mu}) \;=\; H_c^{2\mathrm{ht}(\nu-\mu)}(S_\nu \cap T_\mu, k) \;\cong\; \Bbbk[Irr(S_\nu \cap T_\mu)].$$

2.1.2. In this way we get maps of functors $F_\mu \to F_\nu$, but only for $\mu \leq \nu$. These are precisely the maps between weight functors that one keeps when the weight functors are extended from $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$ to all of $D(\mathcal{G})$.

2.2. **Algebra $\dot{U}_+$.** Let $\dot{U}_+ = \underset{\mu,\nu \in X_*(T)}{\oplus} \dot{U}_+(\mu, \nu)$ for

$$\dot{U}_+(\mu,\nu) \overset{\mathrm{def}}{=} \mathrm{Hom}(\Bbbk_{S_\nu}[2\mathrm{ht}(\nu)], \Bbbk_{S_\mu}[2\mathrm{ht}(\mu)]) \;\cong\; \Bbbk[Irr(S_\mu \cap T_\nu)] \cong\; \mathrm{Hom}(\Bbbk_{T_\mu}[-2\mathrm{ht}(\mu)], \Bbbk_{T_\nu}[2\mathrm{ht}(\nu)]).$$

2.2.1. *Canonical basis.* So $\dot{U}_+(\mu,\nu)$ has a canonical basis $\dot{\mathbb{B}}(\mu,\nu) \overset{\mathrm{def}}{=} Irr(T_\nu \cap S_\mu)$, and $\dot{\mathbb{B}}_+ = \sqcup_\eta \dot{\mathbb{B}}(\mu,\nu)$ is a canonical basis of $\dot{U}_+$.

2.2.2. *$\check{Q}_+$-grading.* This is obviously an algebra graded by $\check{Q}_+$ via $\dot{U}_+(\eta) \overset{\mathrm{def}}{=} \underset{\mu-\nu=\eta}{\oplus} \dot{U}_+(\mu,\nu)$. Observe that $\dot{U}_+(0)$ is a semi-simple algebra $\underset{\mu}{\oplus} \Bbbk \cdot e_\mu = \mathcal{O}[X_*(T)]$ for orthogonal idempotents $e_\mu \in \dot{\mathbb{B}}(\mu,\mu)$, $\mu \in X_*(T)$.

2.2.3. *Abstract formulation of the dot-construction ??* We want to localize a module M for a semigroup S on a space X where S acts. The localization $\dot{M}$ will be a module for the action semigroupoid $\mathcal{S}$ defined by S:

$$\dot{M} \stackrel{\text{def}}{=} \mathcal{O}_X \overset{L}{\underset{S}{\otimes}} M \;=\; \mathcal{O}_X \underset{\mathcal{S}\uparrow}{\otimes} M^0, \quad for \quad M^0 \stackrel{\text{def}}{=} \mathcal{O}_X \underset{\Bbbk}{\otimes} M.$$

Consider the case when M is an S-graded $\Bbbk$-module (it does not mean that S acts on M?) $M = \oplus_{s\in S} M(s)$, and X is the group associated to S with the left multiplication action, and assume that S embeds into X. If X is discrete $\dot{M}$ is the $\Bbbk$-module $\dot{M} = \oplus_{(u,v)\in X} M_{v,u}$ with $M_{v,u} = M(s)$ if $v = s{\cdot}u$ for some (unique) $s \in S$ and otherwise $M_{v,u} = 0$. In particular, $\dot{M}$ is again S-graded by $\dot{M}(s) \stackrel{\text{def}}{=} \oplus_{(u,v)\in X,\ v=s\cdot u} M_{v,u}$.

If A is an S-graded algebra, localization $\dot{A}$ is again an S-graded algebra, and it is an $\mathcal{O}(X)$-ring for the commutative algebra $\mathcal{O}(X) = \dot{A}(0) = \oplus_{u\in X} M_{u,u} = \oplus_{u\in X} \Bbbk 1_u$ (semi-simple if X is discrete, without unit if X is discrete and infinite).

In our case $S = \check{Q}_+$, hence $X = \check{Q}$.

