GROUP ALGEBRA

Not for distribution

This is a sketch of the construction of the dual group over integers, the part not covered by our (Mirković-Vilonen) announcement. A new element is the explicit construction of the group algebra of the dual group.

The ingredients that allow the arithmetic result:
(0) the cohomology decomposes into a sum of local cohomologies on semi- infinite orbits.
(1) a canonical basis of the cohomology of the standard sheaves,
(2) the coincidence of the standard !-sheaf with the IC-sheaf over integers, (3) convolution of standard !-sheaves has a filtration by such sheaves (over integers).

These all flow from the basic observation (0). Another consequence of (0) is a construction of $U\check{\mathfrak{n}}$ [Feigin, Finkelberg, Kuznetsov, Mirković].

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Date: Long Long Time Ago in a Land Far Away .

We will consider the categories of sheaves on the loop Grassmannian $\mathcal{G}$ for a reductive group G, with coefficients in modules over a commutative ring $\Bbbk$, noetherian and of finite global dimension.

1. **Equality of compactly supported and local cohomology on dual strata**

Here we prove the following topological lemma.

1.0.1. *Lemma.* Let V be finite dimensional representation of a torus $T = R{\times}S$ with $R = G_m = S$, such that the weights of R are positive. Let $V = V_+ \oplus V_-$ be a decomposition into T-invariant subspaces such that the weights of S are positive in V_+ and negative in V_-. Then for any T-monodromic $\mathcal{A} \in D(V)$, canonical map

$$H^*_{V_-}(V, \mathcal{A}) \to H^*_c(V_+, \mathcal{A}),$$

is an isomorphism

1.0.2. **Canonical map.** Denote $0 \overset{k_\pm}{\hookrightarrow} V_\pm \overset{p\pm}{\twoheadrightarrow} 0$, $0 \overset{k}{\hookrightarrow} V$, and $V_\pm \overset{i_\pm}{\hookrightarrow} V$.

We use functors on T-monodromic sheaves: compactly supported cohomology on V_+

$$C = (p_+)_!(i_+)^* = (k_+)^!(i_+)^*,$$

and local cohomology along V_-

$$L = (p_-)_*(i_-)^! = (k_-)^*(i_-)^!,$$

(equalities come from say, the R-contraction of $V\pm$ to 0).

The canonical map $L{\to}C$ comes from

$$L{\to}L{\circ}(i_+)_*(i_+)^* = (p_-)_*\ (i_-)^!{\circ}(i_+)_*\ (i_+)^* \overset{Base\ Change}{\cong}$$

$$= (p_-)_*\ (k_+)_*(k_+)^!\ (i_+)^* = (k_-)^!\ (i_+)^* = C,$$

and also symmetrically, $L \cong C{\circ}(i_-)_!(i_-)^!{\to}C$.

1.0.3. **Reduction to the case** $(i_-)^!\mathcal{A} = 0$**.** For $\mathcal{A}$ supported on either of $V_\pm$, the claim follows from base change. So for $j : V - V_- {\hookrightarrow} V$, the claim is true for the first term in the exact triangle

$$(i_-)_!(i_-)^!\mathcal{A}{\to}\mathcal{A}{\to}j_*j^*\mathcal{A},$$

hence it is equivalent for the last two.

So we can suppose that $j_*j^*\mathcal{A} \overset{\cong}{\leftarrow} \mathcal{A}$. Then LHS is zero and we need to kill the RHS. To calculate the RHS we use restriction $\rho : V_+ - 0{\hookrightarrow}V_+$ of j and the exact triangle

$$\rho_!\rho^!(\mathcal{A}|V_+) \to \mathcal{A}|V_+ \ {\to}(k_-)_*(k_-)^*(\mathcal{A}|V_+), \quad i.e.,$$

$$\rho_!(\mathcal{A}|V_+ - 0) \to \mathcal{A}|V_+ \ {\to}(k_-)_*(\mathcal{A}|0),$$

which we write as

$$\mathcal{A}|V_+ \ {\to}(k_-)_*(\mathcal{A}|0) \overset{\alpha}{\to} \rho_!(\mathcal{A}|V_+ - 0)[1].$$

It remains to show that the map $a \overset{\text{def}}{=} H^*_c(V_+,\alpha)$, is an isomorphism:

$$\begin{array}{ccccc} H^*_c[V_+,\mathcal{A}|V_+] & \longrightarrow & H^*_c[V_+,(k_-)_*(\mathcal{A}|0)] & \xrightarrow{\alpha} & H^*_c[V_+,\rho_!(\mathcal{A}|V_+-0)]\ [1] \\ \downarrow = & & \downarrow = & & \downarrow = \\ H^*_c(V_+,\mathcal{A}) & \longrightarrow & \mathcal{A}|0 & \xrightarrow{\alpha} & H^*_c(V_+-0,\mathcal{A})\ )[1] \end{array} .$$

1.0.4. **Passage to a sphere.** Now let $\mathbb{R}^*_+$ act on V via $\mathbb{R}^*_+ \subseteq G_m = R$. Choose a hermitian inner product on V, invariant under the compact part of T. Then for the sphere $\mathcal{S}$ of radius one around 0, map $S \to (V-0)/\mathbb{R}^*_+$ is an isomorphism. Moreover, T action descends to $(V-0)/\mathbb{R}^*_+$ and $\mathcal{A}$ descends to a T-meromorphic complex of sheaves on $(V-0)/\mathbb{R}^*_+$ which one can identify with $\mathcal{A}|\mathcal{S}$.

Now we interpret a in terms of the restriction of $\mathcal{A}$ to the sphere $\mathcal{S}$,

$$\mathcal{A}|0 \cong H^*(V,\mathcal{A}) \overset{\cong}{\to} H^*[V, j_*(\mathcal{A}|V-V_-0)] = H^*(V-V_-,\mathcal{A}) \cong H^*(\mathcal{S}-V_-,\mathcal{A})$$

and

$$H^*_c(V_+-0,\mathcal{A})[1] \cong H^*_c(\mathcal{S}\cap V_+,\mathcal{A}).$$

In this way a is identified with the canonical map

$$H^*(\mathcal{S}-V_-,\mathcal{A}) \xrightarrow{b} H^*_c(\mathcal{S}\cap V_+,\mathcal{A})$$

(restriction of cohomology to a closed subspace).

1.0.5. **Homotopy.** Sheaf $\mathcal{B} = \mathcal{A}|\mathcal{S}-V_-$ is an S-monodromic sheaf on $X = \mathcal{S}-V_-$ and $Y = \mathcal{S}\cap V_+$ is an S-invariant closed subspace with the property that for each $x \in X$, $\underset{z\to 0}{lim}\ z{\cdot}x$ exists and lies in Y. This implies that the restriction of cohomology map $H^*(X,\mathcal{B}) \to H^*(Y,\mathcal{B})$ is an isomorphism.

1.0.6. **Remarks.** (1) This would be a most standard result if the sheaf would be compatible with the projection to one of $V_\pm$. (2) One would like to say that a proof in the algebraic category should work the same with $\mathcal{S}$ replaced by a weighted projective space $(V-0)//R$ or a stack $(V-0)/R$.

2. Integrals over the semi-infinite strata

2.0.7. **The opposite semi-infinite stratifications.** Let N and N_- be opposite unipotent radicals in G and $= \cdot L_\nu$ and $T_\nu = N_-(\mathcal{K}) \cdot L_\nu$. These are intuitively opposite stratifications of$\mathcal{G}$.

2.0.8. *Lemma.* **Intersections with the $G(\mathcal{O})$-orbits.**

$$dim(\cap) = \text{ ht}(\lambda+\nu)$$

and the intersection is of pure dimension.

2.0.9. *Lemma.* **Estimates based on perversity.** For $\mathcal{A}\in\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$

$$H_c^*(,\mathcal{A})\in D^{\le 2\mathrm{ht}(\nu)} \quad \text{and} \quad H^*(\mathcal{G},\mathcal{A})\in D^{\ge -2\mathrm{ht}(\nu)}.$$

2.0.10. *Lemma.* **Integrals over the dual strata.** Let T be a Cartan subgroup of G and $T_a\overset{\mathrm{def}}{=}T\times G_m$ where G_m is the group of rotations. For any T_a-monodromic sheaf $\mathcal{A}$ on the ind-variety $\mathcal{G}$, one has a canonical isomorphism

$$H^*(\mathcal{G},\mathcal{A})\overset{\cong}{\to} H_c^*(T_\nu,\mathcal{A}).$$

Proof. Observe that $X_*(T)\subseteq G(\mathcal{K})$ commutes with T and the action of the rotation group G_m on $T{\cdot}X_*(T)$ becomes trivial modulo the subgroup T. Therefore we can use a translation by $X_*(T)$ to reduce the claim to the case $\nu=0$.

Let K be the the congruence subgroup $\mathrm{Ker}[G(\mathbb{C}[z^{-1}])\xrightarrow{z\mapsto\infty}G]$ of $G(\mathcal{K})$. It is an ind-group subscheme of the ind-group scheme $G(\mathcal{K})$. It lies in a group scheme $\hat{K}\overset{\mathrm{def}}{=}\mathrm{Ker}[G(\mathbb{C}[[z^{-1}]])\xrightarrow{z\mapsto\infty}G]$. Moreover, ind-scheme $\mathcal{G}$ lies in a scheme $\hat{\mathcal{G}}$ which has an open subscheme $\mathcal{U}$ isomorphic to $\hat{K}$ via $\hat{K}\ni k\mapsto k{\cdot}L_0\in\mathcal{U}$ for the origin $L_0\in\mathcal{G}$. Now $S_0=[N(\mathbb{C}[z^{-1}])\cap K]{\cdot}L_0\subseteq\mathcal{U}$, and $N(\mathbb{C}[z^{-1}])\cap K\overset{\cong}{\to}S_0$ via $N(\mathbb{C}[z^{-1}])\cap K\ni k\mapsto k{\cdot}L_0\in S_0$. Also,

$$S_0=[N(\mathbb{C}[[z^{-1}]])\cap\hat{K}]{\cdot}L_0\ \cap\ \mathcal{G}=[B(\mathbb{C}[[z^{-1}]])\cap\hat{K}]{\cdot}L_0\ \cap\ \mathcal{G},$$

and similarly for T_0 and N_-.

