1. SEMIINFINITE FLAGS. I. CASE OF GLOBAL CURVE $\mathbb{P}^1$
♡
2. SEMIINFINITE FLAGS. II. LOCAL AND GLOBAL INTERSECTION COHOMOLOGY OF QUASIMAPS' SPACES

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Part 1. Semiinfinite flags. I. Case of global curve $\mathbb{P}^1$

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Summary

0.1. **Chapter 1. The spaces of (based) quasimaps into a flag variety.** The basic objects are the spaces $\mathcal{Q}^\alpha_C \subseteq \mathcal{Z}^\alpha_C$ of (based) quasimaps from a curve C to the flag manifold $\mathcal{B}$. We first define these for a smooth projective curve C and later we introduce their "quasimap" versions $\mathbf{Q} = \mathcal{Q}_d$ and $\widetilde{\mathbf{Q}} = \mathcal{Q}_{d^*}$ for the local curves d, d^*.

For based quasimaps $\mathcal{Z}_C$ we also introduce the Plucker model $\mathfrak{Z}$ and the global Grassmannian model $\mathbf{Z} \subseteq \mathcal{G}_{\mathcal{H}_C}$ called zastava space. The three models coincide when $C = \mathbb{A}^1$.

The quasimap model readily extends to affine groups. The zastava model has an evident local property and it is the most general one. The data to define the zastava space are a central extension T_Q of the loop group $T_\mathcal{K}$ of a torus T and a trivialization $T \xrightarrow{\cong} {G_m}^I$. The plucker model is the least abstract and convenient for calculations.

Here is a more detailed summary of the first chapter taken from 2.0.

In section 3 we define the spaces $\mathcal{Q}^\alpha \subseteq \mathcal{Z}^\alpha$ of (based) quasimaps from a curve C to a flag manifold. We define the shifts of quasimaps by finite subschemes. The basic stratification of quasimaps is by shifts of maps. The moduli of maps are smooth and all quasimaps are shifts of maps.

In section 4 we introduce the local versions $\mathbf{Q} \subseteq \widetilde{\mathbf{Q}}$ of quasimap spaces for the local curves d and d^*. (These we call "quasimaps". Precisely, these spaces are certain torsors (for groups $H_\mathcal{O}$ and H_{d^*}) over the true local quasimap stacks, which happen to suffice for our purposes.) We also define the restriction map (a closed embedding) $\mathcal{Q} \hookrightarrow \mathbf{Q}$ of global $\mathbb{P}^1$-quasimaps into local d-"quasimaps".

In section 5 we introduce the Plucker model $\mathfrak{Z}^\alpha$ of space of based quasimaps.

In section 6 we introduce the *zastava* model $\mathbf{Z}^\alpha$ of based quasimaps, in terms of the global loop Grassmannian. In this model the locality property of based quasimaps is obvious.[1]

Finally, in section 7 we explain the equivalence of three constructions of space of based quasimaps, which we now call *zastava spaces.*

0.2. **Chapter 2. The category $\mathcal{PS}$ of semiinfinite perverse sheaves on ...** For the curve $C=\mathbb{P}^1$ with the distinguished points $a=0$ and $b=\infty$ we define the progressively finer stratifications

$$\underline{\underline{\mathcal{S}}}_0 \ \le\ \underline{\underline{\mathcal{S}}}_{0,G_m}\ \le \underline{\underline{\mathcal{S}}}_{0,G_m,\mathbf{I}}$$

called coarse, fine and Schubert stratifications. We formulate a conjecture on smoothness of Schubert strata and establish it for the "sufficiently dominant" strata.[2]

0.3. **Chapter 3. Convolution with affine Grassmannian.**

A. Questions

0.4. **The doubling from $\mathcal{G}$ to $\mathcal{Q}$.** This is the result of the second paper SF2.

It is really interesting (impressive) how a "brutal" realization of $U\check{\mathfrak{n}}$ on $\mathcal{G}$ (using Ext of standard semiinfinite sheaves) becomes a sophiticated realization on $\mathcal{Q}$ (Ext of IC-scheaves).

Question. What is the mechanism for this? (We know a proof of the fact and it should lead us us.)

0.4.1. *Rouquier categorigication of the doubling?* To start with, where is $U\check{\mathfrak{g}}$?

0.4.2. *Extending the correspondence construction of e_i, d_i actions to all types?*

0.5. **Twisted quasimaps as C-points.**

0.5.1. *The notion of* P*-twisted maps into flag variety as C-points of the* P*-flag variety* P/B*.* A G-torsor P on C has a flag variety $\mathsf{P}/B\cong \mathsf{P}\times_G\mathcal{B} =\ \mathsf{P}\mathcal{B}$. The notion of twisted maps from C to ${}^{\mathsf{P}}\mathcal{B}$ is really the notion of C-points of the P-flag variety P/B.

Example. When C is a point the moduli of C-points of $\mathcal{B}$ is $\mathbb{B}$ itself.

[1] Locality property implies that the construction is of local nature so that we do not need $\mathbb{P}^1$ but only $\mathcal{H}_C$!

[2] The case of sufficiently dominant strata suffices for all of our purposes. However, the general conjecture would make the arguments more natural.

Remark. The connected components of C points in $\mathcal{B}$ are parametrized by the positive (effective) cone $\mathbf{N}[\mathbf{I}]$ in $H_2(\mathcal{B},\mathbb{Z})$. However, for C-points of ${}^{\mathsf{P}}\mathcal{B}$ we get another cone $H^2(\mathcal{B},\mathbb{Z})_{\mathsf{P}}$ in $H^2(\mathcal{B},\mathbb{Z})$.

Question. What does this cone depend on. For T-fixed points $\mathcal{T}=\mathbf{L}_\eta$ is this something like $eta+\mathbf{N}[\mathbf{I}]$ when $\mathcal{T}\in\mathcal{G}_\eta$? So, it is not constant on GO-orbits in $\mathcal{G}$. What about Iwahori or semiinfinite orbits?

Question. Could this be a geometric characterization of some class of orbits?

0.5.2. *Quasimaps.* This involves the compactification $\overline{\mathcal{B}}\stackrel{\text{def}}{=}[(G/N)^{aff}]/H$ of the flag variety. Then the P-twisted quasimaps are the generic C-points of ${}^{\mathsf{P}}\overline{\mathcal{B}}\stackrel{\text{def}}{=}(\mathsf{P}/N)^{aff}/H$. The non-generic points seems to be discarded because they are stacky.

Remark. However, if we would like to consider maps into the spaces of C-points, we would again need to reintroduce the stacky part.

0.5.3. *Drinfeld compactification $\overline{\mathcal{B}}=[(G/N)^{aff}]/H$ of $\mathcal{B}$.* One would like to explain this better.

It is related to

0.5.4. *The semigroup closure $\overline{H}$ of H.* I do nor remember when does it exist?

It can be defined as the closure of $H=B/N$ in $\overline{\mathcal{B}}$.

0.5.5. *Affinization.* It does the job but it has not been said well how that happens.

Question. Once we are in $Y=G/N$ it is natural (say, "algebro-geometric" and functorial) to complete it to its affinization Y^{aff}. However, what is geometrically involved in completing a qusiaffine scheme Y into an affine Y^{aff}?

0.5.6. *The mechanism of "P-torsors as a correspondence between torsors for G and for a quotient of P".* This appears in SF3.

0.6. **Vinberg.** The Vinberg semigroup seems essential for the SF1 paper since one constructs restriction ("convolution") functors $\mathcal{P}(\overline{\mathcal{G}_\eta},I)\xrightarrow{\mathbf{c}}\mathcal{PS}$ for one η at the time.

0.6.1. *The η-untwisting map.* It is defined for quasimaps twisted by $\mathcal{T}$ from $\overline{\mathcal{G}_\eta}$. Remembering η with $\overline{\mathcal{G}_\eta}$ is clearly the feature of the loop Grassmannian of the the Vinberg semigroup V_G.

0.6.2. *Compatibility in η.* For $\eta' \leq \eta$ we have $\overline{\mathcal{G}_{\eta'}} \subseteq \overline{\mathcal{G}_\eta}$ and

0.7. **The convolution functor in A_1.** Here, G-torsor P twists the representation $V = L(1)$ of G to a rank two vector bundle over C and ${}^{\mathsf{P}}\mathcal{B}$ is the corresponding twisted $\mathbb{P}^1$-bundle $\mathbb{P}({}^{\mathsf{P}}V)$.

Over $C = \mathbb{P}^1$ this means that P is isomorphic to one of $\mathbb{P}^1$-bundles F_n over $\mathbb{P}^1$. The P-wisted quasimaps are the quasisections of F_n.

Without a twist the degree of a map into $\mathbb{P}^1$ appears as d such that $\mathcal{B} = \mathbb{P}^1$ is represented as $\mathcal{P}(V \otimes \mathcal{O}_C(n)$. For a torsor P induced from a Cartan subgroup T we have ${}^{\mathsf{P}}\mathcal{B} = \mathcal{P}[\mathcal{O}_C(a) \oplus \mathcal{O}_C(b)]$. What is the degree?

Remark. The spaces of quasimaps into $\mathbb{P}^1$ were the odd projective spaces. Now we get all projective spaces as $\mathbb{P}[\Gamma\big(C,\ \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)\big) = \mathbb{P}[L(a) \oplus L(b)]$.

0.7.1. *Dependence on the opposite stratification by the type of the G-torsor.* It appears as the G-torsor is used to twist the flag variety!

For isomorphic P_i the twisted flag varieties ${}^{\mathsf{P}_i}\mathcal{B}$ will be isomorphic. So, the their theories of quasimaps will again be isomorphic.

0.8. **Richardson varieties.** We are tensoring IC sheaves The use of $\mathcal{G}_\lambda$ and $\mathcal{G}^\lambda$ (here incarnated as $\widehat{\mathcal{G}}^\lambda$ seems to be drawing in the Richardson varieties. Moreover, the quasimaps themselves – or at least the closed Zastava spaces – are again the semiinfinite Richardson setting.

So, we may be having a double Richardson here?

Question. Is there a Fourier transform here alike MUV? Or a horocycle transform?

Question. Is Poisson structure a feature of Richardson? One would want the Richardson cells to be exactly the symplectic leaves. This is precisely true on $\mathcal{G}$ for $G_{\mathcal{O}}$-orbits and their duals.

0.9. **The convolution (restriction) functor c.** How does it see the difference of a sub or a quotient $0 \to V' \to V \to V'' \to 0$?

0.9.1.

Question. The open zastava space gives the based quasimap spaces. Can one see the whole quasimap spaces in this group theoretic picture?

0.10. **What happens in FS1 when one replaces the open zastava space with the closed one?** Seemingly one recovers the global Grassmannian. This was also my model of 2d Grassmannian at the critical level.

This seems to be the key example for my semiinfinite constructions.

0.11. **Parameter $\chi \in Y$.** We could define more zastava subspaces of the global Grassmannian rather than define fewer and then formally add parameter χ.

Notation List.

- Section **2.** Groups and configurations of points on a curve.
  (1) $G,\ W_f,\ V_\lambda$ and $\mathbf{X} = \mathcal{B}$.
  (2) $\mathcal{H}_C \supseteq \mathcal{H}_C^\alpha = C^\alpha$.
- Section **3.** Quasimaps.
  (1) The based quasimaps $\mathcal{Z}^\alpha = QMap[(\mathbb{P}^1,\infty),(\mathcal{B},\mathfrak{g}_-)] = Q\Gamma[(\mathbb{P}^1,\infty),(\mathcal{B},\mathfrak{g}_-)]$ contains zzz maps $\overset{\circ}{\mathcal{Z}}{}^\alpha = Map[(\mathbb{P}^1,\infty),(\mathcal{B},\mathfrak{g}_-)] = \Gamma[(\mathbb{P}^1,\infty),(\mathcal{B},\mathfrak{g}_-)]$.
  (2) The quasimap $\mathcal{Q}^\alpha = QMap(\mathbb{P}^1,\mathcal{B}) = Q\Gamma(\mathbb{P}^1,\mathcal{B})$ contain maps $\overset{\circ}{\mathcal{Q}}{}^\alpha = Map(\mathbb{P}^1,\mathcal{B}) = \Gamma(\mathbb{P}^1,\mathcal{B})$.
- Section **4.** Local flag space $\mathbf{Q}$.
  (1) Instead of the true space $\mathcal{Q}(d)$ of d-quasimaps we consider first an $(H_a)_\mathcal{O}$-torsor $\mathcal{S}$ over $\mathcal{Q}(d)$ called the "vector d-quasimaps" and then the space of "line d-quasimaps" $\mathbf{Q} = \mathcal{S}/H_a$ which is a torsor over $\mathcal{Q}(d)$ for the congruence subgroup $K(H_a)$ of $(H_a)_\mathcal{O}$.
  The space of line d-quasimaps is also called "local flag space $\mathbf{Q}$".
  (2) Instead of the true space $\mathcal{Q}(d^*)$ of d^*-quasimaps we consider first an $(H_a)_\mathcal{K}$-torsor $\widetilde{\mathcal{S}}$ over $\mathcal{Q}(d^*)$ called the "vector d^*-quasimaps" and then the space of "line d^*-quasimaps" $\widetilde{\mathbf{Q}} = \mathcal{S}/H_a$ which is a torsor over $\mathcal{Q}(d^*)$ for $(H_a)_\mathcal{K}/H_a$.
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- Section **5.** Plucker models
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- Section **6.** Zastava space
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- Section **7.** Equivalence of three constructions of zastava spaces.
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- Section **8.** Schubert stratification of $\mathcal{Z}^\alpha \subseteq \mathcal{Q}^\alpha$.
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- Section **9.** Category $\mathcal{PS}$ of perverse sheaves on $\mathcal{Z}$.
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- Section **10.** Plucker models, twisted quasimaps and parity of stalks of IC sheaves
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- Section **11.** Convolution correspondence $\mathcal{G}\mathcal{Q}^\alpha_\eta$ between $\overline{\mathcal{G}_\eta}$ and $\mathcal{Q}^{\alpha+\eta}$.
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- Section **12.** Convolution
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- Section **13.** Examples of convolution.
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Abstract. *The Semiinfinite Flag Space appeared in the works of B.Feigin and E.Frenkel, and under different disguises was found by V.Drinfeld and G.Lusztig in the early 80-s. Another recent discovery (Beilinson-Drinfeld Grassmannian) turned out to conceal a new incarnation of Semiinfinite Flags. We write down these and other results scattered in folklore. We define the local semiinfinite flag space attached to a semisimple group G as the quotient $G((z))/HN((z))$ (an ind-scheme), where H and N are a Cartan subgroup and the unipotent radical of a Borel subgroup of G. The global semiinfinite flag space attached to a smooth complete curve C is a union of Quasimaps from C to the flag variety of G. In the present work we use $C = P^1$ to construct the category PS of certain collections of perverse sheaves on Quasimaps spaces, with factorization isomorphisms. We construct an exact convolution functor from the category of perverse sheaves on affine Grassmannian, constant along Iwahori orbits, to the category PS. Conjecturally, this functor should correspond to the restriction functor from modules over quantum group with divided powers to modules over the small quantum group.*

1. **Introduction**

1.1. **The origins of the idea of semi-infinite flag spaces.** We learned of the *Semi-infinite Flag Space* from B.Feigin and E.Frenkel in the late 80-s. Since then we tried to understand this remarkable object. It appears that it was essentially constructed, but under different disguises, by V.Drinfeld and G.Lusztig in the early 80-s. Another recent discovery (*Beilinson-Drinfeld Grassmannian*) turned out to conceal a new incarnation of Semiinfinite Flags. We write down these and other results scattered in the folklore.