2.3. **Algebra $\boldsymbol{U_+}$.** Let $U_+ = \underset{\eta\geq 0}{\oplus} U_+(\eta)$ for

$$U_+(\eta) \stackrel{\text{def}}{=} \mathrm{Hom}(\Bbbk_{S_0}, \Bbbk_{S_\eta}[2\mathrm{ht}(\eta)]) \;\cong\; \Bbbk[Irr(S_\eta \cap T_0)] \cong\; \mathrm{Hom}(\Bbbk_{T_\eta}[-2\mathrm{ht}(\eta)], \Bbbk_{T_0}).$$

So $U_+(\eta)$ has a canonical basis $B(\eta) \stackrel{\text{def}}{=} Irr(T_0 \cap S_\eta)$, and $B_+ = \sqcup_\eta B(\eta)$ is a canonical basis of U_+.

2.3.1. *A realization of algebra on a global curve X.* Using

2.4. **Lemma.** glob we can place the algebra on the BD-Grassmannian

$$U_+(\eta_1 + \cdots + \eta_n) \;\cong\; \mathrm{Ext}^{2\mathrm{ht}(\sum \eta_i)}(\Bbbk_{S_{(0,\dots,0)}}, \Bbbk_{S_{(\eta_1,\dots,\eta_n)}}).$$

2.4.1. *Theorem.* (a) U_+ is a Hopf algebra isomorphic to $U(\check{\mathfrak{n}})$.

(b) U_+ acts faithfully on the fiber functor F, hence $\check{\mathfrak{n}}$ embeds into the Lie algebra $\tilde{\mathfrak{g}}$ of the group $\tilde{G} \stackrel{\text{def}}{=} Aut(F)$.

2.5. **Algebra structure.** Periodicity $\alpha(z) : (\mathcal{G}, S_\nu) \xrightarrow{\cong} (\mathcal{G}, S_{\nu+\alpha})$, gives $\alpha(z)_* \;:\; \mathrm{Ext}^{2\mathrm{ht}(\nu-\mu)}(\Bbbk_{S_\nu}, \Bbbk_{S_\mu}) \xrightarrow{\cong} \mathrm{Ext}^{2\mathrm{ht}([\nu+\alpha]-[\mu+\alpha])}(\Bbbk_{S_{\nu+\alpha}}, \Bbbk_{T_{\mu+\alpha}})$. So the composition of maps in the category of projective systems over $D(\mathcal{G})$ gives an associative (since $\alpha(z)_*$ is an automorphism of the category $D(\mathcal{G})$) algebra structure

$$U_+(\alpha)\otimes U_+(\beta) \overset{\beta(z)_*\otimes 1}{\cong} \mathrm{Ext}^{2\mathrm{ht}(\alpha)}(\Bbbk_{S_\beta}, \Bbbk_{S_{\alpha+\beta}})\otimes \mathrm{Ext}^{2\mathrm{ht}(\beta)}(\Bbbk_{S_0}, \Bbbk_{S_\beta}) \to \mathrm{Ext}^{2\mathrm{ht}(\alpha+\beta)}(\Bbbk_{S_0}, \Bbbk_{S_{\alpha+\beta}}) = U_+(\alpha+\beta).$$

2.6. **Coalgebra structure.** In order to define $\Delta_{\alpha_1,\dots,\alpha_n}: U_+(\alpha_1+\cdots+\alpha_n)\to\otimes_i U_+(\alpha_i)$, we will use the decomposition $\alpha=\alpha_1+\cdots+\alpha_n$ to reformulate $U_+(\alpha)$ globally, i.e., on the BD-Grassmannian $\mathcal{G}_{X^n}$ for say, $X=G_a$. Then the comultiplication is given by the restriction to a generic point of X^n.

So we choose distinct points a_i in X and we follow

$$U_+(\alpha)=\mathrm{Ext}^{2\mathrm{ht}(\alpha)}(k_{S_0},k_{S_\alpha})\xleftarrow{\cong} Ext^{2\mathrm{ht}(\alpha)}(k_{S_{0,\dots,0}},k_{S_{\alpha_1,\dots,\alpha_n}}),$$

with the restriction of sheaves to the fiber of $\mathcal{G}_{X^n}$ above $(a_1,\dots,a_n)$:

$$Ext^*(k_{S_{0,\dots,0}},k_{S_{\alpha_1,\dots,\alpha_n}})\to Ext^*(k_{S_{0,\dots,0}}|a_1\times\cdots\times a_n,k_{S_{\alpha_1,\dots,\alpha_n}}|a_1\times\cdots\times a_n)\cong\otimes_i Ext^*(k_{S_0},k_{S_{\alpha_i}})=\otimes_i U_+(\alpha_i$$

The resulting map $\Delta_{\alpha_1,\dots,\alpha_n}$ is independent of the choice of points a_i (**??**). This comultiplication is clearly cocommutative and coassociative.