Let us replace $\mathcal{U}$ by $\hat{K}$ and then also with its Lie algebra $\hat{\mathfrak{k}}$ via $\exp:\hat{\mathfrak{k}}\overset{\mathrm{def}}{=}\mathfrak{k}_{z^{-1}\mathbb{C}[[-1]]}\overset{\cong}{\to}\hat{K}$. Our sheaf $\mathcal{A}$ will then be supported in a finite dimensional vector subspace $V\subseteq\mathfrak{k}\subseteq\hat{\mathfrak{k}}$, invariant under T_a. Let $R=G_m$ be the rotation group, and let $S=G_m$ act via the co-character $2\check{\rho}\in X_*(T)$, so that it acts on $\mathfrak{b}_{z^{-1}\mathbb{C}[[-1]]}$ with nonnegative weights and on $(\mathfrak{n}_-)_{z^{-1}\mathbb{C}[[-1]]}$ with negative weights. The problem now reduces to a case of the above lemma with $V_+=V\cap\mathfrak{b}_{z^{-1}\mathbb{C}[[-1]]}$ and $V_-=V\cap(\mathfrak{n}_-)_{z^{-1}\mathbb{C}[[-1]]}$.

2.0.11. *Lemma.* **Two kinds of integrals over $N(\mathcal{K})$-orbits.** For $\mathcal{A}\in\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$ and the longest element w_0 in the Weyl group of G, one has

$$H^*(\mathcal{G},\mathcal{A})=H_c^*(T_\nu,\mathcal{A})=H_c^*(S_{w_0\cdot\nu},w_0^*\mathcal{A})=H_c^*(S_{w_0\cdot\nu},\mathcal{A}).$$

2.0.12. *Theorem.* **Integrals over $N(\mathcal{K})$-orbits are concentrated in one degree.** For $\mathcal{A}\in\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$,

$H_c^*(,\mathcal{A})$ is concentrated in the degree $2\mathrm{ht}(\nu)$,

$H^*(\mathcal{G},\mathcal{A})$ is concentrated in the degree $-2\mathrm{ht}(\nu)$.

3. **Consequences**

Denote the "fiber functor" by $F \stackrel{\text{def}}{=} H^*(\mathcal{G}, -)$ and for any coweight ν let $F_\nu \stackrel{\text{def}}{=} H^*_c(, \mathcal{A})$.

3.0.13. *Lemma.* (a) $F \cong \oplus_{\nu \in X_*(T)} F_\nu$, i.e.,

$$H^*(\mathcal{G}, \mathcal{A}) \cong \oplus_{\nu \in X_*(T)} H^*_c(, \mathcal{A}), \quad \mathcal{A} \in \mathcal{P}_{G(\mathcal{O})}(\mathcal{G}).$$

The same is true for the T-equivariant cohomology.

(b) F_ν and F appear in one parity (on each connected component of $\mathcal{G}$). In particular, they are exact on $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$.

(c) Dual $\check{T}$ of the Cartan subgroup embeds into the group $\tilde{G}$ given by the Tannakian formalism.

3.0.14. *Lemma.* [**Canonical Basis**] The cohomology of the standard perverse sheaves on a $G(\mathcal{O})$-orbit $\mathcal{G}_\lambda$ has canonical basis

$$F[\mathcal{I}_!(\lambda)] \cong \Bbbk[Irr(\overline{\mathcal{G}}_\lambda \cap S_\nu)] \cong F[\mathcal{I}_*(\lambda)].$$

3.0.15. *Remarks.* (a) The bases here are not the ones constructed by Lusztig (irreducibles as a basis of a K-group, good algebraic properties and characterization), conjecturally they coincide with the bases Nakajima found.

(b) Over $\mathbb{Q}$ we get two (dual) construction of bases of irreducibles, lower and upper bases.

3.0.16. *Corollary.* (a) $\mathcal{I}_!(\lambda, \Bbbk) \cong \mathcal{I}_!(\lambda, \mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$ and $\mathcal{I}_*(\lambda, \Bbbk) \cong \mathcal{I}_*(\lambda, \mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$.

(b) $\mathbb{D}\, \mathcal{I}_!(\lambda) \cong \mathcal{I}_*(\lambda)$.

3.0.17. *Theorem.* The absence of torsion in $\mathbb{Z}$ gives

$$\mathcal{I}_!(\lambda, \mathbb{Z}) \xrightarrow{\cong} \mathcal{I}_{!*}(\lambda, \mathbb{Z}).$$

Proof. We give a convoluted argument: $\mathbb{D}\, \mathcal{I}_!(\lambda, \mathbb{Z}) \cong \mathcal{I}_*(\lambda, \mathbb{Z}) \cong \mathbb{D}\, \mathcal{I}_{!*}(\lambda, \mathbb{Z})$!

The first isomorphism comes from the canonical bases, and the second from the fact that all three sheaves coincide over $\mathbb{C}$ [Lusztig, Ginzburg]. The second ingredient uses the decomposition theorem which we hope to avoid some day. The second isomorphism follows from a general fact:

3.0.18. *Lemma.* Let $U \overset{j}{\hookrightarrow} X$ be open and $\mathcal{A} \in \mathcal{P}(U, \mathbb{Z})$. Suppose that
(a) $\mathbb{D}\mathcal{A}$ is perverse,
(b) map ${}^p j_! \mathcal{A} \to j_{!*}\mathcal{A}$ becomes an isomorphism over $\mathbb{C}$,
then

$$\mathbb{D}(j_{!*}\mathcal{A}) \overset{\cong}{\to} {}^p j_* \mathcal{A}.$$

Alternatively, if (b) is replaced by: (b') map $j_{!*}\mathcal{A} \to {}^p j_* \mathcal{A}$ is an isomorphism over $\mathbb{C}$, the conclusion is: $\mathbb{D}\ j_{!*}(\mathbb{D}\mathcal{A}) \overset{\cong}{\to} {}^p j_! \mathcal{A}$.

Proof. (1) $\mathbb{D}\mathcal{A}$ is perverse iff $\mathcal{A}$ has no torsion subsheaves. So if $\mathbb{D}\mathcal{A}$ is perverse, so are also $\mathbb{D}(j_{!*}\mathcal{A})$ and $\mathbb{D}({}^p j_* \mathcal{A})$.

(2) Since $\mathbb{D}(j_{!*}\mathcal{A})|U = \mathcal{A}$, there is a canonical map $\mathbb{D}(j_{!*}\mathcal{A}) \to {}^p j_*(\mathbb{D}\mathcal{A})$. Over complex numbers this is the dual of the map ${}^p j_! \mathcal{A} \to j_{!*}\mathcal{A}$, hence an isomorphism. Therefore the kernel and the cokernel are torsion sheaves supported off U:

$$0 \to K \to \mathbb{D}(j_{!*}\mathcal{A}) \to {}^p j_*(\mathbb{D}\mathcal{A}) \to C \to 0.$$

(3) Since the dual of $\mathbb{D}(j_{!*}\mathcal{A})$ is perverse, torsion subsheaf K is zero:

$$0 \to \mathbb{D}(j_{!*}\mathcal{A}) \to {}^p j_*(\mathbb{D}\mathcal{A}) \to C \to 0.$$

(4) Since $\mathbb{D}C[1]$ is perverse, turning the exact triangle $\mathbb{D}(C) \to \mathbb{D}[{}^p j_*(\mathbb{D}\mathcal{A})] \to j_{!*}\mathcal{A}$ gives an exact sequence of perverse sheaves

$$0 \to \mathbb{D}[{}^p j_*(\mathbb{D}\mathcal{A})] \to j_{!*}\mathcal{A} \to \mathbb{D}(C)[1] \to 0.$$

However, $j_{!*}\mathcal{A}$ has no quotients supported off U, so $C = 0$.

4. **Construction of projectives that represent the fiber functor**

4.0.19. *Lemma.* Let $Z \subseteq \mathcal{G}$ be a finite closed union of $G(\mathcal{O})$-orbits. Functor F_ν restricted to $\mathcal{P}_{G(\mathcal{O})}(Z)$ is represented by a projective object $P_Z(\nu)$ of $\mathcal{P}_{G(\mathcal{O})}(Z)$.

Proof. For $\mathcal{A}$ in $\mathcal{P}_{G(\mathcal{O})}(Z)$ one has

$$F_\nu(\mathcal{A}) \overset{\text{def}}{=} H^*_c(S_\nu, \mathcal{A})[2\text{ht}(\nu)] \cong H^*_{T_\nu}(\mathcal{G}, \mathcal{A})[2\text{ht}(\nu)] = \text{Ext}^*(k_{T_\nu}, \mathcal{A})[2\text{ht}(\nu)] = \text{Ext}^*(k_{T_\nu \cap Z}[-2\text{ht}(\nu)], \mathcal{A}).$$

Now choose $n >> 0$ so that the $G(\mathcal{O})$-action on Z factors thru the action of $G(\mathcal{O}_n)$ for $\mathcal{O}_n = \mathcal{O}/z^{n+1}$. Then the forgetful functor $\mathcal{F}^{G(\mathcal{O}_n)}_{B(\mathcal{O}_n)} : D_{G(\mathcal{O}_n)}(Z) \to D_{B(\mathcal{O}_n)}(Z)$ has a left adjoint $\gamma^{G(\mathcal{O}_n)}_{B(\mathcal{O}_n)}$ (see the construction bellow).