1.2. **$\mathcal{S}$ as an indscheme $\widetilde{\mathbf{Q}} = \mathbf{G}((z))/\mathbf{HN}((z))$ and as systems $\mathcal{Q}$ and $\mathcal{Z}$; the category $\mathcal{PS}$ of perverse sheaves on $\mathcal{Z}$.** Let $\mathbf{G}$ be an almost simple simply-connected group with a Cartan datum $(I, \cdot)$ and a simply-connected simple root datum $(Y, X, \dots)$ of finite type as in [**?**], 2.2. We fix a Borel subgroup $\mathbf{B} \subset \mathbf{G}$, with a Cartan subgroup $\mathbf{H} \subset \mathbf{B}$, and the unipotent radical $\mathbf{N}$. B.Feigin and E.Frenkel define the Semiinfinite Flag Space $\mathcal{Z}$ as the quotient of $\mathbf{G}((z))$ modulo the connected component of $\mathbf{B}((z))$ (see [**?**]). Then they study the category $\mathcal{PS}$ of perverse sheaves on $\mathcal{Z}$ equivariant with respect to the Iwahori subgroup $\mathbf{I} \subset \mathbf{G}[[z]]$.

In the first two chapters we are trying to make sense of this definition. We encounter a number of versions of this space. In order to give it a structure of an ind-scheme, we define the (local) semiinfinite flag space as $\widetilde{\mathbf{Q}} = \mathbf{G}((z))/\mathbf{HN}((z))$ (see section 4). The (global) semiinfinite space attached to a smooth complete curve C is the system of varieties $\mathcal{Q}^\alpha$ of "quasimaps" from C to the flag variety of $\mathbf{G}$ — the Drinfeld compactifications of the degree α maps. In the present work we restrict ourselves to the case $C = \mathbb{P}^1$.

The main incarnation of the semiinfinite flag space in this paper is a collection $\mathcal{Z}$ (for *zastava*) of (affine irreducible finite dimensional) algebraic varieties $\mathcal{Z}^\alpha_\chi \subseteq \mathcal{Q}^\alpha$, together with certain closed embeddings and *factorizations*. Our definition of $\mathcal{Z}$ follows the scheme suggested by G.Lusztig in [?], §11: we approximate the "closures" of Iwahori orbits by their intersections with the transversal orbits of the opposite Iwahori subgroup. However, since the set-theoretic intersections of the above "closures" with the opposite Iwahori orbits can not be equipped with the structure of algebraic varieties, we postulate $\mathcal{Z}^\alpha_\chi$ for the "correct" substitutes of such intersections.

Having got the collection of $\mathcal{Z}^\alpha_\chi$ with factorizations, we imitate the construction of [?] to define the category $\mathcal{PS}$ (for *polubeskrajni snopovi*) of certain collections of perverse sheaves with $\mathbb{C}$-coefficients on $\mathcal{Z}^\alpha_\chi$ with *factorization isomorphisms*. It is defined in chapter 2; this category is the main character of the present work.

1.3. $\mathcal{P}(\mathcal{G},\mathbf{I})$ **and the principal block** $\mathfrak{C}^0 = PB(\mathfrak{U}, X)$ **of the divided powers quantum group at a root of unity** q**.** If $\mathbf{G}$ is of type A, D, E we set $d = 1$; if $\mathbf{G}$ is of type B, C, F we set $d = 2$; if $\mathbf{G}$ is of type G_2 we set $d = 3$. Let q be a root of unity of sufficiently large degree ℓ divisible by $2d$. Let $\mathfrak{u}$ be the small (finite-dimensional) quantum group associated to q and the root datum $(Y, X, \dots)$ as in [?]. Let $\mathcal{C}$ be the category of X-graded $\mathfrak{u}$-modules as defined in [?]. Let $\mathcal{C}^0$ be the block of $\mathcal{C}$ containing the trivial $\mathfrak{u}$-module. B.Feigin and G.Lusztig conjectured (independently) that the category $\mathcal{C}^0$ is equivalent to $\mathcal{PS}$.

Let $\mathfrak{U} \supset \mathfrak{u}$ be the quantum group with divided powers associated to q and the root datum $(Y, X, \dots)$ as in [?]. Let $\mathfrak{C}$ be the category of X-graded finite dimensional $\mathfrak{U}$-modules, and let $\mathfrak{C}^0$ be the block of $\mathfrak{C}$ containing the trivial $\mathfrak{U}$-module. The works [?], [?] and [?] establish an equivalence of $\mathfrak{C}^0$ and the category $\mathcal{P}(\mathcal{G},\mathbf{I})$. Here $\mathcal{G}$ denotes the affine Grassmannian $\mathbf{G}((z))/\mathbf{G}[[z]]$, and $\mathcal{P}(\mathcal{G},\mathbf{I})$ stands for the category of perverse sheaves on $\mathcal{G}$ with finite-dimensional support constant along the orbits of $\mathbf{I}$.

1.4. **The convolution (restriction) functor** $\mathbf{c}_\mathcal{Z} : \mathcal{P}(\mathcal{G},\mathbf{I}) \longrightarrow \mathcal{PS}$**.** The chapter 3 is devoted to the construction of the *convolution* functor $\mathbf{c}_\mathcal{Z} : \mathcal{P}(\mathcal{G},\mathbf{I}) \longrightarrow \mathcal{PS}$ which is the geometric counterpart of the restriction functor from $\mathfrak{C}^0$ to $\mathcal{C}^0$, as suggested by V.Ginzburg (cf. [?] §4). One of the main results of this chapter is the Theorem **??** which is the sheaf-theoretic version of the classical Satake isomorphism. Recall that one has a *Frobenius homomorphism* $\mathfrak{U} \longrightarrow U(\mathfrak{g}^L)$ (see [?]) where $U(\mathfrak{g}^L)$ stands for the universal enveloping algebra of the Langlands dual Lie algebra $\mathfrak{g}^L$. Thus the category of finite dimensional $\mathbf{G}^L$-modules is naturally embedded into $\mathfrak{C}$ (and in fact, into $\mathfrak{C}^0$). On the geometric level this corresponds to the embedding $\mathcal{P}(\mathcal{G},\mathbf{G}[[z]]) \subset \mathcal{P}(\mathcal{G},\mathbf{I})$. The Theorem **??** gives a natural interpretation (suggested by V.Ginzburg) of the weight spaces of $\mathbf{G}^L$-modules in terms of the composition

$$\mathbf{G}^L - mod \simeq \mathcal{P}(\mathcal{G},\mathbf{G}[[z]]) \subset \mathcal{P}(\mathcal{G},\mathbf{I}) \xrightarrow{\mathbf{c}_\mathcal{Z}} \mathcal{PS}.$$

1.5. **Conjectural realization of principal blocks of** $\check{G}_{\mathbb{F}_p}$ **and its Frobenius kernel.** Let us also mention here the following conjecture which might be known to specialists (characteristic p analogue of conjecture in 1.3). Let $\mathbf{G}^L$ stand for the Langlands dual Lie group. Let p be a prime number bigger than the Coxeter number of $\mathfrak{g}^L$, and let $\overline{\mathbb{F}}_p$

be the algebraic closure of finite field $\mathbb{F}_p$. Let $\mathfrak{C}_p$ be the category of algebraic $\mathbf{G}^L(\overline{\mathbb{F}}_p)$-modules, and let $\mathfrak{C}_p^0$ be the block of $\mathfrak{C}_p$ containing the trivial module. Let $\mathcal{C}_p$ be the category of graded modules over the Frobenius kernel of $\mathbf{G}^L(\overline{\mathbb{F}}_p)$, and let $\mathcal{C}_p^0$ be the block of $\mathcal{C}_p$ containing the trivial module (see [?]). Finally, let $\mathcal{PS}_p$ be the category of snops *with coefficients in* $\overline{\mathbb{F}}_p$, and let $\mathcal{P}(\mathcal{G},\mathbf{I})_p$ be the category of perverse sheaves on $\mathcal{G}$ constant along $\mathbf{I}$-orbits *with coefficients in* $\overline{\mathbb{F}}_p$. Then the categories $\mathcal{C}_p^0$ and $\mathcal{PS}_p$ are equivalent, the categories $\mathfrak{C}_p^0$ and $\mathcal{P}(\mathcal{G},\mathbf{I})_p$ are equivalent, and under these equivalences the restriction functor $\mathfrak{C}_p^0 \longrightarrow \mathcal{C}_p^0$ corresponds to the convolution functor $\mathcal{P}(\mathcal{G},\mathbf{I})_p \longrightarrow \mathcal{PS}_p$ (cf. 1.4). The equivalence $\mathcal{P}(\mathcal{G},\mathbf{I})_p \xrightarrow{\sim} \mathfrak{C}_p^0$ should be an extension of the equivalence between $\mathcal{P}(\mathcal{G},\mathbf{G}[[z]])_p \subset \mathcal{P}(\mathcal{G},\mathbf{I})_p$ and the subcategory of $\mathfrak{C}_p^0$ formed by the $\mathbf{G}^L(\overline{\mathbb{F}}_p)$-modules which factor through the Frobenius homomorphism $Fr: \mathbf{G}^L(\overline{\mathbb{F}}_p) \longrightarrow \mathbf{G}^L(\overline{\mathbb{F}}_p)$. The latter equivalence is the subject of forthcoming paper of K.Vilonen and the second author.

1.6. **Zastava space $\mathcal{Z}$ as the total structure of transverse slices in $\mathcal{S}$.** The Zastava space $\mathcal{Z}$ organizing all the "transversal slices" $\mathcal{Z}_\chi^\alpha$ may seem cumbersome. At any rate the existence of various models of the slices $\mathcal{Z}_\chi^\alpha$ (chapter 1), is undoubtedly beautiful by itself. Some of the wonderful properties of $\mathcal{Q}^\alpha$ and $\mathcal{Z}_\chi^\alpha$ are demonstrated in [?], [?], [?] in the case $\mathbf{G} = SL_n$. We expect all these properties to hold for the general $\mathbf{G}$.

1.7. **The key points of the paper.** To guide the patient reader through the notation, let us list the key points of this paper.

- The Theorem 7.3 identifies the different models of the zastava space $\mathcal{Z}_\chi^\alpha$ (all essentially due to V.Drinfeld) and states the factorization property.
- The exactness of the convolution functor $\mathbf{c}_\mathcal{Z}: \ \mathcal{P}(\mathcal{G},\mathbf{I}) \longrightarrow \mathcal{PS}$ is proved in the Theorem **??** and Corollary **??**.
- The Theorem **??** computes the value of the convolution functor on $\mathbf{G}[[z]]$-equivariant sheaves modulo the parity vanishing conjecture 10.9.

1.8. **Desiderata:** In the next parts we plan to study

- D-modules on the local variety $\widetilde{\mathbf{Q}}$ (local construction of the category $\mathcal{PS}$, global sections as modules over affine Lie algebra $\hat{\mathfrak{g}}$, action of the affine Weyl group by Fourier transforms),
- the relation of the local and global varieties (local and global Whittaker sheaves, a version of the convolution functor twisted by a character of $N((z))$), and
- the sheaves on Drinfeld compactifications of maps into partial flag varieties.

1.9. **Thanks.** The present work owes its very existence to V.Drinfeld. It could not have appeared without the generous help of many people who shared their ideas with the authors. Thus, the idea of *factorization* (section 9) is due to V.Schechtman. A.Beilinson and V.Drinfeld taught us the *Plücker* picture of the (Beilinson-Drinfeld) affine Grassmannian (sections 6 and 10). G.Lusztig has computed the local singularities of the Schubert strata closures in the spaces $\mathcal{Z}^\alpha_\chi$ (unpublished, cf [**?**]). B.Feigin and V.Ginzburg taught us their understanding of the Semiinfinite Flags for many years (in fact, we learned of Drinfeld's Quasimaps' spaces from V.Ginzburg in the Summer 1995). A.Kuznetsov was always ready to help us whenever we were stuck in the geometric problems (in fact, for historical reasons, the section 3 has a lot in common with [**?**] §1). We have also benefited from the discussions with R.Bezrukavnikov and M.Kapranov. Parts of this work were done while the authors were enjoying the hospitality and support of the University of Massachusetts at Amherst, the Independent Moscow University and the Sveučilište u Zagrebu. It is a great pleasure to thank these institutions.

2. **Notations**

2.1. **Group G and its Weyl group $\mathcal{W}_f$.** We fix a Cartan datum $(I,\cdot)$ and a simply-connected simple root datum $(Y, X, \dots)$ of finite type as in [**?**], 2.2.

Let $\mathbf{G}$ be the corresponding simply-connected almost simple Lie group with the Cartan subgroup $\mathbf{H}$ and the Borel subgroup $\mathbf{B} \supset \mathbf{H}$ corresponding to the set of simple roots $I \subset X$. We will denote by $\mathcal{R}^+ \subset X$ the set of positive roots. We will denote by $2\rho \in X$ the sum of all positive roots.

Let $\mathbf{B}_+ = \mathbf{B}$ and let $\mathbf{B}_- \supset \mathbf{H}$ be the opposite Borel subgroup. Let $\mathbf{N}$ (resp. $\mathbf{N}_-$) be the radical of $\mathbf{B}$ (resp. $\mathbf{B}_-$). Let $\mathbf{H}_a = \mathbf{B}/\mathbf{N} = \mathbf{B}_-/\mathbf{N}_-$ be the abstract Cartan group. The corresponding Lie algebras are denoted, respectively, by $\mathfrak{b}, \mathfrak{b}_-, \mathfrak{n}, \mathfrak{n}_-, \mathfrak{h}$.

Let $\mathcal{B}$ be the flag manifold $\mathbf{G}/\mathbf{B}$, and let $\mathbf{A} = \mathbf{G}/\mathbf{N}$ be the principal affine space. We have canonically $H_2(\mathcal{B}, \mathbb{Z}) = Y;\ H^2(\mathcal{B}, \mathbb{Z}) = X$.

For $\nu \in X$ let $\mathbf{L}_\nu$ denote the corresponding $\mathbf{G}$-equivariant line bundle on $\mathcal{B}$.

Let $\mathcal{W}_f$ be the Weyl group of $\mathbf{G}$. We have a canonical bijection $\mathcal{B}^{\mathbf{H}} = \mathcal{W}_f$ such that the neutral element $e \in \mathcal{W}_f = \mathcal{B}^{\mathbf{H}} \subset \mathcal{B}$ forms a single $\mathbf{B}$-orbit.

We have a Schubert stratification of $\mathcal{B}$ by $\mathbf{N}$- (resp. $\mathbf{N}_-$-)orbits: $\mathcal{B} = \sqcup_{w\in\mathcal{W}_f}\mathcal{B}_w$ (resp. $\mathcal{B} = \sqcup_{w\in\mathcal{W}_f}\mathcal{B}^w$) such that for $w \in \mathcal{W}_f = \mathcal{B}^{\mathbf{H}} \subset \mathcal{B}$ we have $\mathcal{B}^w \cap \mathcal{B}_w = \{w\}$.

We denote by $\overline{\mathcal{B}}_w$ (resp. $\overline{\mathcal{B}}^w$) the Schubert variety — the closure of $\mathcal{B}_w$ (resp. $\mathcal{B}^w$). Note that $\overline{\mathcal{B}}_w = \sqcup_{y\le w}\mathcal{B}_y$ while $\overline{\mathcal{B}}^w = \sqcup_{z\ge w}\mathcal{B}^z$ where $\le$ denotes the standard Bruhat order on $\mathcal{W}_f$.

Let $e \in \mathcal{W}_f$ be the shortest element (neutral element), let $w_0 \in \mathcal{W}_f$ be the longest element, and let $s_i,\ i \in I$, be the simple reflections in $\mathcal{W}_f$.

2.2. **Irreducible representations V_λ of G.** We denote by X^+ the cone of positive weights (highest weights of finite dimensional $\mathbf{G}$-modules). The fundamental weights $\omega_i:\ \langle i, \omega_j\rangle = \delta_{ij}$ form the basis of X^+.