2.7. **The action of U_+ on the fiber functor F.** The action of the algebra U_+ on F is obvious

$$U_+(\alpha)\otimes F_\nu\mathcal{A}\overset{\nu(z)_*\otimes 1}{\cong}\mathrm{Hom}(\Bbbk_{S_\nu}[2\mathrm{ht}(\nu)],\Bbbk_{S_{\alpha+\nu}}[2\mathrm{ht}(\nu+\alpha)])\otimes H_c^{2\mathrm{ht}(\beta)}(\mathcal{G},\Bbbk_{S_\nu}[2\mathrm{ht}(\nu)]\otimes\mathcal{A})$$

$$\to H_c^{2\mathrm{ht}(\alpha+\beta)}(\mathcal{G},\Bbbk_{S_{\alpha+\nu}}[2\mathrm{ht}(\nu+\alpha)]\otimes\mathcal{A})=F_{\alpha+\nu}(\mathcal{A}).$$

[[[Observe that a similar use of $U_+(\eta)\cong\mathrm{Hom}(\Bbbk_{T_\eta}[-2\mathrm{ht}(\eta)],\Bbbk_{T_0})$ yields a right action!]]]

2.8. **U_+ acts on the fiber functor as a bialgebra.** For $u\in U_+(\eta)$ and $\eta=\alpha+\beta$, we want

$$\begin{array}{ccc} F_{\phi+\psi}(\mathcal{A}*\mathcal{B}) & \xrightarrow{u} & F_{\phi+\psi+\eta}(\mathcal{A}*\mathcal{B}) \\ \subseteq\uparrow & & \uparrow\subseteq \\ F_\phi(\mathcal{A})\otimes F_\psi(\mathcal{B}) & \xrightarrow{\Delta_{\alpha,\beta}u} & F_{\phi+\alpha}(\mathcal{A})\otimes F_{\psi+\beta}(\mathcal{B}) \end{array}.$$

2.9. **Primitive elements.** U_+ is a cocommutative bialgebra - comultiplication is based on the restriction functors and the contraction isomorphisms of functors, and these are compatible with the composition of maps in categories, which is used to define the multiplication.

Let $\tilde{\mathfrak{n}}$ be the Lie algebra of primitive elements in U_+. Since the torus $\tilde{H}\subseteq\tilde{G}$ defined by $F=\oplus\ F_\nu$ ($\tilde{H}\cong\check{H}$), acts on U_+, it also acts on $\mathfrak{n}=\oplus_\eta\ \mathfrak{n}(\eta)$ and one can define a Lie algebra $\tilde{\mathfrak{b}}=\tilde{\mathfrak{n}}\ltimes\tilde{\mathfrak{h}}$.

2.9.1. $u \in U_+(\alpha)$ is primitive iff it is killed by all $\Delta_{\beta,\gamma}$ for $\beta+\gamma=\alpha$ and $\beta,\gamma \geq 0$ but $\beta,\gamma \neq 0$. This is equivalent to the condition that u is killed by the restriction

$$u \in U_+(\alpha) \;\cong\; Ext^*(k_{S_{0,0}}, k_{S_{\beta,\gamma}}) \to \; Ext^*(k_{S_{0,0}}|X^2-\Delta_X, k_{S_{\beta,\gamma}}|X^2-\Delta_X).$$

If $\Delta_{\beta,\gamma}u=0$, i.e., u is killed by restriction to (a,b) for any two distinct $a,b \in X$, then u is killed by restriction to $A\times B$ for any two disjoint open convex $A,B\subseteq X$. One can cover $X^2-\Delta_X$ with such products and use Leray type argument.