Since $\text{Ext}^*_{D(Z)}(k_{T_\nu \cap Z}[-2\text{ht}(\nu)], \mathcal{A})$ lives in one degree one has (for the obvious $B(\mathcal{O}_n)$-structures)

$$F_\nu \mathcal{A} \otimes H^* B(pt, \Bbbk) \cong \text{Ext}^*_{D(Z)}(k_{T_\nu \cap Z}[-2\text{ht}(\nu)], \mathcal{A}) \otimes H^* B(pt, \Bbbk) \cong \text{Ext}^*_{D_{B(\mathcal{O}_n)}(Z)}(k_{T_\nu \cap Z}[-2\text{ht}(\nu)], \mathcal{A})$$

$$= \text{Ext}^*_{D_{B(\mathcal{O}_n)}(Z)}(k_{T_\nu \cap Z}[-2\text{ht}(\nu)], \mathcal{F}^{G(\mathcal{O}_n)}_{B(\mathcal{O}_n)} \mathcal{A}) \cong \text{Ext}^*_{D_{G(\mathcal{O}_n)}(Z)}(\gamma^{G(\mathcal{O}_n)}_{B(\mathcal{O}_n)} k_{T_\nu \cap Z}[-2\text{ht}(\nu)], \mathcal{A}).$$

Therefore,

$$F_\nu\mathcal{A} \;\cong\; \mathrm{Hom}_{D(Z)}(k_{T_\nu\cap Z}[-2\mathrm{ht}(\nu)],\mathcal{A}) \;\cong\; \mathrm{Hom}_{D_{B(\mathcal{O}_n)}(Z)}(k_{T_\nu\cap Z}[-2\mathrm{ht}(\nu)],\mathcal{A})$$
$$\cong\; \mathrm{Hom}_{D_{G(\mathcal{O}_n)}(Z)}(\gamma_{B(\mathcal{O}_n)}^{G(\mathcal{O}_n)} k_{T_\nu\cap Z}[-2\mathrm{ht}(\nu)],\mathcal{A}).$$

Since $\mathcal{F} = \gamma_{B(\mathcal{O}_n)}^{G(\mathcal{O}_n)} k_{T_\nu\cap Z}[-2\mathrm{ht}(\nu)] \in D_{G(\mathcal{O}_n)}(Z)$ represents the exact functor F_ν, we will find that it lies in ${}^pD^{\le 0}_{G(\mathcal{O}_n)}(Z)$. Let d be the highest non-vanishing degree of $\mathcal{F}$ (i.e. with ${}^pH^d(\mathcal{F})\neq 0$). Then

$$0\neq \mathrm{Hom}_{D_{G(\mathcal{O}_n)}(Z)}(\mathcal{F},\ {}^pH^d\mathcal{F}[-d]) \;=\; \mathrm{Ext}^{-d}_{D_{G(\mathcal{O}_n)}(Z)}(\mathcal{F},\ {}^pH^d\mathcal{F}\ )$$
$$\cong\; H^{-d}[F_\nu({}^pH^d\mathcal{F})\otimes H^*_B(pt,\Bbbk)] \;\cong\; F_\nu({}^pH^d\mathcal{F})\otimes H^{-d}_B(pt,\Bbbk),$$

gives $d\le 0$.

Therefore, $P_Z(\nu)\overset{\mathrm{def}}{=}{}^pH^0(\mathcal{F})$ represents F_ν on $\mathcal{P}_{G(\mathcal{O}_n)}(Z)=\mathcal{P}_{G(\mathcal{O})}(Z)$. Since F_ν is exact, $P_Z(\nu)$ is projective.

4.0.20. *Corollary.* $\mathcal{P}_{G(\mathcal{O})}(Z)$ has enough projectives.

Proof. For $\mathcal{A}\in\mathcal{P}_{G(\mathcal{O})}(Z)$ choose finitely generated k-projective covers $f_\nu\to F_\nu(\mathcal{A})$. Then $\mathrm{Hom}(f_\nu\otimes P_Z(\nu),\mathcal{A}) \;\cong\; \mathrm{Hom}_\Bbbk[f_\nu,\mathrm{Hom}(P_Z(\nu),\mathcal{A})] \;\cong\; \mathrm{Hom}_\Bbbk[f_\nu,F_\nu(\mathcal{A})]$ contains a canonical map p_ν such that $F_\nu(p_\nu)$ is surjective. Therefore, $\oplus_\nu\ f_\nu\otimes P_Z(\nu)$ is a projective cover of $\mathcal{A}$.

4.1. **Bernstein's induction functors.** For a subgroup B of a group A, the left adjoint of the forgetful functor $\mathcal{F}_B^A : D_A(Z)\to D_B(Z)$ is given by $\gamma_B^A\mathcal{A}\overset{\mathrm{def}}{=} a_!\tilde{\mathcal{A}}$. Here we use the diagram

$$Z \overset{p}{\leftarrow} A\times Z \overset{\nu}{\to} A\underset{B}{\times} Z \overset{a}{\to} Z$$

with $A\times B$ acting on $A\times Z$ by $(a,b)\cdot(\alpha,z)\overset{\mathrm{def}}{=}(a\cdot\alpha\cdot b^{-1},b\cdot z)$, in order to characterize $\tilde{\mathcal{A}}\in D_A(A\underset{B}{\times}Z)$ by: $\nu^!\tilde{\mathcal{A}} \;\cong\; p^!\mathcal{A}$ in $D_A(A\times Z)$.

Since the forgetful functor is exact, its left adjoint γ_B^A is right exact, i.e., $\gamma_B^A:\ D_B^{\le 0}(Z)\to D_A^{\le 0}(Z)$, and its perverse version ${}^p\gamma_B^A\overset{\mathrm{def}}{=}{}^pH^0[\gamma_B^A -] :\ \mathcal{P}_B(Z)\to\mathcal{P}_A(Z)$ is the left adjoint of the functor between abelian categories $\mathcal{F}_B^A:\ \mathcal{P}_A(Z)\to\mathcal{P}_B(Z)$.

4.1.1. *Example.* For a $\mathcal{B}$-invariant $Y\subseteq Z$ denote by $\alpha:\mathcal{A}\underset{\mathcal{B}}{\times}Y\to Z$ the action, then

$$\gamma_{\mathcal{B}}^{\mathcal{A}}(\Bbbk_Y) \;=\; \alpha_!\ \Bbbk_{\mathcal{A}\underset{\mathcal{B}}{\times}Y}\ [2\dim\mathcal{A}/\mathcal{B}] \qquad\text{and}\qquad \Gamma_{\mathcal{B}}^{\mathcal{A}}(\Bbbk_Y) \;=\; \alpha_*\ \Bbbk_{\mathcal{A}\underset{\mathcal{B}}{\times}Y}.$$

The first part uses $p^!\Bbbk_Y = \Bbbk_{\mathcal{A}}[2\dim\mathcal{A}]\boxtimes\Bbbk_Y = \Bbbk_{\mathcal{A}\times Y}[2\dim\mathcal{B}][2\dim\mathcal{A}/\mathcal{B}] = \nu^!\ \Bbbk_{\mathcal{A}\underset{\mathcal{B}}{\times}Y}[2\dim\mathcal{A}/\mathcal{B}]$, and the second is simpler $p^*\Bbbk_Y = \Bbbk_{\mathcal{A}\times Y} = \nu^*\ \Bbbk_{\mathcal{A}\underset{\mathcal{B}}{\times}Y}$.

4.2. **The explicit construction of** $P_Z(\nu)$**.** By construction, $P_Z(\nu) = {}^pH^0[\ \gamma_{B(\mathcal{O}_n)}^{G(\mathcal{O}_n)}(k_{T_\nu\cap Z}[2\mathrm{ht}(\nu)])\ ]$ with

$$\gamma_{B(\mathcal{O}_n)}^{G(\mathcal{O}_n)}\ k_{T_\nu\cap Z}\ =\ [G(\mathcal{O}_n) \underset{B(\mathcal{O}_n)}{\times} T_\nu \cap Z \xrightarrow{\alpha} Z]_!\ k_{G(\mathcal{O}_n)\underset{B(\mathcal{O}_n)}{\times} T_\nu\cap Z}\ \ [2\dim\ G(\mathcal{O}_n)/B(\mathcal{O}_n)].$$

The fiber of α at $\eta\in Z^T$ is $\mathcal{B}\backslash\{a\in\mathcal{A},\ a\lambda\in Y\cap\mathcal{A}{\cdot}\eta = T_\nu\cap\mathcal{G}_\eta\}$. The dimension of the fiber of α at a dominant η is $d_{\mathcal{A}/\mathcal{B}} - \mathrm{ht}(\nu+\eta)$.

So $P_Z(\nu)$ is the zeroth perverse cohomology of the !-image of a (shift of) a constant sheaf under an "essentially semi-small" map. Here "essentially semi-small" means that the the dimensions of fibers have the correct increment but the generic fiber is not finite.

5. The structure of the projectives

Let $P_Z = \oplus_\nu\ P_Z(\nu)$ be the projective that represents the fiber functor F on $\mathcal{P}_{G(\mathcal{O})}(Z)$.

5.0.1. *Lemma.* (a) P_Z is a projective generator of $\mathcal{P}_{G(\mathcal{O})}(Z)$. It carries a canonical $\tilde{G}$-action.

(b) If $\mathcal{G}_\lambda$ is open in Z and $Y = Z - \mathcal{G}_\lambda$, then

$$\mathcal{P}_Y =\ {}^pH^0(P_Z|Y).$$

(c) There is a canonical exact sequence

$$0{\to}\ \mathcal{I}_!(\lambda)\ \otimes\ F(\mathcal{I}_*(\lambda))^*\ \to\ P_Z\ \to\ \mathcal{P}_Y\ {\to}0.$$

Here $\mathcal{I}_?(\lambda)$ is the standard perverse ?-extension from $\mathcal{G}_\lambda$.