For $\lambda \in X^+$ we denote by V_λ the finite dimensional irreducible representation of $\mathbf{G}$ with highest weight λ.

We denote by $V_\lambda^\vee$ the representation dual to V_λ; the pairing: $V_\lambda^\vee \times V_\lambda \longrightarrow \mathbb{C}$ is denoted by $\langle , \rangle$.

For each $\lambda \in X^+$ we choose a nonzero vector $y_\lambda \in V_\lambda^{\mathbf{N}_-}$. We also choose a nonzero vector $x_\lambda \in (V_\lambda^\vee)^{\mathbf{N}}$ such that $\langle x_\lambda, y_\lambda \rangle = 1$.

2.3. **Configuration spaces C^α of I-colored divisors.** Let us fix $\alpha \in \mathbb{N}[I] \subset Y$, $\alpha = \sum_{i\in I} a_i i$. Given a curve C we consider the configuration space $C^\alpha \stackrel{\text{def}}{=} \prod_{i\in I} C^{(a_i)}$ of colored effective divisors of multidegree α (the set of colors is I). The dimension of C^α is equal to the length $|\alpha| = \sum_{i\in I} a_i$.

2.3.1. *The diagonal stratification of $\mathcal{H}_C^\alpha$.* Define the *multisubsets* of a set S as elements $\Gamma = \sum_{s\in S} [\Gamma : s]{\cdot}s$ of $\mathbb{N}[S]$. If we think of Γ as an element of some symmetric power $S^{(k)}$ then it can be denoted $\{\{s_1, ..., s_k\}\}$ where the sequence contains $\Gamma : s]$ copies of s.

Now onsider $S = \mathbb{N}[I]$ so that Γ is a formal sum $\Gamma = \underline{\sum}_{\beta\in \mathbf{N}[\mathbf{I}]}[\Gamma : \beta]\beta$, i.e., a finite support function $\Gamma : \mathbf{N}[\mathbf{I}] \to \mathbb{N}$. Then its integral is $|\Gamma| \stackrel{\text{def}}{=} \sum_{\beta\in \mathbf{N}[\mathbf{I}]}[\Gamma : \beta]\beta \in \mathbf{N}[\mathbf{I}]$ which is just its actual sum in $\mathbf{N}[\mathbf{I}]$.

A multisubset Γ of $\mathbf{N}[\mathbf{I}]$ defines the"Γ-diagonal" stratum C_Γ in $\mathcal{H}_{C\times I}$. An I-colored configuration of points $D = \sum_{a\in C} D_a{\cdot}a$ with $D_a \in \mathbf{N}[\mathbf{I}]$, lies in C_Γ if the nonzero coefficients D_a form the multiset Γ.

2.3.2. *Again.* Multisubsets of a set S are defined as elements of some symmetric power $S^{(k)}$ and we denote the image of $(s_1, ..., s_k) \in S^k$ in $S^{(k)}$ by $\{\{s_1, ..., s_k\}\}$. We denote by $\mathfrak{P}(\alpha)$ the set of all partitions of α, i.e multisubsets $\Gamma = \{\{\gamma_1, ..., \gamma_k\}\}$ of $\mathbb{N}[I]$ with $\gamma_r \neq 0$ and $\sum_{r=1}^k \gamma_i = \alpha$.

For $\Gamma \in \mathfrak{P}(\alpha)$ the corresponding stratum C_Γ^α is defined as follows. It is formed by configurations D which can be subdivided into m groups of points $D = (D_r)_{1\leq r\leq k}$ so that

(1) the r-th group D_r contains γ_r points;
(2) all the points in one group equal to each other,
(3) the different groups being disjoint.

This means that $D = \sum_1^r \gamma_r{\cdot}a_r$ for distinct points a_r i C where for $\gamma = \sum_{i\in I} \gamma_i i$, $\gamma{\cdot}a$ means that eand we take the γ_i copies of a that are colored by i (i.e., from the i^{th} copy $C{\times}i$ of C.

Example. (0) The main diagonal in C^α is the closed stratum given by partition $\alpha=\alpha$.

(1) The complement to all diagonals in C^α for $\alpha=\sum_{i\in I}\ a_i i$ is the open stratum given by partition

$$\alpha=\sum_{i\in I}\ (\underbrace{i+i+\ldots+i}_{a_i\,\text{times}})$$

Evidently, $C^\alpha=\bigsqcup_{\Gamma\in\mathfrak{P}(\alpha)} C^\alpha_\Gamma$.

CHAPTER 1. Spaces $\mathcal{Z}\subseteq\mathcal{Q}$ of (based) quasimaps from $\mathbb{P}^1$ to the flag variety $\mathcal{B}$

In section 3 we define the spaces $\mathcal{Q}^\alpha\subseteq\mathcal{Z}^\alpha$ of (based) quasimaps from a curve C to a flag manifold. We define the shifts of quasimaps by finite subschemes. The basic stratification of quasimaps is by shifts of maps.(3)

2.0. **A summary of chapter 1.** In section 4 we introduce the local versions $\mathbf{Q}\subseteq\widetilde{\mathbf{Q}}$ of quasimap spaces for the local curves d and d^*.

These we call "quasimaps" since these spaces are certain torsors (for groups $H_{\mathcal{O}}/H$ and H_{d^*}/H) over the true local quasimap stacks. These torsors happen to suffice for our purposes.)

We also define the restriction map (a closed embedding) $\mathcal{Q}\hookrightarrow\mathbf{Q}$ of global $\mathbb{P}^1$-quasimaps into local d-"quasimaps".

In section 5 we introduce the Plucker model $\mathfrak{Z}^\alpha$ of space of based quasimaps.

In section 6 we introduce the *zastava* model $\mathbf{Z}^\alpha$ of based quasimaps, in terms of the global loop Grassmannian. In this model the locality property of based quasimaps is obvious.(4)

Finally, in section 7 we explain the equivalence of three constructions of space of based quasimaps, which we now call *zastava spaces*.

3. Quasimaps from a curve to a flag manifold

3.1. **Maps from a curve to the flag variety: the degree.** We fix a smooth projective curve C and $\alpha\in\mathbb{N}[I]$.

[3] The moduli of maps are smooth and all quasimaps are shifts of maps.

[4] Locality property implies that the construction is of local nature so that we do not need $\mathbb{P}^1$ but only $\mathcal{H}_C$!

3.1.1. *Definition.* An algebraic map $f: C \longrightarrow \mathcal{B}$ has degree α if the following equivalent conditions hold:

a) For the fundamental class $[C] \in H_2(C, \mathbb{Z})$ we have $f_*[C] = \alpha \in Y = H_2(\mathcal{B}, \mathbb{Z})$;

b) For any $\nu \in X$ the line bundle $f^*\mathbf{L}_\nu$ on C has degree $\langle \alpha, \nu \rangle$.

3.2. **Plucker model of maps of degree α.** The Plücker embedding of the flag manifold $\mathcal{B}$ gives rise to the following interpretation of algebraic maps of degree α.

For any irreducible V_λ we consider the trivial vector bundle $\mathcal{V}_\lambda = V_\lambda \otimes \mathcal{O}$ over C.

For any $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ we denote by the same letter the induced morphism $\phi: \mathcal{V}_\lambda \otimes \mathcal{V}_\mu \longrightarrow \mathcal{V}_\nu$.

Then a map of degree α is a collection of *line subbundles* $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda,\ \lambda \in X^+$ such that:

a) $\deg \mathfrak{L}_\lambda = -\langle \alpha, \lambda \rangle$;

b) For any surjective $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ we have $\phi(\mathfrak{L}_\lambda \otimes \mathfrak{L}_\mu) = \mathfrak{L}_\nu$;

c) For any $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\phi(\mathfrak{L}_\lambda \otimes \mathfrak{L}_\mu) = 0$.

Since the surjections $V_\lambda \otimes V_\mu \to V_{\lambda+\mu}$ form one $\mathbb{C}^*$-orbit, systems $\mathcal{L}_\lambda$ satisfying (b) are determined by a choice of $\mathfrak{L}_{\omega_i}$ for the fundamental weights $\omega_i,\ i \in I$.

If we replace the curve C by a point, we get the Plücker description of the flag variety $\mathcal{B}$ as the set of collections of lines $L_\lambda \subseteq V_\lambda$ satisfying conditions of type (b) and (c). Here, a Borel subgroup B in $\mathcal{B}$ corresponds to a system of lines $(L_\lambda,\ \lambda \in X^+)$ if the lines are the fixed points of the unipotent radical N of B, $L_\lambda = (V_\lambda)^N$, or equivalently, if N is the common stabilizer for all lines $N = \bigcap_{\lambda \in X^+} G_{L_\lambda}$.

The space of degree α quasimaps from C to $\mathcal{B}$ will be denoted by $\overset{\circ}{\mathcal{Q}}{}^\alpha$.

3.3. **The spaces $\mathcal{Q}^\alpha = QMap(C, \mathcal{B})$ of quasimaps from a curve C to the flag variety $\mathcal{B}$.** (V.Drinfeld) The space $\mathcal{Q}^\alpha = \mathcal{Q}^\alpha_C$ of *quasimaps* of degree α from C to $\mathcal{B}$ is the space of collections of *invertible subsheaves* $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda,\ \lambda \in X^+$ such that:

a) $\deg \mathfrak{L}_\lambda = -\langle \alpha, \lambda \rangle$;

b) For any surjective $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ we have $\phi(\mathfrak{L}_\lambda \otimes \mathfrak{L}_\mu) = \mathfrak{L}_\nu$;

c) For any $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\phi(\mathfrak{L}_\lambda \otimes \mathfrak{L}_\mu) = 0$.

3.3.1. *Lemma.* a) The evident inclusion $\overset{\circ}{\mathcal{Q}}{}^\alpha \subset \mathcal{Q}^\alpha$ is an open embedding;

b) $\mathcal{Q}^\alpha$ is a projective variety.

Proof. Obvious. $\square$

3.3.2. Here is another version of the Definition, also due to V.Drinfeld. The principal affine space $\mathbf{A} = \mathbf{G}/\mathbf{N}$ is an $\mathbf{H}_a$-torsor over $\mathcal{B}$. We consider its affine closure $\overline{\mathbf{A}}$, that is, the spectrum of the ring of functions on $\mathbf{A}$. Recall that $\overline{\mathbf{A}}$ is the space of collections of vectors $v_\lambda \in V_\lambda,\ \lambda \in X^+$, satisfying the following Plücker relations:

a) For any surjective $\mathbf{G}$-morphism $\phi :\ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$, and $\phi(y_\lambda \otimes y_\mu) = y_\nu$, we have $\phi(v_\lambda \otimes v_\mu) = v_\nu$;

b) For any $\mathbf{G}$-morphism $\phi :\ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\phi(v_\lambda \otimes v_\mu) = 0$.

The action of $\mathbf{H}_a$ extends to $\overline{\mathbf{A}}$ but it is not free anymore. Consider the quotient stack $\hat{\mathcal{B}} = \overline{\mathbf{A}}/\mathbf{H}_a$. The flag variety $\mathcal{B}$ is an open substack in $\hat{\mathcal{B}}$. A map $\hat{\phi} :\ C \to \hat{\mathcal{B}}$ is nothing else than an $\mathbf{H}_a$-torsor Φ over C along with an $\mathbf{H}_a$-equivariant morphism $f :\ \Phi \to \overline{\mathbf{A}}$. The degree of this map is defined as follows.

Let $\lambda :\ \mathbf{H}_a \to \mathbb{C}^*$ be the character of $\mathbf{H}_a$ corresponding to a weight $\lambda \in X$. Let $\mathbf{H}_\lambda \subset \mathbf{H}_a$ be the kernel of the morphism λ. Consider the induced $\mathbb{C}^*$-torsor $\Phi_\lambda = \Phi/\mathbf{H}_\lambda$ over C. The map $\hat{\phi}$ has degree $\alpha \in \mathbb{N}[I]$ if

$$\text{for any } \lambda \in X \quad \text{we have} \quad \deg(\Phi_\lambda) = \langle \lambda, \alpha \rangle.$$

Definition. The space $\mathcal{Q}^\alpha$ is the space of maps $\hat{\phi} :\ C \to \hat{\mathcal{B}}$ of degree α such that the generic point of C maps into $\mathcal{B} \subset \hat{\mathcal{B}}$.

The equivalence of the two versions of Definition follows by comparing their Plücker descriptions.

3.4. **Stratification of quasimaps $\mathcal{Q}^\alpha$ by C^γ shifts of maps.** In this subsection we describe a stratification of $\mathcal{Q}^\alpha$ according to the singularities of quasimaps.

3.4.1. Given $\beta, \gamma \in \mathbb{N}[I]$ such that $\beta + \gamma = \alpha$, we define the proper map $\sigma_{\beta,\gamma} :\ \mathcal{Q}^\beta \times C^\gamma \longrightarrow \mathcal{Q}^\alpha$.

Namely, let $f = (\mathfrak{L}_\lambda)_{\lambda \in X^+} \in\ \mathcal{Q}^\beta$ be a quasimap of degree β; and let $D = \sum_{i\in I} D_i \cdot \cdot i$ be an effective colored divisor of multidegree $\gamma = \sum_{i \in I} d_i i$, that is, $\deg(D_i) = d_i$. We define $\sigma_{\beta,\gamma}(f, D) \overset{\text{def}}{=} f(-D) \overset{\text{def}}{=} (\mathfrak{L}_\lambda(- < D, \lambda >))_{\lambda \in X^+} \in \mathcal{Q}^\alpha$, where we use the pairing $Div^I(C) \underset{\mathbb{Z}}{\otimes} X \to Div(C)$ given by $< D, \lambda >= \sum_{i \in I} < i, \lambda > \cdot \cdot D_i$.

3.4.2. **Theorem.** $\mathcal{Q}^\alpha = \bigsqcup_{0 \le \beta \le \alpha} \sigma_{\beta, \alpha-\beta}(\overset{\circ}{\mathcal{Q}}{}^\beta \times C^{\alpha-\beta})$

Proof. Any invertible subsheaf $\mathfrak{L}_\lambda \subseteq \mathcal{V}_\lambda$ lies in a unique line subbundle $\tilde{\mathfrak{L}}_\lambda \subseteq \mathcal{V}_\lambda$ called the *normalization* of $\mathfrak{L}$. So any quasimap $\mathfrak{L}$ defines a map $\tilde{\mathfrak{L}}$ (called the *normalization* of $\mathfrak{L}$) of degree $\beta \le \alpha$ and an I-colored effective divisor D (called the *defect* of $\mathfrak{L}$) corresponding to the torsion sheaf $\tilde{\mathfrak{L}}/\mathfrak{L}$, such that $\mathfrak{L} = \tilde{\mathfrak{L}}(-D)$. □

3.4.3. *Definition.* Given a quasimap $f=(\mathfrak{L}_\lambda)_{\lambda\in X^+}\in\ \mathcal{Q}^\alpha$, its *domain of definition* $U(f)$ is the maximal Zariski open $U(f)\subset C$ such that for any λ the invertible subsheaf $\mathfrak{L}_\lambda\subset\mathcal{V}_\lambda$ restricted to $U(f)$ is actually a line subbundle.

3.4.4. *Corollary.* For a quasimap $f=(\mathfrak{L}_\lambda)_{\lambda\in X^+}\in\ \mathcal{Q}^\alpha$ of degree α the complement $C-U(f)$ of its domain of definition consists of at most $|\alpha|$ points. □

3.5. **Smoothness of the moduli** $=Map^\alpha(\mathbb{P}^1,\mathcal{B})$ **of maps into the flag variety.** From now on, unless explicitly stated otherwise, $C=\mathbb{P}^1$.

Proposition. (V.Drinfeld) $\overset{\circ}{\mathcal{Q}}{}^\alpha$ is a smooth manifold of dimension $2|\alpha|+\dim(\mathcal{B})$.