2.9.2. *Theorem.* (a) (**Compatibility of canonical bases**.) For any co-weight λ let $C_\lambda(\nu)$ be the canonical basis of the ν-weight space in $IH(\overline{\mathcal{G}_\lambda}, \Bbbk)$. If λ is anti-dominant let $B_\lambda(\lambda)=\{c_\lambda\}$ and denote by $B(\eta,\lambda)\subseteq B(\eta)$ all elements $b \in B(\eta)$ with $b{\cdot}c_\lambda \neq 0$. The action of U_+ on $IH(\overline{\mathcal{G}_\lambda}, \Bbbk)$ gives a bijection

$$B(\eta,\lambda) \ni b \mapsto b{\cdot}c_\lambda \in C_\lambda(\lambda+\eta).$$

(b) Over integers, $U_+(\mathbb{Z})$ is the Kostant's form of $U(\check{\mathfrak{n}})$.

(c) The basis has the same type of parameterization as in Lusztig's parameterization of his canonical basis.

Proof. (a) is the stabilization property of intersections $S_\nu \cap \mathcal{G}_\lambda$ in the Grassmannian. (b) follows.

(c) is a reformulation of *G.Lusztig, An algebro geometric parameterization of the canonical bases.*

2.9.3. *Conjecture.* The above "Grassmannian" canonical bases is the same as the bases in Nakajima's setting.

2.9.4. *Conjecture.* Verdier duality on $\mathcal{G}$ gives an anti-involution ι of U_+ with $\mathbb{D}(e_i) = e_{-w_0\cdot i}$.

[So the U_+^{opp}-action on F obtained as the adjoint of the U_+ action is just the ι-twist of the U_+-action.]

2.10. **Positivity property of the canonical basis.**

2.10.1. *Lemma.* $\mathbb{Z}_+[\mathcal{B}]\subseteq U_+$ is closed for (co)multiplication. (The action of $U(\check{\mathfrak{n}})$ on L_λ has the same positivity property.)

2.10.2. *Remarks.* Jared Anderson proved half of the statement and conjectured the other half in a different formulation.

Proof. Comultiplication is obvious. One intersects $\overline{S_{\nu,X^n}} \cap T_{0,X^n}$ with the open set which is the restriction of $\mathcal{G}_{X^n}$ to the regular part $(X^n)_r$ of X^n. This gives a bundle over $(X^n)_r$ and one then restricts to one point in $(X^n)_r$.

Multiplication goes in the opposite direction. At (a,b) in $(X^n)_r$ one considers $(\overline{S_{\mu,\nu}} \cap T_{0,0})_{(a,b)} \stackrel{\text{def}}{=} \overline{S_{\mu,a}} \cap T_{0,a} \times \overline{S_{\nu,b}} \cap T_{0,b}$. This is then a bundle over $(X^n)_r$ and it extends across the diagonals to $\overline{S_{\mu,\nu}, X^n} \cap T_{(0,0),X^n} \subseteq (\mathcal{G} * \mathcal{G})_{X^2}$ over X^2. Intersection with the fiber on the diagonal gives multiplication. Positive multiplicities (in the canonical basis of the tensor product) come form the properness of the intersection.

The proof for the action is the same.

2.10.3. *Conjecture.* The same positivity holds for the canonical basis of $U(\check{\mathfrak{g}})$. This gives a $\mathbb{Z}_+$-form of the group G, i.e., reductive groups are defined over commutative semirings. (For $\mathbb{R}_+$ this is Lusztig's positivity theory, for the polar semifield there should be a combinatorial meaning.)

2.10.4. *Conjecture.* The above construction of $\dot{U}(\check{\mathfrak{n}})$ is the same as the one in [FFKM], so it extends to a construction of $\dot{U}(\check{\mathfrak{g}})$.

3. **Remarks**

3.0.5. Inversion ι does not seem geometric on U_+ since it is -1 on on the canonical basis elements of simple root spaces $\tilde{\mathfrak{g}}_i$.

Lie algebra $\tilde{\mathfrak{n}}$ - the primitive part of U_+ does not seem geometric since $\tilde{\mathfrak{g}}(\alpha) \subseteq U_+(\alpha)$ need not be compatible with the canonical basis of $U_+(\alpha)$. Say, for $G = SL_3$ and $\alpha = i + j$, $\tilde{\mathfrak{g}}(\alpha)$ is the kernel of the comultiplication map $\Delta_{i,j} :\ U_+(\alpha) \to U_+(il) \otimes U_+(j)$ from $\Bbbk \oplus \Bbbk$ to $\Bbbk$, which is symmetric in the canonical basis of $U_+(\alpha)$.

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