(d) P_Z has a $\tilde{G}$-equivariant filtration with canonical $\tilde{G}$-isomorphisms

$$Gr(P_Z)\ \cong\ \underset{\mathcal{G}_\lambda\subseteq Z}{\oplus}\ \mathcal{I}_!(\lambda)\otimes F[\mathcal{I}_*(\lambda)]^*\cong\ \underset{\mathcal{G}_\lambda\subseteq Z}{\oplus}\ \mathcal{I}_!(\lambda)\otimes F[\mathcal{I}_!(-w_0{\cdot}\lambda)].$$

In particular, $F[\mathcal{P}(\nu)]$ is a free $\Bbbk$-module and

$$Gr[F(P_Z)] = F[Gr(P_Z)] \cong\ \underset{\mathcal{G}_\lambda\subseteq Z}{\oplus}\ F[\mathcal{I}_!(\lambda)]\otimes F[\mathcal{I}_!(-w_0{\cdot}\lambda)]$$

has a canonical basis - the union of products of canonical basis.

(e) $P_Z(\nu,\Bbbk)\cong\ P_Z(\nu,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk$.

Proof. (a) $\tilde{G} = \mathrm{Aut}(F)$ acts on $F|\mathcal{P}_{G(\mathcal{O})}(Z)$, hence also on P_Z.

(b) Clearly, on $\mathcal{P}_{G(\mathcal{O})}(Y)$, complex $P_Z|Y\in\ {}^pD^{\le 0}_{G(\mathcal{O})}(Y)$, still represents F, and then so does ${}^pH^0(P_Z|Y)\in\mathcal{P}_{G(\mathcal{O})}(Y)$.

(c$_1$) Restriction $P_Z(\nu)|\mathcal{G}_\lambda$ is up to a shift a constant sheaf $k_{\mathcal{G}_\lambda}[2\mathrm{ht}(\lambda)]\otimes$?. Moreover, ? is a free $\Bbbk$-module - the basis is given by irreducible components of the fiber at $\lambda\in\mathcal{G}_\lambda$

of of the semi-small map that occurs in the construction of $P_Z(\nu)$ (actually, the union of such over all co-weights ν). This gives a map $\mathcal{I}_!(\lambda)\otimes ? \to P_Z$.

Let $\Bbbk = \mathbb{Z}$ for a moment. Then $\mathcal{I}_!(\lambda)$ coincides with $\mathcal{I}_{!*}(\lambda)$ hence the map $\mathcal{I}_!(\lambda) \otimes ?\to P_Z$ is an embedding. The quotient Q is supported on Y. Since $\mathcal{I}_!(\lambda)|Y \in {}^pD^{<0}(Y)$, exact triangle $\mathcal{I}_!(\lambda)|Y \otimes ? \to P_Z|Y \to Q$, gives $Q = {}^pH^0(Q) = {}^pH^0(P_Z(\nu)|Y) = \mathcal{P}_Y(\nu)$.

So for $\Bbbk = \mathbb{Z}$ we find by the induction in the number of $G(\mathcal{O})$-orbits in Z that $P_Z(\nu)$ has a filtration whose quotients are standard !-sheaves.

(e) Therefore, $\mathcal{I}_!(\lambda, \Bbbk) \cong \mathcal{I}_!(\lambda, \mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$ implies that $P_Z(\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$ is perverse. Since

$$\mathrm{Hom}[P_Z(\Bbbk), P_Z(\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk] \cong F[P_Z(\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk] \cong F[P_Z(\mathbb{Z})]\overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk = F[P_Z(\mathbb{Z})]\underset{\mathbb{Z}}{\otimes} \Bbbk \cong \mathrm{End}[P_Z(\mathbb{Z})]\underset{\mathbb{Z}}{\otimes} \Bbbk,$$

there is a canonical map $P_Z(\Bbbk)\to P_Z(\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$. Again, one can see that this is an isomorphism by the induction since it is clearly an isomorphism over $\mathcal{G}_\lambda$.

One can see the same from our construction of projectives. For $\mathcal{F}(\Bbbk) = \gamma_{B(\mathcal{O}_n)}^{G(\mathcal{O}_n)} k_{T_\nu\cap Z}[-2\mathrm{ht}(\nu)]$, one clearly has $\mathcal{F}(\Bbbk) = \mathcal{F}(\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk$. So $P_Z(\nu,\Bbbk) \cong P_Z(\nu,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk$ follows by applying ${}^pH^0$ to the triangle ${}^p\tau_{<0}\mathcal{F}(\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk\to \mathcal{F}(\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk \to P_Z(\nu,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk$ (since the first term in is in ${}^pD^{<0}$).

(c$_2$) For a general $\Bbbk$, exact sequence $0\to \mathcal{I}_!(\lambda) \otimes ? \to P_Z \to \mathcal{P}_Y \to 0$ is now obtained by tensoring the sequence for $\mathbb{Z}$.

The multiplicity ? is given by

$$? \cong \mathrm{Hom}[P_Z, \mathcal{I}_*(\lambda)]^* \cong F[\mathcal{I}_*(\lambda)]^*.$$

To obtain the first isomorphism we apply $\mathrm{Hom}[-, \mathcal{I}_*(\lambda)]$ to the exact sequence

$$0\to \mathcal{I}_!(\lambda) \otimes ? \to P_Z \to P_Y \to 0.$$

Recall that for a perverse sheaf A supported in $\partial\mathcal{G}_\lambda$ one has $\mathrm{Ext}^p[A, \mathcal{I}_*(\lambda)] = 0$, for $p = 0,\ 1$. (Since $A|\mathcal{G}_\lambda = 0$ gives $\mathrm{Ext}^*[A, (\mathcal{G}_\lambda \hookrightarrow Z)_*\Bbbk_{\mathcal{G}_\lambda}] = 0$, for the cone C of the canonical map $\mathcal{I}_*(\lambda)\to(\mathcal{G}_\lambda\hookrightarrow Z)_*\Bbbk_{\mathcal{G}_\lambda}[\dim\mathcal{G}_\lambda]$ one has $\mathrm{Ext}^p[A, \mathcal{I}_*(\lambda)] \cong \mathrm{Ext}^{p-1}(A, C)$. The claim follows from $C \in {}^pD^{>0}(Z)$.) For $A = P_Y$ this vanishing gives the first isomorphism in

$$\mathrm{Hom}[P_Z, \mathcal{I}_*(\lambda)]\xrightarrow{\cong}\mathrm{Hom}[\mathcal{I}_!(\lambda) \otimes ?, \mathcal{I}_*(\lambda)] \cong ?^*,$$

the second comes from the identification $\mathrm{Hom}[\mathcal{I}_!(\lambda), \mathcal{I}_*(\lambda)] \cong \Bbbk$.

(d) The canonical map $P_Z\to {}^pH^0(P_Z|Y) = P_Y$ comes from $F|\mathcal{P}_{G(\mathcal{O})}(Y)$ being a restriction of $F|\mathcal{P}_{G(\mathcal{O})}(Z)$, so it is $\tilde{G}$-equivariant. Therefore the filtration on P_Z is $\tilde{G}$-equivariant. Moreover, $\tilde{G}$-action on the kernel $\mathcal{I}_!(\lambda)\otimes ?\subseteq P_Z$ is given by the corresponding action on $? = \mathrm{Hom}[\mathcal{I}_!(\lambda), P_Z]$. Finally, isomorphism $?^* \cong F[\mathcal{I}_*(\lambda)]$ is equivariant since $\tilde{G} = \mathrm{Aut}F$.

5.0.2. *Remark.* The proof of (c) over a filed is abstract, but in general we seem to need the fact that the multiplicity of $\mathcal{I}_!(\lambda)$ is $\mathbb{Z}$-free.

5.0.3. *Injectives.* The Verdier dual $I_Z \overset{\text{def}}{=} \mathbb{D}(P_Z)$ is perverse since $\mathcal{P}_Z \in \mathcal{P}^p_{G(\mathcal{O})}(Z)$, i.e., $F[\mathcal{P}(\nu)]$ is a free $\Bbbk$-module. I_Z is clearly an injective object in the category $\mathcal{P}^p_{G(\mathcal{O})}(Z)$ which is closed under the Verdier duality.

6. The algebras $\hat{U}(\tilde{G})$ and $\mathcal{O}(\tilde{G})$ given by the Tannakian formalism

6.1. **Tannakian formalism for an abelian category with a fiber functor.** Consider an abelian category $\mathcal{C}$ with a "fiber functor" $\omega: \mathcal{C} \to \mathfrak{m}(\Bbbk)$, i.e., an exact and fully faithful $\Bbbk$-functor. The Tannakian formalism [P. Deligne, Categories Tannakiens] says that ω can be viewed as an equivalence of $\mathcal{C}$ and the category of $\Bbbk$-modules with additional data of an action of the $\Bbbk$-algebra $\mathrm{End}(\omega)$. If $\mathrm{End}(\omega)$ is $\Bbbk$-projective one can instead describe $\mathcal{C}$ as the category of comodules for the coalgebra $\mathrm{End}(\omega)^*$. Moreover, using a projective generator P of $\mathcal{C}$ one can describe this coalgebra explicitly, $\mathrm{End}(\omega)^* \cong \omega(P)^* \underset{\mathrm{End}_{\mathcal{C}}(P)}{\otimes} \omega(P)$.

All of this is clear if we know a representing object P for ω. It is a projective generator of $\mathcal{C}$ since the functor is exact and fully faithful. Then $\mathrm{End}(\omega)^o \cong \mathrm{End}_{\mathcal{C}}(P) \cong \omega(P)$, hence $\mathrm{End}(\omega)^* \cong \omega(P)^*$ and the coalgebra structure is the dual of the algebra structure on $\omega(P) \cong \mathrm{End}_{\mathcal{C}}(P)$.