Proof. We have to check that at a map $f\in\overset{\circ}{\mathcal{Q}}{}^\alpha$ the first cohomology $H^1(\mathbb{P}^1,f^*\mathcal{TB})$ vanishes (where $\mathcal{TB}$ stands for the tangent bundle of $\mathcal{B}$), and then the tangent space $\Theta_f\overset{\circ}{\mathcal{Q}}{}^\alpha$ equals $H^0(\mathbb{P}^1,f^*\mathcal{TB})$. As $\mathcal{TB}$ is generated by the global sections, $f^*\mathcal{TB}$ is generated by global sections as well, hence $H^1(\mathbb{P}^1,f^*\mathcal{TB})=0$. To compute the dimension of $\Theta_f\overset{\circ}{\mathcal{Q}}{}^\alpha=H^0(\mathbb{P}^1,f^*\mathcal{TB})$ it remains to compute the Euler characteristic $\chi(\mathbb{P}^1,f^*\mathcal{TB})$. To this end we may replace $\mathcal{TB}$ with its associated graded bundle $\oplus_{\theta\in\mathcal{R}^+}\mathbf{L}_\theta$. Then

$$\chi(\mathbb{P}^1,f^*(\bigoplus_{\theta\in\mathcal{R}^+}\mathbf{L}_\theta))=\sum_{\theta\in\mathcal{R}^+}(\langle\alpha,\theta\rangle+1)=\langle\alpha,2\rho\rangle+\sharp\mathcal{R}^+=2|\alpha|+\dim\mathcal{B}\quad\square$$

3.6. **Spaces** $\mathcal{Z}^\alpha=QMap[(\mathbb{P}^1,\infty),(\mathcal{B},\mathfrak{b}_-)]$ **of based quasimaps of degree** α**.** Now we are able to introduce our main character. First we consider the open subspace $U^\alpha\subset\ \mathcal{Q}^\alpha$ formed by the quasimaps containing $\infty\in\mathbb{P}^1$ in their domain of definition (see 3.4.3). Next we define the closed subspace $\mathcal{Z}^\alpha\subset U^\alpha$ formed by quasimaps with value at ∞ equal to $\mathbf{B}_-\in\mathcal{B}$:

$$\mathcal{Z}^\alpha\overset{\text{def}}{=}\ \{f\in U^\alpha|f(\infty)=\mathbf{B}_-\}$$

We will see below that $\mathcal{Z}^\alpha$ is an affine algebraic variety.

3.6.1. It follows from Proposition **??** that $\dim\mathcal{Z}^\alpha=2|\alpha|$.

4. "Quasimap" spaces $\mathbf{Q}\subseteq\widetilde{\mathbf{Q}}$ for local curves d and d^*

In this section we define a d-version $\mathbf{Q}$ of $\mathcal{Q}^\alpha$. Here one replaces the global curve C by the formal neighborhood d of a point however, we will replace quasimaps by "quasimaps" (also called *line quasimaps*), meaning elements of a certain torsor (for $(H_a)_\mathcal{O}/H_a$) over the true space $\mathcal{Q}(d)$ of d-quasimaps. So, $\mathcal{Q}(d)$ is really the stack quotient $\mathbf{Q}/[(H_a)_\mathcal{O}/H_a]$. Similarly for d^* we will consider an $[(H_a)_\mathcal{K}/H_a]$-torsor $\widetilde{\mathbf{Q}}$ over $\mathcal{Q}(d^*)$.

The use of these torsors suffices for the restriction construction. The restriction in question takes quasimaps on $\mathbb{P}^1$ to quasimaps on d but it happens to factor through "quasimaps" on d.

We set $\mathcal{O} = \mathbb{C}[[z]] \xrightarrow{p_n} \mathcal{O}_n = \mathbb{C}[[z]]/z^n, \mathcal{K} = \mathbb{C}((z))$.

4.1. **The vector and line versions $\mathcal{S} \twoheadrightarrow \mathbf{Q} = \mathcal{S}/\mathbf{H}_a$ of d-quasimaps.** Here we define the local quasimap space for the local curve d as a closed subscheme $\mathbf{Q}$ of $\prod_{i\in I} \mathbb{P}(V_{\omega_i} \otimes \mathcal{O})$ given by Plucker equations. We start with a certain $\mathbf{H}_a$-torsor $\mathcal{S} \subseteq \prod_{i\in I} V_{\omega_i} \otimes \mathcal{O}$ over $\mathbf{Q}$.

4.1.1. *Vectors and quasimaps.* The vectors $v \in \prod_{i\in I} (V_i)_{\mathcal{O}}$ that we consider are generators of invertible subsheaves $\mathfrak{L}_i$ of trivial vector bundles $(V_i)_d$ on the disc. In order to pass from v to the quasimap $\mathfrak{L}$ on D we should really divide by $(H_a)_{\mathcal{O}}$ (so that each v_i can be adjusted by an $\mathcal{O}^*$ factor). However, we will only take the quotient by H_a which will keep us in schemes rather than stacks. [It will also suffice for our need to restrict global quasimaps to local ones, see 4.3.]

xxx Inside the scheme $\prod_{i\in I} \mathbb{P}[(V_i)_{\mathcal{O}_n}]$ we will define the projective subscheme $\mathcal{Q}(d_n)$ by Plucker equations. It comes with an H_a-torsor $\mathcal{S}(d_n)$ which is locally closed in $\prod_{i\in I} (V_i)_{\mathcal{O}_n}$.

4.1.2. *The $\mathbb{C}$-projective space of $U_{\mathcal{O}}$.* For a finite dimensional vector space U let $U^* \subseteq U$ be the open subscheme which is a G_m-torsor over $\mathbb{P}(U)$. So, for a commutative algebra $\Bbbk$ let $U^*(\Bbbk)$consists of vectors $u \in U(\Bbbk)$ such that $\Bbbk v$ is a projective $\Bbbk$-module of rank one.

Now, for any n we can pull back via $U_{\mathcal{O}} \to U_{\mathcal{O}_n}$ the open subscheme $U^*_{\mathcal{O}_n}$ of the evctor space $U_{\mathcal{O}_n}$ to an open subscheme $(U_{\mathcal{O}})^*_n$ of $U_{\mathcal{O}}$. This is an incrasing open filtratin of $U_{\mathcal{O}}$ and we let $U^*_{\mathcal{O}} \stackrel{\text{def}}{=} \cup_n \ (U_{\mathcal{O}})^*_n$. This defines the scheme

$$\mathbb{P}(U_{\mathcal{O}}) \stackrel{\text{def}}{=} U^*_{\mathcal{O}}/G_m = \varinjlim (U_{\mathcal{O}})^*_n/G_m\mathbb{P}$$

where the limit is under open inclusions. It arries the G_m-torsor $U^*_{\mathcal{O}}$.

4.1.3. *The vector and line version $\mathcal{S} \twoheadrightarrow \mathbf{Q}$ of d-quasimaps.* Now, for a system of vector spaces V_i we have

$$\prod_{i\in I} \mathbb{P}(V_i \otimes \mathcal{O}) = \prod_{i\in I} (V_i)^*_{\mathcal{O}}/G_m = [\prod_{i\in I} (V_i)^*_{\mathcal{O}}]/H_a = \cup_n [\prod_{i\in I} (V_i)_{\mathcal{O}}]^*_n]/H_a.$$

Inside $\prod_{i\in I} \mathbb{P}(V_i \otimes \mathcal{O})$ there is a subscheme $\mathbf{Q}$ with an $\mathbb{H}_a$-torsor $\mathcal{S}$ which is a closed subscheme of the scheme $\prod_{i\in I} (V_i)^*_{\mathcal{O}}$ given by the Plucker equations.

Old Version.

4.1.4. *Subscheme $\overline{\mathbf{A}}(\mathcal{O})$ of $\prod_{i\in I} V_{\omega_i} \otimes \mathcal{O}$ given by Plücker equations.* We define the scheme $\overline{\mathbf{A}}(\mathcal{O})$ (of infinite type): its points are the collections of vectors $v_\lambda \in V_\lambda \otimes \mathcal{O}, \ \lambda \in X^+$, satisfying the Plücker equations like in 3.3.2. It is a closed subscheme of $\prod_{i\in I} V_{\omega_i} \otimes \mathcal{O}$.

4.1.5. *Open increasing filtration* $\overline{\mathbf{A}}(\mathcal{O})_n \subset \overline{\mathbf{A}}(\mathcal{O})$ *by vectors that survive* $\mathcal{O} \twoheadrightarrow \mathcal{O}_n$. We define the open subscheme $\mathbf{A}(\mathcal{O})_n \subset \overline{\mathbf{A}}(\mathcal{O})$: it is formed by the collections $(v_\lambda)_{\lambda\in X^+}$ such that $p_n(v_{\omega_i}) \neq 0$ for all $i \in I$. Evidently, for $0 \leq n \leq m$, one has $\mathbf{A}(\mathcal{O})_n \subset \mathbf{A}(\mathcal{O})_m$.

4.1.6. *The vector version* $\mathcal{S}$ *of d-quasimaps.* This is the open subscheme $\mathcal{S} \overset{\text{def}}{=} \bigcup_{n\geq 0} \overline{\mathbf{A}}(\mathcal{O})_n \subset \overline{\mathbf{A}}(\mathcal{O})$ of vectors that survive on some finite level. One has $\mathcal{S} = \mathbf{A}(\mathcal{O})$.

4.1.7. *The line version* $\mathbf{Q}$ *of d-quasimaps.* This is the closed subscheme $\mathbf{Q} = \mathcal{S}/\mathbf{H}_a \subseteq \prod_{i\in I} \mathbb{P}(V_{\omega_i} \otimes \mathcal{O})$ given by Plucker equations. The scheme $\mathcal{S}$ is equipped with the free action of $\mathbf{H}_a$: $h(v_\lambda)_{\lambda\in X^+} = (\lambda(h)v_\lambda)_{\lambda\in X^+}$. The quotient scheme $\mathbf{Q} = \mathcal{S}/\mathbf{H}_a$ is a closed subscheme in $\prod_{i\in I} \mathbb{P}(V_{\omega_i} \otimes \mathcal{O})$. It is formed by the collections of lines satisfying the Plücker equations. We denote the natural projection $\mathcal{S} \longrightarrow \mathbf{Q}$ by pr.

4.2. **The local quasimap space $\widetilde{\mathbf{Q}} = \widetilde{\mathcal{S}}/\mathbf{H}_a$ associated to the punctured disc d^*.** Now we define the $\mathcal{K}$-level local quasimap space as a closed subindscheme $\widetilde{\mathbf{Q}} = \mathcal{S}/\mathbf{H}_a$ of the indscheme $\prod_{i\in I} \mathbb{P}[V_{\omega_i})_{\mathcal{K}} \otimes \mathcal{O}]$, given by the Plucker equations. Again, we start with the $\mathbf{H}_a$-torsor $\widetilde{\mathcal{S}}$ over $\widetilde{\mathbf{Q}}$ given by Plucker equations in $\prod_{i\in I}(V_{\omega_i})_{\mathcal{K}} \otimes \mathcal{O}$.

The ind-scheme $\widetilde{\mathbf{Q}}$ is filtered by closed subschemes $\mathbf{Q}^\eta,\ \eta \in Y$ and there is the corresponding stratifications $\overset{\bullet}{\mathbf{Q}}{}^\eta$ of $\mathbf{Q}$. Similarly one has $\overset{\bullet}{\mathcal{S}}{}^\eta \subseteq \mathcal{S}^\eta \subseteq \widetilde{\mathcal{S}}$.

4.2.1. *A decreasing filtration of* $\mathcal{S}$ *by closed subschemes* $\mathcal{S}^{-\eta} \subset \mathcal{S}$ *(*$\eta \in \mathbb{N}[I]$*), given by vectors that vanish to order* η. For $\eta \in \mathbb{N}[I]$ we define the closed subscheme $\mathcal{S}^{-\eta} \subset \mathcal{S}$ formed by the collections $(v_\lambda)_{\lambda\in X^+}$ such that $v_\lambda = 0 \bmod z^{\langle\eta,\lambda\rangle}$. Notice that

$$z^\eta:\ \mathcal{S} \overset{\sim}{\longrightarrow} \mathcal{S}^{-\eta},\ \text{by } (v_\lambda)_{\lambda\in X^+} \mapsto (z^{\langle\eta,\lambda\rangle}v_\lambda)_{\lambda\in X^+}.$$

4.2.2. *The semiinfinite system of schemes* $\mathcal{S}^\chi \subseteq \prod_{i\in I}(V_i)_{\mathcal{K}},\ \chi \in Y$. Now we can extend the definition of $\mathcal{S}^\chi$ to arbitrary $\chi \in Y$. Namely, we define $\mathcal{S}^\chi$ to be formed by the collections $(v_\lambda \in V_\lambda \otimes \mathcal{K})_{\lambda\in X^+}$ such that $(z^{\langle\chi,\lambda\rangle}v_\lambda)_{\lambda\in X^+} \in \mathcal{S}$. Evidently, $\mathcal{S}^\chi \subset \mathcal{S}^\eta$ iff $\chi \leq \eta$, and then the inclusion is the closed embedding. The filtration of S^η $S^\chi,\ \chi < \eta$ gives a stratification of S^η by $\overset{\bullet}{\mathcal{S}}{}^\chi,\ \chi \leq \eta$ where $\overset{\bullet}{\mathcal{S}}{}^\chi$ is the open subscheme $\mathcal{S}^\chi - \bigcup_{\chi'<\chi} \mathcal{S}^{\chi'}$ in $\mathcal{S}^\chi$.

4.2.3. *The ind-scheme* $\widetilde{\mathcal{S}} = \lim\limits_{\to\ \eta\in Y} \mathcal{S}^\eta$. The ind-scheme $\widetilde{\mathcal{S}}$ is equipped with the natural action of the proalgebraic group $\mathbf{G}(\mathcal{O})$ (coming from the action on $\prod_{i\in I} V_{\omega_i} \otimes \mathcal{K}$.

Lemma. The orbits $\mathbf{G}(\mathcal{O})$ in $\mathcal{S}$ are exactly the strata $\overset{\bullet}{\mathcal{S}}{}^\eta,\ \eta \in Y$. □

4.2.4. *Closed subschemes* $\mathbf{Q}^\eta = \mathcal{S}^\eta/\mathbf{H}_a,\ \eta \in Y,$ *of the ind-scheme* $\prod_{i\in I} \mathbb{P}(V_{\omega_i} \otimes \mathcal{K})$. All the above (ind)schemes are equipped with the free action of $\mathbf{H}_a$, and taking quotients we obtain the schemes $\mathbf{Q}^\eta = \mathcal{S}^\eta/\mathbf{H}_a,\ \eta \in Y$. They are all closed subschemes of the ind-scheme $\prod_{i\in I} \mathbb{P}(V_{\omega_i} \otimes \mathcal{K})$.

4.2.5. *The indscheme* $\widetilde{\mathbf{Q}} = \widetilde{\mathcal{S}}/\mathbf{H}_a = \lim_{\to\ \eta\in Y} \mathbf{Q}^\eta$. We have $\mathbf{Q}^\chi \subset \mathbf{Q}^\eta$ iff $\chi \le \eta$, and then the inclusion is the closed embedding. The ind-scheme $\widetilde{\mathbf{Q}} = \widetilde{\mathcal{S}}/\mathbf{H}_a$ is the union $\widetilde{\mathbf{Q}} = \bigcup_{\eta\in Y} \mathbf{Q}^\eta$. The ind-scheme $\widetilde{\mathbf{Q}}$ is equipped with the natural action of the proalgebraic group $\mathbf{G}(\mathcal{O})$ (coming from the action on $\prod_{i\in I} \mathbb{P}(V_{\omega_i} \otimes \mathcal{K})$). The orbits are exactly $\dot{\mathbf{Q}}^\eta \stackrel{\text{def}}{=} \dot{\mathcal{S}}^\eta/\mathbf{H}_a,\ \eta \in Y$.

4.3. **The closed embedding $\mathcal{Q}^\alpha \hookrightarrow \mathbf{Q}$ of $\mathbb{P}^1$-quasimaps (global) into d-quasimaps (local).** The closed embedding is given by restricting a global quasimap $\mathfrak{L}$ on $\mathbb{P}^1$ to a quasimap on d.

This is first done on the level of H_a-torsors $\widehat{\mathcal{Q}}^\alpha \hookrightarrow \mathcal{S}$ over $\mathcal{Q}^\alpha \hookrightarrow \mathbf{Q}$. So, we first construct an H_a-torsor $\widehat{\mathcal{Q}}^\alpha \to \mathcal{Q}^\alpha$ by adding to the quasimap $\mathfrak{L}$ on $\mathbb{P}^1$ a choice of a trivialization v off $\infty \in \mathbb{P}^1$. (then one restricts the trivializations $\prod_{i\in I}(V_i)[z] \ni v \mapsto v|_d \in \prod_{i\in I}(V_i)[[z]]$).

We consider $C = \mathbb{P}^1$ with two marked points $0, \infty \in C$. We choose a coordinate z on C such that $z(0) = 0, z(\infty) = \infty$.

4.3.1. *The* $\mathbf{H}_a$*-torsor* $\widehat{\mathcal{Q}}^\alpha \to \mathcal{Q}^\alpha$. Here we refine the invertible subsheaves $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda$ by adding a *trivialization* v_λ of $\mathfrak{L}_\lambda$ off $\infty\mathbb{P}^1$, i.e., on $\mathbb{A}^1 = \mathbb{P}^1 - \infty$.

For $\alpha \in \mathbb{N}[I]$ we define the space $\widehat{\mathcal{Q}}^\alpha \xrightarrow{pr} \mathcal{Q}^\alpha$ formed by the collections $(v_\lambda \in \mathfrak{L}_\lambda \subset \mathcal{V}_\lambda)_{\lambda\in X^+}$ such that

a) $(\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda)_{\lambda\in X^+} \in \mathcal{Q}^\alpha$;

b) v_λ is a regular non-vanishing section of $\mathfrak{L}_\lambda$ on $\mathbb{A}^1 = \mathbb{P}^1 - \infty$;

c) $(v_\lambda)_{\lambda\in X^+}$ satisfy the Plücker equations like in 3.3.2.

It is easy to see that $\widehat{\mathcal{Q}}^\alpha \xrightarrow{pr} \mathcal{Q}^\alpha$ is a $\mathbf{H}_a$-torsor: $h(v_\lambda, \mathfrak{L}_\lambda) = (\lambda(h)v_\lambda, \mathfrak{L}_\lambda)$.

4.3.2. Taking a formal expansion at $0 \in C$ we obtain the closed embedding $\mathfrak{s}_\alpha :\ \widehat{\mathcal{Q}}^\alpha \hookrightarrow \mathcal{S}$. Evidently, $\mathfrak{s}_\alpha$ is compatible with the $\mathbf{H}_a$-action, so it descends to the same named closed embedding $\mathfrak{s}_\alpha :\ \mathcal{Q}^\alpha \hookrightarrow \mathbf{Q}$.

4.3.3. *Lemma.* Let $\beta \in \mathbb{N}[I]$. Then $\operatorname{codim}_{\mathbf{Q}} \mathbf{Q}^{-\beta} \ge 2|\beta|$.

Proof. Choose $\alpha \ge \beta$, and consider the closed embedding $\mathfrak{s}_\alpha :\ \mathcal{Q}^\alpha \hookrightarrow \mathbf{Q}$. Then $\mathfrak{s}_\alpha^{-1}(\mathbf{Q}^{-\beta}) = \mathcal{Q}^{\alpha-\beta}$ embedded into $\mathcal{Q}^\alpha$ as follows: $(\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda)_{\lambda\in X^+} \mapsto (\mathfrak{L}_\lambda(-\langle\beta,\lambda\rangle 0) \subset \mathcal{V}_\lambda)_{\lambda\in X^+}$. Now $\operatorname{codim}_{\mathbf{Q}} \mathbf{Q}^{-\beta} \ge \operatorname{codim}_{\mathcal{Q}^\alpha} \mathcal{Q}^{\alpha-\beta} = 2|\beta|$. □

4.3.4. *Remark.* One can show that for $\beta \in \mathbb{N}[I]$ we have $\operatorname{codim}_{\mathbf{Q}} \mathbf{Q}^{-\beta} = 2|\beta|$. We postpone the proof till Part II.

5. The Plücker model $\mathfrak{Z}^\alpha$ of based quasimaps (Plücker sections)

In this section we describe another model of the space $\mathcal{Z}^\alpha$ introduced in 3.6.

5.1. **Colored configuration spaces $\mathbb{A}^{(\alpha)}$ as spaces of unitary polynomials.** We fix a coordinate z on the affine line $\mathbb{A}^1 = \mathbb{P}^1 - \infty$. We will also view the configuration space $\mathbb{A}^\alpha \overset{\text{def}}{=} (\mathbb{A}^1)^\alpha$ (see 2.3) as the space of collections of unitary polynomials $(Q_\lambda)_{\lambda\in X^+}$ in z, such that (a) $\deg(Q_\lambda) = \langle\alpha, \lambda\rangle$, and (b) $Q_{\lambda+\mu} = Q_\lambda Q_\mu$.

5.2. **Plucker sections of degree α.** Recall the notations of 2.2. For each $\lambda \in X^+$ we will use the decomposition $V_\lambda = \mathbb{C}y_\lambda \oplus (\mathrm{Ker}x_\lambda) = (V_\lambda)^{\mathbf{N}} \oplus \mathfrak{n}_- V_\lambda$, compatible with the action of $\mathfrak{h} = \mathfrak{b}_- \cap \mathfrak{b}$, i.e., with the weight decomposition. For a section $v_\lambda \in \Gamma(\mathbb{A}^1, \mathcal{V}_\lambda) = V_\lambda \otimes \mathbb{C}[z] \overset{\text{def}}{=} V_\lambda[z]$, we will use a polynomial $Q_\lambda \overset{\text{def}}{=} \langle x_\lambda, v_\lambda\rangle \in \mathbb{C}[z]$, to write down the decomposition $v_\lambda = Q_\lambda \cdot\cdot y_\lambda \oplus v''_\lambda \in \mathbb{C}[z]\cdot\cdot y_\lambda \oplus (\mathrm{Ker}x_\lambda)[z] = V_\lambda[z]$.

Definition. (V.Drinfeld) The space $\mathfrak{Z}^\alpha$ of *Plücker sections* of degree α is the space of collections of sections $v_\lambda \in \Gamma(\mathbb{A}^1, \mathcal{V}_\lambda) = V_\lambda \otimes \mathbb{C}[z] \overset{\text{def}}{=} V_\lambda[z]$, $\lambda \in X^+$; such that for $v_\lambda = Q_\lambda \cdot\cdot y_\lambda \oplus v''_\lambda \in \mathbb{C}[z]\cdot\cdot y_\lambda \oplus (\mathrm{Ker}x_\lambda)[z]$, one has

a) Polynomial Q_λ is unitary of degree $\langle\alpha, \lambda\rangle$;

b) Component v''_λ of v_λ in $(\mathrm{Ker}x_\lambda)[z]$ has degree strictly less than $\langle\alpha, \lambda\rangle$;

c) For any $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ and $\phi^\vee(x_\nu) = x_\lambda \otimes x_\mu$ we have $\phi(v_\lambda \otimes v_\mu) = v_\nu$;

d) For any $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\phi(v_\lambda \otimes v_\mu) = 0$.

5.2.1. Collections $(v_\lambda)_{\lambda\in X^+}$ that satisfy (c), are determined by a choice of v_{ω_i}, $i \in I$. Hence $\mathfrak{Z}^\alpha$ is an affine algebraic variety.

5.2.2. Due to the properties a),c) above, the collection of polynomials Q_λ defined in a) satisfies the conditions of 5.1. Hence we have the map

$$\pi_\alpha : \mathfrak{Z}^\alpha \longrightarrow \mathbb{A}^\alpha$$

6. The global Grassmannian model (zastava model) $\mathbf{Z}^\alpha \subseteq \mathcal{G}_{\mathbb{A}^{(\alpha)}}$ of based quasimaps

In this section we describe yet another model of the space $\mathcal{Z}^\alpha$ introduced in 3.6.

6.1. **The B-type of a rational section of a G-torsor (a measure of singularity).** Let C be an arbitrary smooth projective curve; let $\mathcal{T}$ be a left $\mathbf{G}$-torsor over C, and let τ be a section of $\mathcal{T}$ defined over a Zariski open subset $U \subset C$, i.e., a trivialization of $\mathcal{T}$ over U. We will define a $\mathbf{B}$- (resp. $\mathbf{B}_-$-) type $d(\tau)$ (resp. $d_-(\tau)$): a measure of singularity of τ at $C - U$.

6.1.1. Section τ defines a $\mathbf{B}$-subtorsor $\mathbf{B}\cdots\tau\subseteq\mathcal{T}$. This reduction of $\mathcal{T}$ to $\mathbf{B}$ over U is the same as a section of $\mathbf{B}\backslash\mathcal{T}$ over U. Since $\mathbf{G}/\mathbf{B}$ is proper, this reduction (i.e. section), extends uniquely to the whole C. Thus we obtain a $\mathbf{B}$-subtorsor $\overline{\mathbf{B}\cdots\tau}\subseteq\mathcal{T}$ (the closure of $\mathbf{B}\cdots\tau\subseteq\mathcal{T}|U$ in $\mathcal{T}$), equipped with a section τ defined over U.

Using the projection $\mathbf{B}\longrightarrow\mathbf{H}_a$ we can induce $\overline{\mathbf{B}\cdots\tau}$ to a torsor over C for the abstract Cartan group $\mathbf{H}_a\cong\mathbf{B}/\mathbf{N}$ of $\mathbf{G}$; namely, $\mathcal{T}_{\tau,\mathbf{B}}\overset{\text{def}}{=}\mathbf{N}\backslash\overline{\mathbf{B}\cdots\tau}$, equipped with a section $\tau_{\mathbf{B}}$ defined over U.

The choice of simple coroots (cocharacters of $\mathbf{H}_a$) $I\subset Y$ identifies $\mathbf{H}_a$ with $(\mathbb{C}^*)^I$. Thus the section $\tau_{\mathbf{B}}$ of $\mathcal{T}_{\tau,\mathbf{B}}$ produces an I-colored divisor $d(\tau)$ supported at $C-U$. We will call $d(\tau)$ the $\mathbf{B}$-*type* of τ.

Replacing $\mathbf{B}$ by $\mathbf{B}_-$ in the above construction we define the $\mathbf{B}_-$-*type* $d_-(\tau)$.

6.2. **Subspace $\mathbf{Z}^\alpha$ of the relative Grassmannian $\mathcal{G}_{\mathbb{A}^{(\alpha)}}/\mathbb{A}^{(\alpha)}$.** Recall that A.Beilinson and V.Drinfeld have introduced the *relative Grassmannian* $\mathcal{G}_C^{(n)}$ over C^n for any $n\in\mathbb{N}$ (see [**?**]): its fiber $p_n^{-1}(x_1,\ldots,x_n)$ over an n-tuple $(x_1,\ldots,x_n)\in C^n$ is the space of isomorphism classes of $\mathbf{G}$-torsors $\mathcal{T}$ equipped with a section τ defined over $C-\{x_1,\ldots,x_n\}$.

We will consider a certain finite-dimensional subspace of a partially symmetrized version of the relative Grassmannian.

Definition. (A.Beilinson and V.Drinfeld) $\mathbf{Z}^\alpha$ is the space of isomorphism classes of the following data:

a) an I-colored effective divisor $D\in\mathbb{A}^\alpha$;

b) $\mathbf{G}$-torsor $\mathcal{T}$ over $\mathbb{P}^1$ equipped with a section τ defined over $\mathbb{P}^1-supp(D)$ such that:

i) $\mathbf{B}$-type $d(\tau)=0$;

ii) $\mathbf{B}_-$-type $d_-(\tau)$ is a negative divisor (opposite to effective) such that $d_-(\tau)+D$ is effective.

6.2.1. By the definition, the space $\mathbf{Z}^\alpha$ is equipped with a projection p_α to $\mathbb{A}^\alpha:\ (D,\mathcal{T},\tau)\mapsto D$. For a subset $U\subset\mathbb{A}^1$ we will denote by $\mathbf{Z}^\alpha_U$ the preimage $p_\alpha^{-1}(U)$.

6.2.2. The reader may find another realization of $\mathbf{Z}^\alpha$ in 10.3 below.

6.3. **Factorization (locality) property of spaces $\mathbf{Z}^\alpha$.** We will formulate the *factorization* property of spaces $\mathbf{Z}^\alpha\to C^\alpha$, i.e., of $\mathbf{Z}\to\mathcal{H}_C$ for $C=\mathbb{A}^1$.

- (**Complex geometry.**) For disjoint open $U,V\subseteq C$ there is a canonical isomorphism
$$\mathbf{Z}^\beta|_{U^\beta}\times\mathbf{Z}^\gamma|_{V^\gamma}\cong\mathbf{Z}^{\beta+\gamma}|_{(U^\beta\times V^\gamma)}.$$

- (**Algebraic Geometry.**) For the embedding $C^{\beta,\gamma}\subseteq C^\beta\times C^\gamma$ and the (etale) union map $\sqcup: C^{\beta,\gamma}\to C^{\beta+\gamma}$ there is a canonical isomorphism of pull backs to
$$(\mathbf{Z}^\beta\times\mathbf{Z}^\gamma)|_{C^{\beta,\gamma}}\cong\mathbf{Z}^{\beta+\gamma}|_{C^{\beta,\gamma}}.$$

6.3.1. Recall the following property of the Beilinson-Drinfeld relative Grassmannian $\mathcal{G}_C^{(n)}\xrightarrow{p_n}C^n$ (see [**?**]). Suppose an n-tuple $(x_1,\dots,x_n)\in C^n$ is represented as a union of an m-tuple $(y_1,\dots,y_m)\in C^m$ and a k-tuple $(z_1,\dots,z_k)\in C^k$, $k+m=n$, such that all the points of the m-tuple are disjoint from all the points of the k-tuple. Then $p_n^{-1}(x_1,\dots,x_n)$ is canonically isomorphic to the product $p_m^{-1}(y_1,\dots,y_m)\times p_k^{-1}(z_1,\dots,z_k)$

6.3.2. Suppose we are given a decomposition $\alpha=\beta+\gamma,\ \beta,\gamma\in\mathbb{N}[I]$ and two disjoint subsets $U,\Upsilon\subset\mathbb{A}^1$. Then $U^\beta\times\Upsilon^\gamma$ lies in $\mathbb{A}^\alpha$, and we will denote the preimage $p_\alpha^{-1}(U^\beta\times\Upsilon^\gamma)$ in $\mathbf{Z}^\alpha$ by $\mathbf{Z}_{U,\Upsilon}^{\beta,\gamma}=\mathbf{Z}^\alpha|_{(U^\beta\times\Upsilon^\gamma)}$ (cf. 6.2.1).

The above property of relative Grassmannian immediately implies the following

Factorization structure. There is a canonical factorization isomorphism $\mathbf{Z}_{U,\Upsilon}^{\beta,\gamma}\cong\mathbf{Z}_U^\beta\times\mathbf{Z}_\Upsilon^\gamma$, i.e.,
$$\mathbf{Z}^\alpha|_{(U^\beta\times\Upsilon^\gamma)}\cong\mathbf{Z}^\beta|_{U^\beta}\times\mathbf{Z}^\gamma|_{\Upsilon^\gamma}.$$

6.4. **The fibers of $\mathcal{Z}^\alpha/\mathbb{A}^{(\alpha)}$ as intersections of dual semiinfinite Schubert cells in the loop Grassmannian.** Let us describe the fibers of p_α in terms of the normal slices to the semiinfinite Schubert cells in the loop Grassmannian.