6.1.1. *Lemma.* $F: \mathcal{P}_{G(\mathcal{O})}(Z) \to \mathfrak{m}(\Bbbk)$ gives an equivalence of $\mathcal{P}_{G(\mathcal{O})}(Z)$ with the right modules for the algebra $F(P_Z) \cong \mathrm{End}(P_Z) \cong \mathrm{End}(\omega)^0$. We can also view these as the comodules for the coalgebra

$$\mathrm{End}(\omega)^* \cong \mathrm{End}(P_Z)^* \cong F(P_Z)^* \cong F(I_Z).$$

6.1.2. *Passage to $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$.* Since $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$ is a union of abelian sub-categories $\mathcal{P}_{G(\mathcal{O})}(Z)$, we get a pro-projective object $P = \lim_{\leftarrow Z} P_Z$ that pro-represents the fiber functor F, and an ind-injective object $I = \lim_{\rightarrow Z} I_Z$. Now F gives an equivalence between $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$ and the category of modules of the pro-algebra $F(P) \overset{\text{def}}{=} \lim_{\leftarrow} F(P_Z)$ (a complete topological algebra), which can be seen also as the category of comodules for the ind-coalgebra $F(I) \overset{\text{def}}{=} \lim_{\rightarrow} F(I_Z)$ (a union of sub-coalgebras of finite rank).

6.2. **Tannakian category with a fiber functor.** The subcategory $\mathcal{P}^p_{G(\mathcal{O})}(\mathcal{G}) \subseteq \mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$ consisting of sheaves A with $F(A)$ a projective $\Bbbk$-module, is a Tannakian category with a fiber functor F. Filtration by sub-categories $\mathcal{P}^p_{G(\mathcal{O})}(\lambda) \overset{\text{def}}{=} \mathcal{P}^p_{G(\mathcal{O})}(\overline{\mathcal{G}_\lambda})$ indexed by dominant co-weights λ is compatible with convolution: $\mathcal{P}^p_{G(\mathcal{O})}(\lambda) * \mathcal{P}^p_{G(\mathcal{O})}(\mu) \subseteq \mathcal{P}^p_{G(\mathcal{O})}(\lambda+\mu)$, and with duality (it sends $\mathcal{P}^p_{G(\mathcal{O})}(\lambda)$ to $\mathcal{P}^p_{G(\mathcal{O})}(-w_0 \cdot \lambda)$). This makes I into an ind-Hopf object $I =$

$\lim\limits_{\to} I(\lambda)$, with the multiplication given by the corresponding maps $I(\lambda)*I(\mu)\to I(\lambda+\mu)$. In consequence, $F(I)$ is a Hopf algebra. The Tannakian formalism says that this is the group algebra $\mathcal{O}(\tilde{G})$ of the group scheme $\tilde{G}\stackrel{\text{def}}{=}\text{Aut}(F)$ of automorphisms of the fiber functor.

As an algebra it has an increasing filtration by $\mathcal{O}(\tilde{G})_\lambda \stackrel{\text{def}}{=} F(I_\lambda)$ (λ a dominant co-character), with

$$Gr_\lambda(\mathcal{O}(\tilde{G})) \;\cong\; F[\mathcal{I}_*(\lambda)]\otimes F[\mathcal{I}_*(-w_0\cdot\lambda)].$$

(This is dual of the above formula for $Gr[F(P_Z)]$.) The dual Hopf algebra is the topological Hopf algebra $\hat{U}(\tilde{G}) = F(P)$.

6.3. **Equivalences of categories.** By the definition of $\tilde{G}$, we can view the fiber functor F as a functor between the Tannakian categories $\mathcal{P}^p_{G(\mathcal{O})}(\mathcal{G})$ and $\mathfrak{m}^p(\tilde{G})$, the category of algebraic $\tilde{G}$-modules which are $\Bbbk$-projective. The Tannakian formalism guarantees that this is an equivalence.

6.3.1. *Lemma.* The equivalence of Tannakian categories $\mathcal{P}^p_{G(\mathcal{O})}(\mathcal{G})\cong\mathfrak{m}^p(\tilde{G}_\Bbbk)$ extends to a canonical equivalence of abelian categories $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})\cong\mathfrak{m}(\tilde{G}_\Bbbk)$.

Proof. The main point is that any object A in $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$ is a quotient of an object P in $\mathcal{P}^p_{G(\mathcal{O})}(\mathcal{G})$, but for $A\in\mathcal{P}_{G(\mathcal{O})}(Z)$ we saw that we can choose such P which is even projective in $\mathcal{P}_{G(\mathcal{O})}(Z)$.

6.4. **$\tilde{G}_\Bbbk$ is flat over $\Bbbk$.**

6.4.1. *Lemma.* (a) $\mathcal{O}(\tilde{G}_\mathbb{Z})$ is a free $\mathbb{Z}$-module and $U(\tilde{G}_\mathbb{Z})$ has no torsion.

(b) $\mathcal{O}(\tilde{G}_\mathbb{Z})$ has no zero divisors.

(c) $\mathcal{O}(\tilde{G}_\Bbbk)\cong\mathcal{O}(\tilde{G}_\mathbb{Z})\underset{\mathbb{Z}}{\otimes}\Bbbk$.

(d) Lie algebra $\tilde{\mathfrak{g}}_\mathbb{Z}$ has no torsion and it is a $\mathbb{Z}$-form of $\check{\mathfrak{g}}$.

Proof. (a) follows from the description of $Gr[\mathcal{O}(\tilde{G})]$, as a free module with a canonical basis.

(b) follows since now $\mathcal{O}(\tilde{G}_\mathbb{Z})\subseteq\mathcal{O}(\tilde{G},\mathbb{Z})\underset{\mathbb{Z}}{\otimes}\mathbb{C} \;=\; \mathcal{O}(\tilde{G},\mathbb{C}) \;\cong\; \mathcal{O}(\check{G},\mathbb{C})$ and the last algebra is integral.

(c) The base change claim follows from such claim for the basic projectives - canonical isomorphisms $P_Z(\nu,\Bbbk)\cong P_Z(\nu,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk$.

(d) In particular, the primitive part $\tilde{\mathfrak{g}}_\mathbb{Z}$ of the topological Hopf algebra $U(\tilde{G}_\mathbb{Z})$ has no torsion, while $U(\tilde{G}_\mathbb{Z})\subseteq U(\tilde{G}_\mathbb{C})$ gives $\tilde{\mathfrak{g}}_\mathbb{Z}\underset{\mathbb{Z}}{\otimes}\mathbb{C}\cong\;\tilde{\mathfrak{g}}_\mathbb{C}\;\cong\;\check{\mathfrak{g}}$.

7. The calculation of $\mathcal{O}(\tilde{G})$

7.0.2. *Lemma.* (a) The algebra structure on $Gr(\mathcal{O}(\tilde{G})) \cong \underset{\lambda}{\oplus} F[\mathcal{I}_*(\lambda)]\otimes F[\mathcal{I}_*(-w_0\cdot\lambda)]$, is the quotient of the algebra $\underset{\lambda}{\oplus} F[\mathcal{I}_*(\lambda)] \otimes \underset{\mu}{\oplus} F[\mathcal{I}_*(\mu)] = A\otimes A$. Here

$$A \overset{\text{def}}{=} \underset{\lambda}{\oplus} F[\mathcal{I}_*(\lambda)]$$

is given the algebra structure thru the canonical maps $F[\mathcal{I}_*(\lambda)]\otimes F[\mathcal{I}_*(\mu)] \to F[\mathcal{I}_*(\lambda+\mu)]$.

(b) Filtration on $\mathcal{O}(\tilde{G})$ is compatible with the coalgebra structure and each summand $F[\mathcal{I}_*(\lambda)]\otimes F[\mathcal{I}_*(-w_0\cdot\lambda)] \cong F[\mathcal{I}_*(\lambda)]\otimes F[\mathcal{I}_!(\lambda)]^*$ of $Gr[\mathcal{O}(\tilde{G})]$ is a sub-coalgebra. Its dual is the associative algebra structure (possibly without a unit) on $F[\mathcal{I}_*(\lambda)]^*\otimes F[\mathcal{I}_!(\lambda)] \cong \mathrm{Hom}[F\mathcal{I}_*(\lambda), F\mathcal{I}_!(\lambda)]$, defined via the canonical map $F[\mathcal{I}_!(\lambda)] \to F[\mathcal{I}_*(\lambda)]$.

Proof. (a) One checks that the maps $I(\lambda) * I(\mu) \to I(\lambda+\mu)$ defined by $\mathcal{P}^p_{G(\mathcal{O})}(\lambda) \times \mathcal{P}^p_{G(\mathcal{O})}(\mu) \overset{*}{\to} \mathcal{P}^p_{G(\mathcal{O})}(\lambda+\mu)$, induce on the graded pieces $Gr_\lambda I_\lambda = \mathcal{I}_*(\lambda)]\otimes F[\mathcal{I}_*(-w_0\cdot\lambda)]$ the maps $Gr_\lambda I(\lambda) * Gr_\mu I(\mu) \to Gr_{\lambda+\mu} I(\lambda+\mu)$ given in each factor by the canonical maps $\mathcal{I}_*(\lambda) * \mathcal{I}_*(\mu) \to \mathcal{I}_*(\lambda+\mu)$.

7.0.3. Part (b) will not be needed.

7.0.4. *Corollary.* $\mathcal{O}(\tilde{G})$ is finitely generated, hence $\tilde{G}$ is noetherian.

Proof. The maps $F[\mathcal{I}_*(\lambda)]\otimes F[\mathcal{I}_*(\mu)] \to F[\mathcal{I}_*(\lambda+\mu)]$ are surjective, hence A is generated by the sum of fundamental representations $F[\mathcal{I}_*(\omega_i)]$.

7.0.5. *Lemma.* Let $\Bbbk$ be a closed field.

(a) $\tilde{G}_\Bbbk$ is connected.

(b) If $\tilde{G}_\Bbbk$ is reduced then $\tilde{G}_\Bbbk \cong \check{G}_\Bbbk$.

" *Proof.* " (a) For any field $\Bbbk$, noetherian group scheme $\tilde{G}_\Bbbk$ is connected. Otherwise it would have a finite group scheme quotient and its representations would form a tensor subcategory of $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G})$ supported in a compact subvariety. Since $\dim \mathrm{supp}(\mathcal{A} * \mathcal{B}) = \dim \mathrm{supp}(\mathcal{A}) + \dim \mathrm{supp}(\mathcal{B})$ such quotient is trivial.

(b) $\tilde{G}_\Bbbk = U \ltimes R$ for a reductive group R and a unipotent group U (?).

Now irreducible representations of $\tilde{G}_\Bbbk$ and R are the same (?).

Since the parameterization of irreducibles is given by the same cone in the same lattice as in $\check{G}_\Bbbk$ we see that $\tilde{T}_\Bbbk$ is a maximal torus of $\tilde{G}_\Bbbk$ and R, and the root systems of R and $\tilde{G}_\Bbbk$ coincide.

Since $Gr[\mathcal{O}(\tilde{G}_\Bbbk)] \cong Gr[\mathcal{O}(\check{G}_\Bbbk)]$ as a module for $R \cong \check{G}_\Bbbk$ we see that $U = 1$.

7.0.6. *Lemma.* $\mathcal{I}_!(\lambda) * \mathcal{I}_!(\mu)$ has a canonical filtration with

$$Gr[\mathcal{I}_!(\lambda) * \mathcal{I}_!(\mu)] \cong \ \oplus_\eta \ M^\eta_{\lambda,\mu} \otimes \mathcal{I}_!(\eta)$$

for some free modules $M^\eta_{\lambda,\mu}$ with a canonical basis.

The same is true for standard $*$-sheaves.

Proof. The general case follows from the case $\Bbbk = \mathbb{Z}$ for !-sheaves. The proof of this case is very similar to the one for the structure of the projectives (

7.1. **Lemma.** filt). The convolution is given by the direct image under a stratified semi-small map and one peels one by one the layers corresponding to open orbits in the remainder. This is formalized bellow in the notion of "f-semi-small maps" (

7.2. **Lemma.** f).

7.2.1. *Lemma.* (a) Over an integral ring $\Bbbk$, the pairings given by the canonical maps $F[\mathcal{I}_*(\lambda)] \otimes F[\mathcal{I}_*(\mu)] \to F[\mathcal{I}_*(\lambda+\mu)]$ have no zero divisors.

(b) For any closed field $\Bbbk$, $\tilde{G}_\Bbbk$ is a connected algebraic group.

Proof. Let us start with $\Bbbk = \mathbb{C}$. Over $\mathbb{C}$ any noetherian group scheme is always reduced, so $\tilde{G}_\mathbb{C} \cong \tilde{G}_\mathbb{C}$ by

7.3. **Lemma.** ident. Now (a) is clear over $\mathbb{C}$ since for a reductive group these maps are given by tensoring the sections of line bundles on a (connected) flag variety.

This implies (a) for $\Bbbk = \mathbb{Z}$. The general case follows since the kernel of $F[\mathcal{I}_*(\lambda)] \otimes F[\mathcal{I}_*(\mu)] \to F[\mathcal{I}_*(\lambda+\mu)]$ is defined over integers by

7.4. **Lemma.** filt.

Applying (a) to $G \times G$ shows that $Gr[\mathcal{O}(\tilde{G}_\Bbbk)]$ has no zero divisors, and then the same is true for $\mathcal{O}(\tilde{G}_\Bbbk)$. So over a field $\Bbbk$, $\tilde{G}_\Bbbk$ is an algebraic group.

7.4.1. *Lemma.* $\tilde{G}_\mathbb{Z}$ is smooth.

Proof. $\tilde{G}_\mathbb{Z}$ is flat over $\mathrm{Spec}(\mathbb{Z})$ since $\mathcal{O}(\tilde{G}_\mathbb{Z})$ is a free $\mathbb{Z}$-module. So smoothness is equivalent to the smoothness of the fibers over closed points, but these are algebraic groups.

7.4.2. *Theorem.* $\tilde{G}_\mathbb{Z} \cong \check{G}_\mathbb{Z}$.

Proof. $\tilde{G}_\mathbb{Z}$ is a reductive group since $\tilde{G}_\Bbbk$ is for any closed field $\Bbbk$ by

7.5. **Lemma.** reduced and

7.6. **Lemma.** ident. $\tilde{G}_\mathbb{Z}$ is split since for all dominant λ the Weyl module has a $\tilde{G}_\mathbb{Z}$-form.

8. **Duality**

Verdier duality corresponds to the composition of the duality for $\tilde{G}$-modules with the involution $-w_0 \in \mathrm{Aut}(\tilde{G})$ (more precisely, a representative of this involution is chosen canonically by our fixed choice of the pinning $\tilde{\mathfrak{b}}, \tilde{\mathfrak{h}}, \tilde{e}$ - the last one is the regular nilpotent given by the hyperplane section.

9. **Appendix A. Semi-small maps**

A will denote some coefficient ring (commutative, with a unit, noetherian and of finite global dimension).

9.1. **Semi-small maps.** A map $\pi : X\to Y$ defines constructible subsets $Y_k(X) \stackrel{\mathrm{def}}{=} \{y \in Y, \dim(X \cap \pi^{-1}y) = k\}$ and $X_k \stackrel{\mathrm{def}}{=} X \cap \pi^{-1}Y_k(X),\ k \in \mathbb{N}$. For any real number k we denote $X_{\geq k} \stackrel{\mathrm{def}}{=} \underset{p\geq k}{\cup} X_p$, and $Y_{\geq k}(X) \stackrel{\mathrm{def}}{=} \underset{p\geq k}{\cup} Y_p(X)$. We use the same notation for any subvariety $S\subseteq X$ and the restriction of the map to S.

Map $\pi : X\to Y$ is said to be **dimensionally semi-small** if X is irreducible and $\mathrm{codim}_X X_{\geq k} \geq k,\ k \in \mathbb{N}$.

An irreducible subvariety S of X is said to be dimensionally semi-small relative to the map $\pi : X\to Y$ if $\pi|S$ is dimensionally semi-small. An A-Stratification $\mathcal{S}$ of X is said to be dimensionally semi-small relative to the map π if all strata satisfy this property. We say that $\mathcal{S}$ is **semi-small** relative to π if it is dimensionally semi-small and for any stratum $S \in \mathcal{S}$, restriction $\pi|\bar{S}$ is proper.

9.1.1. **Lemma.** (a) If X has a stratification dimensionally semi-small for $\pi : X\to Y$, then π is dimensionally semi-small.

(b) Let $\pi : X\to Y$ be dimensionally semi-small. Any irreducible subvariety S of X which is "transversal" to the filtration $X_{\geq k}$ of X in the sense that

$$\mathrm{codim}_S S \cap X_{\geq k} \geq k,\ k \in \mathbb{N};$$

is dimensionally semi-small for π and meets X_0.

(c) For a dimensionally semi-small $S\subseteq X$ one has $\dim Y_{\geq k}(S) \leq \dim S - 2k$, $k \in \mathbb{R}$.

Proof. (a) If $\mathcal{S}$ is a stratification dimensionally semi-small for π, then

$$\dim X_k = \underset{S\in\mathcal{S}}{max}\ \dim S \cap X_k \leq \underset{S\in\mathcal{S}}{max}\ \dim S - k \leq \dim X - k.$$

(b) Since $S_k \subseteq \underset{p\geq k}{\cup} S \cap X_p$, transversality condition above implies that for $k \in \mathbb{N}$,

$$\dim S_k \leq \underset{p\geq k}{max}\ \dim S \cap X_p \leq \underset{p\geq k}{max}\ \dim S - p = \dim S - k;$$

hence $\pi|S$ is dimensionally semi-small. S meets X_0 since $S - X_0 = S \cap X_{\geq 1}$ has codimension ≥ 1.

(c) For $k \in \mathbb{N}$, $\dim Y_k(S) \leq \dim S_k - k \leq \dim S - 2k = \dim \pi(S) - 2k$, hence also $\dim Y_{\geq k}(S) \leq \dim \pi(S) - 2k$. The same claim for real numbers $k \geq 0$ follows.

9.1.2. **Lemma.** (a) Let $\mathcal{S}$ be a stratification of X, dimensionally semi-small for the map $\pi : X \to Y$. Then the functor $\pi_! : D_{\mathcal{S}}(X, A) \to D(Y, A)$ is right exact (i.e. it preserves ${}^pD^{\leq 0}$), and $\pi_* : D_{\mathcal{S}}(X, A) \to D(Y, A)$ is left exact (preserves ${}^pD^{\geq 0}$).

(b) If $\mathcal{S}$ is semi-small for $\pi : X \to Y$, direct image preserves perversity, i.e.

$$\pi_* : \mathcal{P}_{\mathcal{S}}(X, A) \to \mathcal{P}(Y, A).$$

Proof. (b) is a consequence of (a). Part (a) is the claim that for any perverse sheaf $\mathcal{F}$ in $\mathcal{P}_{\mathcal{S}}(X, A)$, one has $\pi_!\mathcal{F}$ is in $D^{\leq 0}(Y)$ and $\pi_*\mathcal{F}$ is in $D^{\geq 0}(Y)$, i.e. for any integer d, constructible sets

$$Y_d^*(\mathcal{F}) = \{y \in Y,\ H^i(\pi_!\mathcal{F})_y \neq 0 \text{ for some } i \geq -d\}$$

and

$$Y_d^!(\mathcal{F}) = \{y \in Y,\ H^i(\pi_*\mathcal{F})_y^! \neq 0 \text{ for some } i \leq d\},$$

have dimension $\leq d$.