6.4.1. Let $\mathcal{G}$ be the usual affine Grassmannian $\mathbf{G}((z))/\mathbf{G}[[z]]$. It is naturally identified with the fiber of $\mathcal{G}_{\mathbb{P}^1}^{(1)}$ over the point $0\in\mathbb{P}^1$. Due to the Iwasawa decomposition in p-adic groups, there is a natural bijection between Y and the set of orbits of the group $\mathbf{N}((z))$ (resp. $\mathbf{N}_-((z))$) in $\mathcal{G}$; for $\gamma\in Y$ we will denote the corresponding orbit by S_γ (resp. T_γ). We will denote by $\overline{T}_\gamma$ the "closure" of T_γ, that is, the union $\cup_{\beta\geq\gamma}T_\gamma$.

It is proved in [**?**] that the intersection $\overline{T}_\gamma\cap S_\beta$ is not empty iff $\gamma\leq\beta$. Then it is an affine algebraic variety, a kind of a normal slice to T_β in $\overline{T}_\gamma$. Let us call it $TS_{\gamma,\beta}\stackrel{\text{def}}{=}\overline{T}_\gamma\cap S_\beta$ for short. If $\operatorname{rank}(\mathbf{G})>1$ then $TS_{\gamma,\beta}=\overline{T}_\gamma\cap S_\beta$ is not necessarily irreducible. But it is always equidimensional of dimension $|\beta-\gamma|$. There is a natural bijection between the set of irreducible components of $TS_{\gamma,\beta}=\overline{T}_\gamma\cap S_\beta$ and the canonical basis of $U_{\beta-\gamma}^+$ (the weight $\beta-\gamma$ component of the quantum universal enveloping algebra of $\mathfrak{n}$) (see [**?**] for the definition of canonical basis of U^+).

6.4.2. Recall the diagonal stratification of $\mathbb{A}^\alpha$ defined in 2.3 and the map $p_\alpha:\mathbf{Z}^\alpha\to\mathbb{A}^\alpha$. We consider a partition $\Gamma:\ \alpha=\sum_{k=1}^m\gamma_k$ and a divisor D in the stratum $\mathbb{A}_\Gamma^\alpha$. The interested reader will check readily the following

Claim. $p_\alpha^{-1}(D)$ is isomorphic to the product $\prod_{k=1}^m TS_{-\gamma_k,0}=\prod_{k=1}^m\overline{T}_{-\gamma_k}\cap S_0\cong\prod_{k=1}^m\overline{T}_0\cap S_{\gamma_k}$.

In particular, the fiber over a point in the closed stratum is isomorphic to $TS_{-\alpha,0} = \overline{T}_{-\alpha} \cap S_0 \cong \overline{T}_0 \cap S_\alpha$, while the fiber over a generic point is isomorphic to the product of affine lines $TS_{-i,0} \cong \overline{T}_0 \cap S_{-i} \cong \mathbb{A}^1$, that is, the affine space $\mathbb{A}^{|\alpha|}$.

6.4.3. *Corollary.* $\mathbf{Z}^\alpha$ is irreducible.

7. Equivalence of three constructions of zastava spaces: $\mathcal{Z}^\alpha \cong \mathfrak{Z}^\alpha \cong \mathbf{Z}^\alpha$

7.1. **Quasimaps and Plucker sections: $\mathcal{Z}^\alpha \cong \mathfrak{Z}^\alpha$.** In this subsection we construct an isomorphism $\varpi : \mathcal{Z}^\alpha \xrightarrow{\sim} \mathfrak{Z}^\alpha$, i.e., from the subsheaves $\mathfrak{L}_\lambda \subseteq \mathcal{V}_\lambda$ we construct the sections $v_\lambda \in \Gamma(\mathbb{A}^1, \mathcal{V}_\lambda)$.

7.1.1. Let $f \in \mathcal{Z}^\alpha$ be a quasimap given by a collection $(\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda = V_\lambda \otimes \mathcal{O}_{\mathbb{P}^1})_{\lambda \in X^+}$. Since $\mathfrak{L}_\lambda|_{\mathbb{A}^1}$ is trivial, it has a unique up to proportionality section v_λ generating it over $\mathbb{A}^1$.

We claim that the pairing $\langle x_\lambda, v_\lambda \rangle$ does not vanish identically. In effect, since $\deg(f) = \alpha$, the meromorphic section $\frac{v_\lambda}{z^{\langle \alpha,\lambda \rangle}}$ of $\mathcal{V}_\lambda$ is regular non-vanishing at $\infty \in \mathbb{P}^1$. Moreover, since $f(\infty) = \mathbf{B}_-$, we have $\frac{v_\lambda}{z^{\langle \alpha,\lambda \rangle}}(\infty) \in V_\lambda^{\mathbf{N}_-}$. Thus, $\langle x_\lambda, \frac{v_\lambda}{z^{\langle \alpha,\lambda \rangle}} \rangle(\infty) \neq 0$.

Now we can normalize v_λ (so far defined up to a multiplication by a constant) by the condition that $\langle x_\lambda, v_\lambda \rangle$ is a unitary polynomial. Let us denote this polynomial by Q_λ. It has degree $d_\lambda \leq \langle \alpha, \lambda \rangle$ since $\deg(f) = \alpha$. Since $\frac{v_\lambda}{z^{\langle \alpha,\lambda \rangle}}(\infty) \in V_\lambda^{\mathbf{N}_-}$, we see that $\deg\langle e, v_\lambda \rangle < d_\lambda$ for any $e \perp y_\lambda$. Moreover, since $\deg(f) = \alpha$ we must then have $d_\lambda = \langle \alpha, \lambda \rangle$.

Thus we have checked that the collection $(v_\lambda)_{\lambda \in X^+}$ satisfies the conditions a),b) of the Definition 5.2. The conditions c),d) of *loc. cit.* follow from the conditions b),c) of the Definition 3.3. In other words, we have defined the Plücker section

$$\varpi(f) \stackrel{\text{def}}{=} (v_\lambda)_{\lambda \in X^+} \in \mathfrak{Z}^\alpha$$

7.1.2. Here is the inverse construction. Given a Plücker section $(v_\lambda)_{\lambda \in X^+} \in \mathfrak{Z}^\alpha$ we define the corresponding quasimap $f = (\mathfrak{L}_\lambda)_{\lambda \in X^+} \in \mathcal{Z}^\alpha$ as follows.

We can view v_λ as a regular section of $\mathcal{V}_\lambda(\langle \alpha, \lambda \rangle \infty)$ over the whole $\mathbb{P}^1$. It generates an invertible subsheaf $\mathfrak{L}'_\lambda \subset \mathcal{V}_\lambda(\langle \alpha, \lambda \rangle \infty)$. We define

$$\mathfrak{L}_\lambda \stackrel{\text{def}}{=} \mathfrak{L}'_\lambda(-\langle \alpha, \lambda \rangle \infty) \subset \mathcal{V}_\lambda$$

7.1.3. It is immediate to see that the above constructions are inverse to each other, so that $\varpi : \mathcal{Z}^\alpha \longrightarrow \mathfrak{Z}^\alpha$ is an isomorphism.

7.1.4. *Remark.* Note that the definition of the space $\mathcal{Z}^\alpha$ depends only on the choice of Borel subgroup $\mathbf{B}_- \subset \mathbf{G}$, while the definition of $\mathfrak{Z}^\alpha$ depends also on the choice of the opposite Borel subgroup $\mathbf{B} \subset \mathbf{G}$ or, equivalently, on the choice of the Cartan subgroup $\mathbf{H} \subset \mathbf{B}_-$.

We want to stress that the projection $\pi_\alpha: \mathcal{Z}^\alpha = \mathfrak{Z}^\alpha \longrightarrow \mathbb{A}^\alpha$ *does depend* on the choice of $\mathbf{B}$. Let us describe $\pi_\alpha\varpi(f)$ for a genuine map (as opposed to quasimap) $f \in \mathcal{Z}^\alpha$. To this end recall (see 2.1) that the $\mathbf{B}$-invariant Schubert varieties $\overline{\mathcal{B}}_{s_iw_0},\ i \in I$, are divisors in $\mathcal{B}$. Their formal sum may be viewed as an I-colored divisor $\mathfrak{D}$ in $\mathcal{B}$. Then $f^*\mathfrak{D}$ is a well defined I-colored divisor on $\mathbb{P}^1$ since $f(\mathbb{P}^1) \not\subset \mathfrak{D}$ since $f(\infty) = \mathbf{B}_- \in \mathcal{B}_{w_0}$. For the same reason the point ∞ does not lie in $f^*\mathfrak{D}$, so $f^*\mathfrak{D}$ is really a divisor in $\mathbb{A}^1$. It is easy to see that $f^*\mathfrak{D} \in \mathbb{A}^\alpha$ and $f^*\mathfrak{D} = \pi_\alpha\varpi(f)$.

7.2. **Relative Grassmannian and Plucker sections: $\mathfrak{Z}^\alpha \cong \mathbf{Z}^\alpha$.** In this subsection we construct an isomorphism $\xi:\ \mathfrak{Z}^\alpha \xrightarrow{\sim} \mathbf{Z}^\alpha$, so from a system of sections υ_λ we construct a $\mathbf{G}$-torsor $\mathcal{T}$ with a section τ and an I-colored divisor D.

7.2.1. *Lemma.* (*The Plücker picture of* $\mathbf{G}$.) The map $\psi:\ \ g \mapsto (gx_\lambda, gy_\lambda)_{\lambda\in X^+}$ is a bijection between $\mathbf{G}$ and the space of collections $\{(u_\lambda \in V_\lambda^\vee, \upsilon_\lambda \in V_\lambda)_{\lambda\in X^+})\}$ satisfying the following conditions:

a) For any $\mathbf{G}$-morphism $\phi:\ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda+\mu$ and $\phi^\vee(x_\nu) = x_\lambda \otimes x_\mu$ we have $\phi(\upsilon_\lambda \otimes \upsilon_\mu) = \upsilon_\nu$;

b) For any $\mathbf{G}$-morphism $\phi:\ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda+\mu$ we have $\phi(\upsilon_\lambda \otimes \upsilon_\mu) = 0$;

c) For any $\mathbf{G}$-morphism $\varphi:\ V_\lambda^\vee \otimes V_\mu^\vee \longrightarrow V_\nu^\vee$ such that $\nu = \lambda+\mu$ and $\varphi(x_\lambda \otimes x_\mu) = x_\nu$ we have $\varphi(u_\lambda \otimes u_\mu) = u_\nu$;

d) For any $\mathbf{G}$-morphism $\varphi:\ V_\lambda^\vee \otimes V_\mu^\vee \longrightarrow V_\nu^\vee$ such that $\nu < \lambda+\mu$ we have $\varphi(u_\lambda \otimes u_\mu) = 0$;

e) $\langle u_\lambda, \upsilon_\lambda\rangle = 1$.

Proof. We are considering the systems $(v,u) = (v_\lambda \in V_\lambda,\ u_\lambda \in V_\lambda^\vee,\ \lambda \in X^+)$ such that both v and u are Plücker sections and $< v, u >= 1$, i.e., $< v_\lambda, u_\lambda >= 1$ for each λ.

These form a $\mathbf{G}$-torsor and we have fixed its element (y,x), which we will use to think of this torsor as a Plücker picture of $\mathbf{G}$.

The stabilizers $\mathbf{G}_v$ and $\mathbf{G}_u$ are the unipotent radicals of the opposite Borel subgroups, for instance $\mathbf{G}_x = \mathbf{N}$ and $\mathbf{G}_y = \mathbf{N}_-$. So this torsor canonically maps into the open $\mathbf{G}$-orbit in $\mathcal{B}\times\mathcal{B}$ and the fiber at $(\mathbf{B}', \mathbf{B}'')$ is a torsor for a Cartan subgroup $\mathbf{B}' \cap \mathbf{B}''$. □

7.2.2. Given a Plücker section $(\upsilon_\lambda)_{\lambda\in X^+}$, the collection of meromorphic sections $(x_\lambda \in \mathcal{V}_\lambda^\vee, \frac{\upsilon_\lambda}{Q_\lambda} \in \mathcal{V}_\lambda)$ evidently satisfies the conditions a)–e) of the above Lemma, and hence defines a meromorphic function $g:\ \mathbb{A}^1 \longrightarrow \mathbf{G}$. Let us list the properties of this function.

a) By the definition 7.2.1 of the isomorphism ψ, since g fixes the Plücker section x the function g actually takes values in $\mathbf{N} \subset \mathbf{G}$;

b) The argument similar to that used in 7.1.1 shows that g can be extended to $\mathbb{P}^1$, is regular at ∞, and $g(\infty) = 1 \in \mathbf{N}$ (since $\frac{v_\lambda}{Q_\lambda}(\infty) = y_\lambda$, $g(\infty)$ stabilizes x_λ and y_λ so it lies in $\mathbf{N} \cap \mathbf{N}_-$);

c) Let $D = \pi_\alpha(v_\lambda)$ (see 5.2.2) be the I-colored divisor supported at the roots of Q_λ. Then g is regular on $\mathbb{P}^1 - D$.

7.2.3. We define $\xi(v_\lambda) = (D, \mathcal{T}, \tau) \in \mathbf{Z}^\alpha$ as follows: $D = \pi_\alpha(v_\lambda)$; $\mathcal{T}$ is the trivial $\mathbf{G}$-torsor; the section τ is given by the function g. Let us describe the corresponding $\mathbf{H}_a$-torsor $\mathcal{T}_{\tau,\mathbf{B}_-}$ with meromorphic section $\tau_{\mathbf{B}_-}$. To describe an $\mathbf{H}_a$-torsor $\mathfrak{L}$ with a section s it suffices to describe the induced $\mathbb{C}^*$-torsors $\mathfrak{L}_\lambda$ with sections s_λ for all characters $\lambda: \mathbf{H}_a \longrightarrow \mathbb{C}^*$. In fact, it suffices to consider only $\lambda \in X^+$. Then $\mathfrak{L}_\lambda$ is given by the construction of 7.1.2, and $s_\lambda = \frac{v_\lambda}{Q_\lambda}$.

Thus, the conditions i),ii) of the Definition 6.2 are evidently satisfied.

7.2.4. To proceed with the inverse construction, we will need the following

Lemma. Suppose $(D, \mathcal{T}, \tau) \in \mathbf{Z}^\alpha$. Then $\mathcal{T}$ is trivial and has a canonical section ς.

Proof. By the construction 6.1.1, $\mathcal{T}$ is induced from the $\mathbf{B}$-torsor $\overline{\mathbf{B}\cdots\tau}$. By the Definition 6.2, the induced $\mathbf{H}_a$-torsor $\mathcal{T}_{\tau,\mathbf{B}}$ is trivial, that is, $\overline{\mathbf{B}\cdots\tau}$ can be further reduced to an $\mathbf{N}$-torsor. But any $\mathbf{N}$-torsor over $\mathbb{P}^1$ is trivial since $H^1(\mathbb{P}^1, \mathbf{V}) = 0$ for any unipotent group $\mathbf{V}$ (induction in the lower central series). □

7.2.5. According to the above Lemma, we can find a unique section ς of $\mathcal{T}$ defined over the whole $\mathbb{P}^1$ and such that $\varsigma(\infty) = \tau(\infty)$. Hence a triple $(D, \mathcal{T}, \tau) \in \mathbf{Z}^\alpha$ canonically defines a meromorphic function

$$g \stackrel{\text{def}}{=} \tau\varsigma^{-1} : \mathbb{P}^1 \longrightarrow \mathbf{G},$$

i.e., $g(x)\cdots\varsigma(x) = \tau(x)$, $x \in \mathbb{P}^1$. One sees immediately that g enjoys the properties 7.2.2a)–c). Now we can apply the Lemma 7.2.1 in the opposite direction and obtain from g a collection $\psi^{-1}(g) = (x_\lambda, \tilde{v}_\lambda)_{\lambda \in X^+}$ with $\tilde{v}_\lambda$ a certain meromorphic sections of $\mathcal{V}_\lambda$. According to 5.1, the divisor D defines a collection of unitary polynomials $(Q_\lambda)_{\lambda \in X^+}$, and we can define $v_\lambda \stackrel{\text{def}}{=} Q_\lambda \tilde{v}_\lambda$. One checks easily that $(v_\lambda) \in \mathfrak{Z}^\alpha$, and $(D, \mathcal{T}, \tau) = \xi(v_\lambda)$.