Let y be a point in Y and S a stratum in $\mathcal{S}$. Denote

$$\begin{array}{ccccc} S_y \overset{\text{def}}{=} p^{-1}y & \xrightarrow{\rho} & S & \xrightarrow{k} & X \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle \pi}\downarrow \\ y & \xrightarrow{i} & Y & = & Y. \end{array}$$

Any $\mathcal{F}$ in $\mathcal{P}_{\mathcal{S}}(X, A)$ is an extension of the sheaves $k_!k^*\mathcal{F}$, $S \in \mathcal{S}$; hence $(\pi_!\mathcal{F})_y$ is an extension of the terms $(\pi_!(k_!k^*\mathcal{F}))_y$, corresponding to the various strata S in X. So $Y_d^*(\mathcal{F}) \subseteq \bigcup_{S \in \mathcal{S}} Y_d^*(k_!k^*\mathcal{F})$ and similarly $Y_d^!(\mathcal{F}) \subseteq \bigcup_{S \in \mathcal{S}} Y_d^!(k_*k^!\mathcal{F})$. So it suffices to see that

$$Y_d^*(k_!k^*\mathcal{F}) \subseteq Y_{\geq \frac{1}{2}(\dim S - d)}(S) \supseteq Y_d^!(k_!k^*\mathcal{F}),$$

the estimates we need will then follow from

$$\dim Y_{\geq \frac{1}{2}(\dim S - d)}(S) \leq \dim S - 2\frac{1}{2}(\dim S - d) = d.$$

To check the first inclusion we estimate the stalk at a point y in Y. The base change gives

$$[\pi_!(k_!k^*\mathcal{F})]_y = [p_!k^*\mathcal{F}]_y = H_c^*(S_y, k^*\mathcal{F}).$$

Since $\mathcal{F}$ is perverse and $\mathcal{S}$-constructible, $k^*\mathcal{F}$ is in the degrees $\leq -\dim S$, hence $H_c^*(S_y, k^*\mathcal{F})$ is in the degrees $\leq -\dim S + 2\dim S_y$. So $y \in Y_d^*(k_!k^*\mathcal{F})$ implies $-d \leq -\dim S + 2\dim S_y$, i.e. $\dim S_y \geq \frac{1}{2}(\dim S - d)$.

The costalk at $y \in Y$ is

$$[\pi_*(k_* k^! \mathcal{F})]^!_y \overset{\text{def}}{=} i^! [p_* k^! \mathcal{F}] = q_* \rho^! (k^! \mathcal{F}) = H^*[S_y, (k\rho)^! \mathcal{F}].$$

Since $\mathcal{F}$ is in $\mathcal{P}_{\mathcal{S}}(X, A)$, $\mathcal{L} \overset{\text{def}}{=} k^! \mathcal{F}$ is a smooth complex of sheaves on S concentrated in the degrees $\geq -\dim S$, and $\rho^! \mathcal{L}$ is therefore in the degrees $\geq -\dim S + 2 \cdot \text{codim}_S S_y$. [To check this, choose a stratification $\mathcal{C}$ of S_y. Then $\rho^! \mathcal{L}$ is an extension of all $\sigma_* \sigma^! (\rho^! \mathcal{L})$ for the strata $C \overset{\sigma}{\subset} S_y$ in $\mathcal{C}$. Since $\mathcal{L}$, S and C are smooth, $(\rho\sigma)^! \mathcal{L}$ is in the degrees of $\mathcal{L}$ shifted up by $2 \cdot \text{codim}_S C \geq 2 \cdot \text{codim}_S S_y$, i.e. in the degrees $\geq -\dim S + 2 \cdot \text{codim}_S S_y$.]

For $y \in Y^!_d(k_* k^! \mathcal{F})$, we get $d \geq -\dim S + 2 \cdot \text{codim}_S S_y = \dim S - 2 \dim S_y$, hence again $\dim S_y \geq \frac{1}{2}(\dim S - d)$.

9.1.3. **Lemma.** Let $\mathcal{S}$ be a stratification of X and let $\pi : X \to Y$ be a proper map. Then the following is equivalent:

(a) $\mathcal{S}$ is π-semi-small,

(b) $\pi_* : \mathcal{P}_{\mathcal{S}}(X, A) \to \mathcal{P}(Y, A)$,

(c) $\pi_* IC(S, A)$ is perverse for each stratum $S \in \mathcal{S}$.

In particular, validity of (b) or (c) is independent of the the choice of the ring A.

Proof. We only have to show that if for some choice of the coefficient ring A one has

$$(*) \qquad \pi_* IC(S, A) \in \mathcal{P}(Y, A) \text{ for all strata } S \in \mathcal{S},$$

then any stratum S in $\mathcal{S}$ is dimensionally semi-small for π.

Partition $\partial S \overset{i}{\hookrightarrow} \bar{S} \overset{j}{\hookleftarrow} S$ of $\bar{S}$, gives a Mayer-Vietoris triangle for the sheaf $\mathcal{L} = IC(S, A)$:

$$j_!(\mathcal{L}|S) \to \mathcal{L} \to i_* i^* \mathcal{L}.$$

We apply π_* and get another exact triangle

$$\pi_* \ j_!(\mathcal{L}|S) \to \pi_* \mathcal{L} \to \pi_* \ i_* i^* \mathcal{L}.$$

Observe that the restriction $\pi_S \overset{\text{def}}{=} \pi|\partial S : \partial S \to Y$ is proper and with the stratification $\mathcal{S}|\partial S$ of $\partial S = \bar{S} - S$, it satisfies $(*)$. Therefore, by an induction assumption $\mathcal{S}|\partial S$ is semi-small for π_S. Now, since $\mathcal{L}|\partial S$ is in strictly negative perverse degrees, so is (by the preceding lemma) $\pi_*(\mathcal{L}|\partial S)$. By the assumption $(*)$, $\pi_* \mathcal{L}$ is in perverse degrees zero, so from a triangle above we deduce that $\pi_* j_!(\mathcal{L}|S)$ is in the non-negative perverse degrees.

Therefore, for any integer d one has $\dim \mathcal{Y}(d) \leq d$ for the constructible set $\mathcal{Y}(d)$ consisting of all $y \in Y$, such that for some $i \geq -d$ cohomology group

$$H^i[\pi_* \ j_!(\mathcal{L}|S)]_y = H^i_c(S \cap \pi^{-1} y, A[\dim S])$$

does not vanish. However, for $y \in \pi(S_k)$, the highest nontrivial cohomology is in the degree $-\dim S + 2 \cdot \dim S \cap \pi^{-1} y = -\dim S + 2k$. So $\pi(S_k) \subseteq \mathcal{Y}(\dim S - 2k)$ and therefore $\dim S_k \leq \dim \pi(S_k) + k \leq \dim S - 2k + k = \dim S - k$.

9.2. **Remarks.** **(1)** The proof of the lemma actually proves the following.

For a stratification $\mathcal{S}$ of X and a map $\pi : X \to Y$, the following is equivalent:
(a) $\mathcal{S}$ is π dimensionally semi-small for π,
(b) $\pi_!$ is right exact on $D_{\mathcal{S}}(X, A)$, i.e. $\pi_! : \mathcal{P}_{\mathcal{S}}(X, A) \to D^{\leq 0}(Y, A)$,
(c) $\pi_! IC(S, A)$ is in $D^{\leq 0}(Y, A)$ for each stratum $S \in \mathcal{S}$.

(2) Implications 2⇒3 and 3⇒2 are proved in lemmas 3 and 4, however, one can probably prove both at the same time?

(3) The converse in lemma 2a holds, but it is not true that for a semi-small π any stratification $\mathcal{S}$ which is a refinement of $X = \cup X_k$ is semi-small: $\mathcal{S} = \{X_0, X_1\}$ for $X = \tilde{\mathcal{N}} \to \mathcal{N} = Y$ in sl_2 is a counterexample.

9.2.1. *Lemma.* If $\pi : X \to Y$ is semi-small there is a semi-small stratification $\mathcal{S}$ of X, hence in particular, $\pi_* IC(Y)$ is perverse.

9.3. **Functor $\pi_{!*}$.**

9.3.1. *Dimensionally semi-small maps.* Let $\mathcal{S}$ be a stratification of X, dimensionally semi-small for $\pi : X \to Y$. On $\mathcal{P}_{\mathcal{S}}(X, A)$, map of functors $\pi_! \mathcal{A} \to \pi_* \mathcal{A}$ factors into a diagram (self-dual if A is a field)

$$\pi_! \to {}^pH^0 \pi_! \xrightarrow{\iota} {}^pH^0 \pi_* \to \pi_*,$$

since $\pi_!$ maps into $D^{\leq 0}(Y, A)$ and π_* into $D^{\geq 0}(Y, A)$. We define a functor (self-dual if A is a field)

$$\pi_{!*} : \mathcal{P}_{\mathcal{S}}(X, A) \to \mathcal{P}(Y, A), \quad \pi_{!*} \stackrel{\text{def}}{=} Im[{}^pH^0 \pi_! \xrightarrow{\iota} {}^pH^0 \pi_*].$$

9.3.2. *Lemma.* For mixed sheaves, $\pi_{!*}$ preserves weight, hence it satisfies the decomposition theorem. If $\mathcal{T}$ is a stratification of Y that makes π into a stratified map, then for a local system $\mathcal{L}$ on a stratum $S \in \mathcal{S}$

$$\pi_{!*} IC(S, \mathcal{L}) \cong \bigoplus_{T \in \mathcal{T}_S} IC(T, \mathcal{L}_T),$$

here $\mathcal{T}_S \subseteq \mathcal{T}$ consists of S-relevant strata, i.e, such that the fiber $S_y = \pi^{-1} y \cap S$ has dimension $\frac{dim(S) - \dim(T)}{2}$ (maximal possible), and the stalk of the local system $\mathcal{L}_T$ at $y \in T$ has a canonical basis consisting of all irreducible components of S_y of maximal dimension.