In particular, $\xi: \mathfrak{Z}^\alpha \longrightarrow \mathbf{Z}^\alpha$ is an isomorphism.

7.3. **Based quasimaps spaces $\mathcal{Z}^\alpha$: a summary (locality ...)** We conclude that $\mathcal{Z}^\alpha, \mathfrak{Z}^\alpha, \mathbf{Z}^\alpha$ are all the same and all maps to $\mathbb{A}^\alpha$ coincide. We preserve the notation $\mathcal{Z}^\alpha$ for this *Zastava* space, and π_α for its projection onto $\mathbb{A}^\alpha$. We combine the properties 3.6.1, 5.2.1, 6.3.2, 6.4.3 into the following

Theorem. a) $\mathcal{Z}^\alpha$ is an irreducible affine algebraic variety of dimension $2|\alpha|$;

b) For any decomposition $\alpha=\beta+\gamma,\ \beta,\gamma\in\mathbb{N}[I]$, and a pair of disjoint subsets $U,\Upsilon\subset\mathbb{A}^1$, we have the *factorization property* (notations of 6.2.1 and 6.3.2):

$$\mathcal{Z}^{\beta,\gamma}_{U,\Upsilon}=\mathcal{Z}^\beta_U\times\mathcal{Z}^\gamma_\Upsilon$$

CHAPTER 2. The category $\mathcal{PS}$ of semiinfinite perverse sheaves on ...

8. Schubert stratification of (based) quasimap spaces $\mathcal{Z}^\alpha \subseteq \mathcal{Q}^\alpha$ for the curve $\mathbb{P}^1$

We consider fine Schubert stratifications of quasimaps $\mathcal{Q} = QMap(\mathbb{P}^1, \mathcal{B})$ and based quasimaps $\mathcal{Z}_{\mathbb{P}^1,\infty}$. These strata are obviously smooth in the case of quasimaps and conjecturally also for based quasimaps. the latter is proved (in this paper) only for the strata $\overset{\circ}{\mathcal{Z}}{}^\gamma_w$ such that the parameter γ is sufficiently dominant (this is still sufficient though less elegant). It is convenient to consider a diamond of stratifications:

$$coarse = \underline{\underline{\mathcal{S}}}_0 \ \leq fine = \underline{\underline{\mathcal{S}}}_{0,G_m}, \ Schubert = \underline{\underline{\mathcal{S}}}_{0,I} \ \leq \ fine\ Schubert = \underline{\underline{\mathcal{S}}}_{0,G_m,I}.$$

Remark. Since the based quasimaps are defined at ∞ so we are only interested in how they behave over $\mathbb{A}^1$.

8.1. The diamond of stratifications of $\mathcal{Z}^\alpha_{\mathbb{P}^1,\infty} \subseteq \mathcal{Q}^\alpha_{\mathbb{P}^1}$: (i) *coarse* $\underline{\underline{\mathcal{S}}}_0$, (ii) *fine* $\underline{\underline{\mathcal{S}}}_{0,G_m}$, (iii) *Schubert* $\underline{\underline{\mathcal{S}}}_{0,\mathbf{I}}$, (iv) *fine Schubert* $\underline{\underline{\mathcal{S}}}_{0,G_m,\mathbf{I}}$.

We are interested in a "true" stratification, i.e., where the strata are smooth. This is true only on the last level, i.e., the fine Schubert strata are smooth.

The fine stratification is the usual stratification of quasimaps by *singularities on* $\mathbb{A}^1$ that are caused by shifting genuine maps by finite subschemes, The new aspect (the coarse strata) is that it is sometimes useful to keep less information and only consider the singularity at the distinguished point $0 \in \mathbb{A}^1$.

The final ingredient concerns the "Iwahori orbits" or "Schubert cells" in quasimaps (defined at 0). These are given by the cell $\mathcal{B}_w$ that contains the value of a quasimap at the distinguished point $0 \in \mathbb{A}^1$.

8.1.1. We denote by $\mathcal{Q}^\alpha \supseteq \overset{\bullet}{\mathcal{Q}}{}^\alpha \supseteq \overset{\circ}{\mathcal{Q}}{}^\alpha$, respectively the variety of all quasimaps of degree α and the subvarieties of the quasimaps defined at 0 and of genuine maps. In the same way we denote the varieties of based quasimaps $\mathcal{Z}^\alpha \supseteq \overset{\bullet}{\mathcal{Z}}{}^\alpha \overset{\text{def}}{=} \mathcal{Z}^\alpha \cap \overset{\bullet}{\mathcal{Q}}{}^\alpha \supseteq \overset{\circ}{\mathcal{Z}}{}^\alpha \overset{\text{def}}{=} \mathcal{Z}^\alpha \cap \overset{\circ}{\mathcal{Q}}{}^\alpha =$ based maps of degree α.

8.1.2. <u>*Step 1.*</u> [Coarse stratification $\underline{\underline{\mathcal{S}}}_0$.] *Strata* $\overset{\bullet}{\mathcal{Z}}{}^\beta$ *of* $\mathcal{Z}^\alpha$ *according to singularity at* 0. It follows immediately from the Theorem 3.4.2 that

$$\mathcal{Z}^\alpha \cong \bigsqcup_{0 \leq \beta \leq \alpha} \overset{\bullet}{\mathcal{Z}}{}^\beta.$$

The closed embedding of a stratum $\overset{\bullet}{\mathcal{Z}}{}^\beta$ into $\mathcal{Z}^\alpha$ will be denoted by $\sigma_{\beta,\alpha-\beta}$.

8.1.3. <u>*Step 2.*</u> [Fine stratification $\underline{\underline{\mathcal{S}}}_{0,G_m}$.] *Strata* $\overset{\circ}{\mathcal{Z}}{}^{\beta}\times(G_m)^{\alpha-\beta}$ *of* $\overset{\bullet}{\mathcal{Z}}{}^{\alpha}$ *(quasimaps defined at* 0*) by singularities on* G_m. The consideration of quasimaps on G_m adds the diagonal stratification of the configuration space $(G_m)^{\delta} = \bigsqcup_{\Gamma\in\mathfrak{P}(\delta)}(G_m)^{\delta}_{\Gamma}$. Again, it follows immediately from the Theorem 3.4.2 that

$$\overset{\bullet}{\mathcal{Z}}{}^{\alpha} \cong \bigsqcup_{0\le\beta\le\alpha} \overset{\circ}{\mathcal{Z}}{}^{\beta}\times(G_m)^{\alpha-\beta}.$$

8.1.4. <u>*Step 3.*</u> [Schubert Stratification $\underline{\underline{\mathcal{S}}}_{0,\mathbf{I}}$]. *Strata* $\overset{\bullet}{\mathcal{Z}}{}^{\alpha}_{w}$ *of* $\overset{\bullet}{\mathcal{Z}}{}^{\alpha}$ *("defined at* 0*") according to the Bruhat cell that contains the value at* 0*,* This refinement comes from the decomposition of the flag variety $\mathcal{B}$ into the $\mathbf{B}$-invariant Schubert cells. It is defined on $\overset{\bullet}{\mathcal{Z}}{}^{\alpha}$ as it uses the value of a quasimap at 0.

For $w\in\mathcal{W}_f$, we define the locally closed subvariety (*Schubert strata*) $\overset{\bullet}{\mathcal{Z}}{}^{\alpha}_{w}\subset\overset{\bullet}{\mathcal{Z}}{}^{\alpha}$ as the set of quasimaps f such that $f(0)\in\mathcal{B}_w$. The closure of $\overset{\bullet}{\mathcal{Z}}{}^{\alpha}_{w}$ in $\mathcal{Z}^{\alpha}$ will be denoted by $\overline{\mathcal{Z}}{}^{\alpha}_{w}$. Evidently, these are a filtration and the corresponding stratification of $\overset{\bullet}{\mathcal{Z}}{}^{\alpha}$:

$$\overset{\bullet}{\mathcal{Z}}{}^{\alpha} = \bigsqcup_{w\in\mathcal{W}_f} \overset{\bullet}{\mathcal{Z}}{}^{\alpha}_{w}.$$

Beware that $\overset{\bullet}{\mathcal{Z}}{}^{\alpha}_{w}$ may happen to be empty: e.g. for $\alpha=0$ and $w\neq w_0$.

8.1.5. <u>*Step 4.*</u> [Fine Schubert stratification $\underline{\underline{\mathcal{S}}}_{0,G_m,\mathbf{I}}$.] *This is the intersection of the Schubert and fine stratification s.* for the (*fine Schubert strata*) $\overset{\circ}{\mathcal{Z}}{}^{\alpha}_{w}\subset\overset{\circ}{\mathcal{Z}}{}^{\alpha}$ as quasimaps $f\overset{\circ}{\mathcal{Z}}{}^{\alpha}$ such that $f(0)\in\mathcal{B}_w$. Evidently,

$$\overset{\circ}{\mathcal{Z}}{}^{\alpha} = \bigsqcup_{w\in\mathcal{W}_f} \overset{\circ}{\mathcal{Z}}{}^{\alpha}_{w}.$$

8.1.6. *Summary of stratifications of* $\mathcal{Z}^{\alpha}_{\mathbb{P}^1,\infty}$. Altogether, we obtain the following stratifications of $\mathcal{Z}^{\alpha}$:

$$\mathcal{Z}^{\alpha} \cong \bigsqcup_{\alpha\ge\beta} \overset{\bullet}{\mathcal{Z}}{}^{\beta} \quad (\textit{coarse stratification})$$

$$\cong \bigsqcup_{\Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\ge\beta\ge\gamma} \overset{\circ}{\mathcal{Z}}{}^{\gamma}\times(\mathbb{A}^1)^{\beta-\gamma}_{\Gamma} \quad (\textit{fine stratification})$$

$$\cong \bigsqcup_{w\in\mathcal{W}_f,\ \Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\ge\beta\ge\gamma} \overset{\circ}{\mathcal{Z}}{}^{\gamma}_{w}\times(G_m)^{\beta-\gamma}_{\Gamma} \quad (\textit{fine Schubert stratification}).$$

8.1.7. *Stratifications of* $\mathcal{Q}^\alpha_{\mathbb{P}^1}$. Similarly, we have the *fine stratification* (resp. *fine Schubert stratification*) of quasimaps $\mathcal{Q}^\alpha$ on $\mathbb{P}^1$:

$$\mathcal{Q}^\alpha = \bigsqcup_{\Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\geq\beta\geq\gamma} \overset{\circ}{\mathcal{Q}}{}^\gamma \times (\mathbb{P}^1-0)^{\beta-\gamma}_\Gamma = \bigsqcup_{w\in\mathcal{W}_f,\ \Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\geq\beta\geq\gamma} \overset{\circ}{\mathcal{Q}}{}^\gamma_w \times (\mathbb{P}^1-0)^{\beta-\gamma}_\Gamma$$

Here $\overset{\circ}{\mathcal{Q}}{}^\gamma_w \subset \overset{\circ}{\mathcal{Q}}{}^\gamma$ denotes the locally closed subspace of maps $\mathbb{P}^1 \to \mathcal{B}$ taking value in $\mathcal{B}_w \subset \mathcal{B}$ at $0 \in \mathbb{P}^1$.

Notice that once the quasimaps are not required to be based, the singularities may now appear also at ∞.

Lemma. The strata $\overset{\circ}{\mathcal{Q}}{}^\gamma_w \times (\mathbb{P}^1-0)^{\beta-\gamma}_\Gamma$ are smooth.

Proof. $\overset{\circ}{\mathcal{Q}}{}^\gamma$ is the space of maps f of degree γ from $\mathbb{P}^1$ to $\mathcal{B}$ so we know that it is smooth. The condition that $f(0)$ is in $\mathcal{B}_w$ does not cause a problem because of the G symmetry of $\overset{\circ}{\mathcal{Q}}{}^\gamma$. □

8.2. **Smoothness of fine Schubert strata (conjecture and the stable case).** While the fine Schubert strata $\overset{\circ}{\mathcal{Q}}{}^\gamma_w$ strata in quasimaps are evidently smooth, this is not so obvious for based quasimaps because imposing the condition $f(\infty) = \mathfrak{b}_-$ breaks the G-symmetry of values at 0.

Remark. The coarse Schubert strata $\overset{\bullet}{\mathcal{Z}}{}^\alpha_w$ are not necessarily smooth in general (for instance for $\mathbf{G} = SL_3$, α the sum of simple coroots and $w = w_0$). The understanding of the "fine Schubert strata" $\overset{\circ}{\mathcal{Z}}{}^\gamma_w \times (G_m)^\beta_\Gamma$ reduces to the varieties $\overset{\circ}{\mathcal{Z}}{}^\gamma_w$.

Conjecture. For $\gamma \in \mathbb{N}[I], w \in \mathcal{W}_f$ the variety $\overset{\circ}{\mathcal{Z}}{}^\gamma_w$ is smooth. Hence the "fine Schubert stratification" is really a stratification.

8.2.1. *Lemma.* For γ sufficiently dominant (e.g. $\langle\gamma, i'\rangle > 10$) and arbitrary $w \in \mathcal{W}_f$ the variety $\overset{\circ}{\mathcal{Z}}{}^\gamma_w$ is smooth.

Proof. Let us consider the map $\varrho_\gamma : \overset{\circ}{\mathcal{Q}}{}^\gamma \longrightarrow \mathcal{B}\times\mathcal{B},\ f \mapsto (f(0), f(\infty))$. We have $\overset{\circ}{\mathcal{Z}}{}^\gamma = \varrho_\gamma^{-1}(\mathcal{B}_w, \mathbf{B}_-)$. It suffices to prove that ϱ_γ is smooth and surjective. Recall that the tangent space Θ_f to $\overset{\circ}{\mathcal{Q}}{}^\gamma$ at $f \in \overset{\circ}{\mathcal{Q}}{}^\gamma$ is canonically isomorphic to $H^0(\mathbb{P}^1, f^*\mathcal{TB})$. Let us interpret $\mathcal{B}$ as a variety of Borel subalgebras of $\mathfrak{g}$. We denote $f(0)$ by $\mathfrak{b}_0$, and $f(\infty)$ by $\mathfrak{b}_\infty$. So we have to prove that the canonical map $H^0(\mathbb{P}^1, f^*\mathcal{TB}) \longrightarrow \mathcal{T}_{\mathfrak{b}_0}\mathcal{B} \oplus \mathcal{T}_{\mathfrak{b}_\infty}\mathcal{B}$ is surjective. To this end it is enough to have $H^1(\mathbb{P}^1, f^*\mathcal{TB}(-0-\infty)) = 0$. This in turn holds whenever γ is sufficiently dominant. □

8.2.2. *Lemma.* For γ sufficiently dominant we have $\dim \overset{\circ}{\mathcal{Z}}{}^{\gamma}_{w} = 2|\gamma| - \dim \mathcal{B} + \dim \mathcal{B}_w$.