9.3.3. *Lemma.* If $\pi : X \to Y$ is dimensionally semi-small there is a semi-small stratification $\mathcal{S}$ of X. Therefore, $\pi_! IC(Y) \in D^{\leq 0}(Y, A)$, $\pi_* IC(Y) \in D^{\geq 0}(Y, A)$, and $\pi_{!*} IC(Y)$ is defined and perverse.

9.3.4. *Lemma.* $(\pi\tau)_{!*} \cong \pi_{!*} \tau_{!*}$.

9.4. **f-semi-small maps.** In this section we will weaken the above definitions to allow for a generic fiber to have dimension $f_{X\to Y}\stackrel{\text{def}}{=}\dim X-\dim\pi(X)$.

9.4.1. *f-semi-small maps.* Map $\pi:X\to Y$ is said to be **f-semi-small** if X is irreducible, $f=f_{S\to Y}$ and

$$\text{codim}_X X_{\geq k+f}\geq k,\ \ k\in\mathbb{N},$$

i.e.,

$$\text{codim}_X X_{\geq p}\geq p-f,\ \ k\in\mathbb{N}.$$

An irreducible subvariety S of X is said to be f-semi-small relative to the map $\pi:X\to Y$ if $\pi|S$ is f-semi-small. An A-Stratification $\mathcal{S}$ of X is said to be f-semi-small relative to the map π if all strata satisfy this property (so denote $f_{S\to Y}$ by $f_{\mathcal{S}\to Y}$).

9.4.2. **Lemma.** (a) Let $\mathcal{S}$ be a stratification of X, f-semi-small for the map $\pi:X\to Y$. Then $\pi_!:\ \mathcal{P}_{\mathcal{S}}(X,A)\to {}^pD^{\leq f}(Y,A)$, and $\pi_*:\ \mathcal{P}_{\mathcal{S}}(X,A)\to {}^pD^{\geq -f}(Y,A)$.

(b) Suppose that moreover, $\mathcal{T}$ is a stratification of Y such that $\pi:(X,\mathcal{S})\to(Y,\mathcal{T})$ is a stratified map. For a stratum $S\in\mathcal{S}$ consider the highest perverse cohomology ${}^pH^f[\pi_!\ \mathcal{I}_!(S)]$ of the !-direct image of the standard !-sheaf for S. It has a filtration such that the subquotients are the perverse sheaves $\mathcal{I}_T,\ T\in\mathcal{T}$, of the following form.

Let $\mathcal{L}_T$ be the local system on T with stalks $(\mathcal{L}_T)_y=\ \Bbbk[Irr(S_y)]$, ($Irr(S_y)$ denotes the set of all irreducible components of the fiber S_y of $\pi|S$ at y, that have the (maximal possible) dimension). Then $\mathcal{I}_T$ is between the standard !-sheaf and the IC-sheaf: $\mathcal{I}_!(T,\mathcal{L}_T)\to\!\!\to \mathcal{I}_T\to\!\!\to\mathcal{I}_{!*}(T,\mathcal{L}_T)$.

(c) Suppose moreover, that π is proper. Then ${}^pH^f[\pi_!\ IC(S,\Bbbk)]\ \cong\ \oplus_{T\in\mathcal{T}}\ IC(T,\mathcal{L}_T)$. Similarly for ${}^pH^{-f}(\pi_*IC(S,\Bbbk))$.

9.4.3. *Sublemma.* (a) Divide X into an open and a closed part $U\stackrel{j}{\hookrightarrow}X\stackrel{i}{\hookleftarrow}Y=X-U$. For $\mathcal{A}\in\mathcal{P}_{\mathcal{S}}(X)$ one has

$$0\to Im[{}^pj_!(\mathcal{A}|U)\to\mathcal{A}]\to\mathcal{A}\to i_*({}^pi^*\mathcal{A})\to 0,$$

and $\mathcal{I}=Im[{}^pj_!(\mathcal{A}|U)\to\mathcal{A}]$ is between ${}^pj_!(\mathcal{A}|U)$ and $j_{!*}(\mathcal{A}|U)$: ${}^pj_!(\mathcal{A}|U)\to\!\!\to\mathcal{I}\to\!\!\to j_{!*}(\mathcal{A}|U)$.

(b) For $\mathcal{A}\in\mathcal{P}_{\mathcal{T}}(Y)$ and a stratum $T\in\mathcal{T}$, $\mathcal{A}_T=\ H^{-\dim T}(\mathcal{A}|T)$ is a local system on T. Perverse sheaf $\mathcal{A}$ has a filtration with $Gr\mathcal{A}\cong\ \oplus_{T\in\mathcal{T}}\mathcal{I}_T$ where $\mathcal{I}_T$ is between ${}^pj_!(\mathcal{A}_T)$ and $j_{!*}(\mathcal{A}_T)$.

Proof. (a) Since the canonical map ${}^pj_!(\mathcal{A}|U)\to\mathcal{A}$ is an identity over U, the cokernel $\mathcal{C}$ is supported on Y and the image $\mathcal{I}$ is between ${}^pj_!(\mathcal{A}|U)$ and $j_{!*}(\mathcal{A}|U)$. Therefore ${}^pi^*(\mathcal{I})$ is a quotient of ${}^pi*({}^pj_!(\mathcal{A}|U))=0$, and this implies that $i_*({}^pi^*\mathcal{A})\stackrel{\cong}{\to} i_*({}^pi^*\mathcal{C})=\mathcal{C}$.

(b) Clearly, $\mathcal{A}_T = {}^pH^0\ (T \overset{i_T}{\hookrightarrow} Y)^* \mathcal{A} = {}^p i_T{}^*\ \mathcal{A}$. Replace Y with the support of $\mathcal{A}$, let T be an open stratum and $Y - T \overset{k}{\hookrightarrow} Y$. then (a) gives $0 \to \mathcal{I}_T \to \mathcal{A} \to k_*({}^pk^*\mathcal{A}) \to 0$ for $\mathcal{I}_T = Im[{}^p(i_T)_!(\mathcal{A}_T) \to \mathcal{A}]$. Since for a stratum $S \subseteq Y - T$ one has ${}^p(S \hookrightarrow Y_T)^*\ {}^pk^*\mathcal{A} = {}^p(S \hookrightarrow Y)^*\mathcal{A}$ we have reduced the number of the strata.

9.4.4. *Fast Proof of the lemma.* (a) follows by the kind of estimates that were used in the basic lemma for semi-small maps. (b) follows from the sublemma by observing that for $\mathcal{A} = {}^pH^f[\pi_!\ \mathcal{I}_!(S)]$ one has $\mathcal{A}_T = \mathcal{L}_T$ - the estimate in (a) for the integral of $\mathcal{I}_!(S)$ over the intersection of the fiber with a stratum different from S is better by two then for arbitrary perverse sheaves, hence only stratum S contributes and others do not subtract.

The proof of (c) is the same - $\mathcal{I}_{!*}(S)$ has estimate better by only 1, so only S contributes, others do not subtract since the exact sequences split by the decomposition theorem (?).

9.4.5. *Remarks.* (a) One way f-semi-small maps arise is by restricting a semi-small map to closed subvariety of the target.

(b) It is not true that ${}^pH^f[\pi_!\ \mathcal{I}_!(S)]$ has a filtration by standard !-sheaves. Let X be a minimal resolution of the affine quadric $xy = uv$, and $\mathcal{S} = \{X\}$ while $\mathcal{T}$ consists of the singular point p and $T = Y - p$. Then $\pi_! IC_!(X) = \pi_! \mathcal{C}_X[3] = IC(Y)$ while $\mathcal{I}_!(T) = \mathbb{C}_T[3]$ is larger: $0 \to IC(Y) \to \mathcal{I}_!(T) \to \mathbb{C}_p \to 0$.

9.5. **Small stratified maps.** Let us consider two complex stratified spaces $(Y, \mathcal{T})$ and $(X, \mathcal{S})$ and a map $f : Y \to X$. We assume that the two stratifications are locally trivial with connected strata and that f is a stratified with respect to the stratifications $\mathcal{T}$ and $\mathcal{S}$, i.e., that for any $T \in \mathcal{T}$ the image $f(T)$ is a union of strata in $\mathcal{S}$ and for any $S \in \mathcal{S}$ the map $f|f^{-1}(S) : f^{-1}(S) \to S$ is locally trivial in the stratified sense. We say that f is a stratified semi-small map if

a) for any $T \in \mathcal{T}$ the map $f|\bar{T}$ is proper

b) for any $T \in \mathcal{T}$ and any $S \in \mathcal{S}$ such that $S \subset f(\bar{T})$ we have

$$\dim(f^{-1}(x) \cap \bar{T}) \leq \frac{1}{2}(\dim f(\bar{T}) - \dim S)$$

for any (and thus all) $x \in S$.

Next the notion of a small stratified map. We say that f is a small stratified map if there exists a (non-trivial) open stratified subset W of Y such that

a) for any $T \in \mathcal{T}$ the map $f|\bar{T}$ is proper

b) the map $f|W : W \to f(W)$ is proper and has finite fibers

c) for any $T \in \mathcal{T}$, $T \subset W$, and any $S \in \mathcal{S}$ such that $S \subset f(\bar{T}) - f(T)$

we have $\dim(f^{-1}(x) \cap \bar{T}) \leq \frac{1}{2}(\dim f(\bar{T}) - \dim S)$

for any (and thus all) $x \in S$.

The result below follows directly from dimension counting:

9.5.1. *Lemma.* If f is a semi-small stratified map then $Rf_*A \in \mathcal{P}_{\mathcal{S}}(X, \Bbbk)$ for all $A \in \mathcal{P}_{\mathcal{T}}(Y, \Bbbk)$. If f is a small stratified map then, with any W as above, and any $A \in \mathcal{P}_{\mathcal{T}}(W, \Bbbk)$, we have $Rf_* j_{!*}A = \tilde{j}_{!*} f_* A$, where $j : W \hookrightarrow Y$ and $\tilde{j} : f(W) \hookrightarrow X$ denote the two inclusions.