Proof. The same as the proof of 8.2.1. □

8.2.3. *Remark.* Unfortunately, one cannot prove the conjecture 8.2 for arbitrary γ the same way as the Lemma 8.2.1: for arbitrary γ the map ϱ_γ is not smooth. The simplest example occurs for $\mathbf{G} = SL_4$ when γ is twice the sum of simple coroots. This example was found by A.Kuznetsov.

9. The category $\mathcal{PS}$ of perverse sheaves on the factorization space $\mathcal{Z}$

This section follows closely §4 of [**?**].

Summary.

A. System of spaces $\mathcal{Z}^\alpha_\chi$.

9.0.1. *Zastava space $\mathcal{Z}(G) = \mathcal{Z}_{C\times I}(G)$, a local space over $C\times I$.* Let C be a smooth curve (here taken to be $\mathbb{A}^1$) and let $\mathcal{H}_{C\times I} = \sqcup_\alpha\, C^\alpha$ be the punctual Hilbert scheme of C.

We start with the zastava space $Z(G)$ which is a local space $\mathcal{Z}^{\mathbf{u}} \to \mathcal{H}_{C\times I}$ over $C\times I$, with the connected components $\mathcal{Z}^\alpha \to C^\alpha,\ \alpha \in \mathbf{N}[\mathbf{I}]$.

9.0.2. *A system $\boldsymbol{\mathcal{Z}}$ of local spaces over $C\times I$.* In 9.1 we introduce a system $\boldsymbol{\mathcal{Z}} = \mathcal{Z}(G,C)$ of local spaces $\boldsymbol{\mathcal{Z}}_\chi$ over a curve C.

- Each $\boldsymbol{\mathcal{Z}}_\chi,\ \chi \in Y$, is a copy of the local space $\mathcal{Z}$ over the I-multiple $C\times I$ of a curve C. So, $\boldsymbol{\mathcal{Z}}_\chi \to \mathcal{H}_{C\times I}$ is a system of $\mathcal{Z}^\alpha_\chi \to X^\alpha$.
- The $\mathbf{N}[\mathbf{I}]$-action on Y lifts to an action on $\boldsymbol{\mathcal{Z}} \to Y$. We use the distinguished point $a \in C$ to make $\gamma \in \mathbf{N}[\mathbf{I}]$ act by the γa twist of quasimaps

$$\mathcal{Z}_\chi \hookrightarrow \mathcal{Z}_{\chi+\gamma} \quad \text{where} \quad \mathcal{Z}^\alpha_\chi \hookrightarrow \mathcal{Z}^{\alpha\gamma}_{\chi+\gamma}, \quad f \mapsto f(-\gamma a).$$

Remark. Passing to the indschemes $\overset{\rightarrow}{\mathcal{Z}}_\circ \overset{\text{def}}{=} \lim_{\rightarrow\, be\in\mathbf{N}[\mathbf{I}]} \mathcal{Z}^\beta_{\chi+\beta}$. one looses the locality property. (Notice that $\mathcal{Z}^\alpha_\chi \hookrightarrow \mathcal{Z}_{\chi-\alpha}$.)

9.0.3. *The intuition: $\boldsymbol{\mathcal{Z}}$ is a visible part of a "semiinfinite" space $\underline{\mathcal{Z}}$.* The system $\boldsymbol{\mathcal{Z}}$ is intuitively what we can see from a semi-infinite space $\underline{\mathcal{Z}}$ which is stratified by strata $\underline{\mathcal{Z}}_\chi,\ \chi \in Y$. We are describing $\underline{\mathcal{Z}}$ by the system $\boldsymbol{\mathcal{Z}}$ of spaces $\mathcal{Z}^\alpha_\chi$ and the meaning of $\mathcal{Z}^\alpha_\chi$ is that it is a normal slice to the stratum $\mathcal{S}_{\chi-\alpha}$ inside $\overline{\mathcal{S}_\chi}$.

Remarks. (0) Intuitively, the inclusions σ reflect the inclusions of closures of strata $\overline{\mathcal{S}_\chi}\subseteq\overline{\mathcal{S}_{\chi+\gamma}}$ in the semiinfinite space $\mathcal{S}$. Here, $\sigma_\chi^{\beta,\gamma}: \ \mathcal{Z}_\chi^\beta \hookrightarrow \mathcal{Z}_{\chi+\gamma}^{\beta+\gamma}$ embeds the slice for $\mathcal{S}_{\chi-\beta}\subseteq\overline{\mathcal{S}_\chi}$ into the slice for $\mathcal{S}_{\chi-\beta}=\mathcal{S}_{(\chi+\gamma)-(\beta+\gamma)}\subseteq\overline{\mathcal{S}_{\chi+\beta}}$.

(1) The identification of objects on C that differ by a multiple of a point a means that one is considering the pair (C,a).

It is a familiar idea that passing from C to (C,a) embeds glues the restrictions of the local space $Z\to\mathcal{H}_C$ to the connected components $\mathcal{H}_C^\alpha$ of $\mathcal{H}_C$. (the scheme Z^α embeds into $Z^{\alpha+\beta}$ by adding βa). However, here there is the extra parameter $\chi\in Y$ which also gets increased to $\chi+\beta$.

B. The category $\mathcal{PS}$ of (finite length) perverse sheaves on the system. The notion of a perverse sheaf on the system $\boldsymbol{\mathcal{Z}}$ is here called a *snop*. We define abelian category $\widetilde{\mathcal{PS}}$ of snops and the subcategory $\mathcal{PS}$ of snops of finite length. The irreducibles are parametrized by pairs of $\chi\in Y$ and $w\in\mathcal{W}_f$. They intuitively correspond to "Iwahori orbits."

- The first level of the construction is an abstract procedure called he *local cohomology* $H_Y^1[Z,\ (\mathcal{P},\mathcal{I})]$ of a class of sheaves $\mathcal{P}$ on a local space Z over C, at $Y\subseteq C$ and with respect to a local sheaf $\mathcal{I}$ on the local space $Z|_{C-Y}$ over $C-Y$.
- The second level is a stabilization with respect to the $\mathbf{N}[\mathbf{I}]$-action on Y.

9.0.4. *The local cohomology $H_Y^1[Z,\ (\mathcal{P},\mathcal{I})]$ at $Y\subseteq C$ of a class of sheaves $\mathcal{P}$ on a local space Z over C and with respect to a local sheaf $\mathcal{I}$ on $Z|_{C-Y}$.* Here, local cohomology is taken in the setting of sheaves over local spaces over C (rather than just sheaves over C). So, its meaning is not to "trivialize over C^*" but to "trivialize in the direction of C^*" where "direction" means a factor in the locality isomorphism.

The setting is given by

- a local space Z over C;
- A class $\mathcal{P}$ of sheaves over Z;
- an open subset V of a curve C;
- a local sheaf $\mathcal{I}$ (in class $\mathcal{P}$) over the restriction $Z|_V$.

Then $H^1[(C,V),(\mathcal{P},\mathcal{I})]$ means that we consider sheaves in $\mathcal{P}$ with $\mathcal{I}$-trivializations in the direction of V.

Let $(C,V)^{\alpha,\beta}\overset{\text{def}}{=}C^{\alpha,\beta}\cap C^\alpha\times V^\beta$, i.e., the moduli of disjoint pairs of $D'\in C^\alpha$ and $D''\in V^\beta$. Then the $\mathcal{I}$-trivialization of a sheaf $\mathcal{K}$ on Z means a compatible system of isomorphisms

$$[\mathcal{K}^\alpha\boxtimes\mathcal{I}^\beta]\big|_{(C,V)^{\alpha,\beta}} \overset{\cong}{\to} \mathcal{K}^{\alpha+\beta}\big|_{(C,V)^{\alpha,\beta}}.$$

9.0.5. *Category* $\widetilde{PS}$ *as* $H^1_a[Z,\ (\mathcal{P},IC)]$. Here we use the point a to define the "local cohomology"

$$\widetilde{PS} \overset{\text{def}}{=} H^1_{a,IC}[C,\ \mathcal{P}]$$

of the class $\mathcal{P}$ of perverse sheaves and with respect to the local sheaf IC on $\mathcal{Z}_{C^*\times I}(G)$ for $C^* = C - a$.[5]

A $(C-a, IC)$-trivialization of a perverse sheaf $\mathcal{K}$ on Z means a compatible system of isomorphisms

$$[\mathcal{K}^\alpha \boxtimes IC_{Z^\beta}]\big|_{(C,V)^{\alpha,\beta}} \xrightarrow{\cong} \mathcal{K}^{\alpha+\beta}\big|_{(C,V)^{\alpha,\beta}}.$$

9.0.6. *Category* $\widetilde{\mathcal{PS}}$*: to the category* $\widetilde{PS}$ *add the parameter* $\chi \in Y$ *and the* $\mathbf{N}[\mathbf{I}]$*-stabilization.* So, the added parameter is in $Y/\mathbf{N}[\mathbf{I}]$. More precisely, to $\mathcal{PS}$ we apply $Y\times_{\mathbf{N}[\mathbf{I}]} -$.

9.0.7. *Category* $\mathcal{PS}$*.* This is the subcategory of $\widetilde{\mathcal{PS}}$ given by sheaves constructible with respect to the Iwahori stratification.

9.1. **Local space $\mathcal{Z}$: a system of varieties $\mathcal{Z}^\alpha_\chi,\ \alpha\in\mathbb{N}[I],\ \gamma\in Y$.** The system $\mathcal{Z}$ is intuitively what we can see from a semi-infinite space $\mathcal{S}$ stratified by strata $\mathcal{S}_\chi,\ \chi\in Y$. We are describing $\mathcal{S}$ by the system $\mathcal{Z}$ of normal slices $\mathcal{Z}^\alpha_\chi$, here $\mathcal{Z}^\alpha_\chi$ is the slice to the stratum $\mathcal{S}_{\chi-\alpha}$ inside $\overline{\mathcal{S}_\chi}$.

9.1.1. *From* $\mathcal{B}$ *to* $\mathcal{B}\times Y$*.* Now we upgrade the flag variety $\mathcal{B}$ to the product $\mathcal{B}\times Y = \sqcup_{\chi\in Y}\ \mathcal{B}_\chi$, i.e., copies of $\mathcal{B}$ indexed by the cocharacter lattice Y. So for arbitrary $\chi\in Y$ and $\alpha\in\mathbb{N}[I]$ we obtain the spaces $\mathcal{Z}^\alpha_\chi$ of based maps into $\mathcal{B}_\chi$ and it makes sense now to add the subscript χ to all the strata (coarse, Schubert, fine) defined in the previous section.

9.2. **The indsystem of based quasimap spaces $\mathcal{Z}^\alpha$ (by twists at $a\in C$).** A distinguished point $a\in C$ defines maps $\gamma a\cup_C - :\ C^\alpha \hookrightarrow C^{\alpha+\gamma}$ and the monoid indscheme $\mathbb{H}_{C,a} = \lim\limits_{\rightarrow}\ \mathcal{H}^\alpha_C$. Notice that the map $\mathcal{H}_{C-a} \rightarrow \mathcal{H}_\gamma a\cup_C -C, a$ by $\mathcal{H}^\alpha_{C-a} \subseteq \mathcal{H}^\alpha_C \hookrightarrow \mathcal{H}_{C,a}$ is a locally closed embedding and a morphism of monoids.

This lifts to an ind-system of quasimap spaces

$$\mathcal{Q}^\alpha_C \hookrightarrow\ (\gamma a\cup_C -)^*\mathcal{Q}^{\alpha+\gamma}_C$$

by $f\mapsto f(-\gamma a)$ and defines the indscheme $\mathcal{Q}_C(-\infty a)$ of quasimaps with arbitrary poles at a.

[5] The key point is that for any local space Z/C the construction IC gives a local perverse sheaf on Z. Some other universal constructions of local sheaves: the constant sheaf Z_X and the dualizing sheaf ω_X.

For $\beta \leq \beta+\gamma$), it restricts to the embedding $\mathcal{Q}^\beta \hookrightarrow \mathcal{Q}^\alpha,\ f \mapsto f(-\gamma \cdot\cdot 0)$, and in particular

$$\mathcal{Z}^\beta \hookrightarrow \mathcal{Z}^\alpha.$$

Remark. (0) Passing to the limit indscheme destroys degrees and locality.

(1) For an open curve C (for instance a local curve) there is no notion of the degree of a quasimap so we do not know how to decompose the space $\mathcal{Q}_C$. There is still a notion of a defect but now it is not continuous since a part of the defect may float to the boundary of C. □

9.2.1. *System $\mathcal{Z}$ of varieties $\mathcal{Z}^\alpha_\chi,\ \alpha,\gamma \in Y$.* 9.2

9.2.2. We will consider a system $\mathcal{Z}$ of varieties $\mathcal{Z}^\alpha_\chi,\ \alpha,\gamma \in Y$, together with two kinds of maps defined for any $\beta,\gamma \in \mathbb{N}[I]$:

- a) closed embeddings,
$$\sigma^{\beta,\gamma}_\chi:\ \mathcal{Z}^\beta_\chi \hookrightarrow \mathcal{Z}^{\beta+\gamma}_{\chi+\gamma},$$
- b) factorization identifications
$$\mathcal{Z}^{\beta,\gamma}_{\chi,U_\varepsilon,\Upsilon_\varepsilon} \ \cong\ \mathcal{Z}^\beta_{\chi,U_\varepsilon} \times \mathcal{Z}^\gamma_{\chi-\beta,\Upsilon_\varepsilon}$$
defined for $\varepsilon > 0$ and $U_\varepsilon \overset{\text{def}}{=} \{z \in \mathbb{C},\ |z| < \varepsilon\}$, and $\Upsilon_\varepsilon \overset{\text{def}}{=} \{z \in \mathbb{C},\ |z| > \varepsilon\}$.

Of course, these are the embeddings from 8.1 and factorizations from 6.3.2.

Remark. Here we use only the very special factorizations based on decomposing $\mathbb{A}^1 = \mathbb{C}$ into an open disc U_ε around 0 and the open complememt Υ_ε (and the circle).

9.3. **Snops.** We will denote by $\mathcal{IC}^\alpha_\chi$ the perverse IC-extension of the constant sheaf at the generic point of $\mathcal{Z}^\alpha_\chi$.

By a *snop* we will mean

(1) A choice of $\chi \in Y$ and a system of perverse sheaves $\mathcal{K}^\alpha_\chi$ on $\mathcal{Z}^\alpha_\chi$, $\alpha \in \mathbb{N}[I]$.
The reason that the support estimate is needed is that intuitively, we are describing a sheaf $\mathcal{K}$ on a semi-infinite space $\mathcal{S}$ supported on the closure of one stratum $\mathcal{S}_\chi$. We are describing $\mathcal{K}$ by a system of its restrictions to normal slices $\mathcal{Z}^\alpha_\chi$, here $\mathcal{K}^\alpha_\chi$ is a restriction to the normal slice $\mathcal{Z}^\alpha_\chi$ to the stratum $\mathcal{S}_{\lambda-\alpha}$.
(2) The perverse sheaves $\mathcal{K}^\alpha_\chi$ on $\mathcal{Z}^\alpha_\chi$ are required to be smooth along the fine Schubert stratification.
(3) $\mathcal{K}$ has a factorization structure which says that it behaves the same as the IC sheaf away from $0 \in \mathbb{A}^1$.
(4) A sheaf supported on $\overline{\mathcal{S}_\chi}$ is also supported on $\overline{\mathcal{S}_{\chi'}}$ for $\chi' \geq \chi$. So we will later add the stabilization procedure that increases χ by adding any $\alpha \in Y_+$.

9.3.1. *A conjectural definition.* The following definition makes sense only modulo the validity of conjecture 8.2.

Definition. A *snop* $\mathcal{K}$ is the following collection of data:

a) $\chi = \chi(\mathcal{K}) \in Y$, called the *support estimate* of $\mathcal{K}$;

b) For any $\alpha \in \mathbb{N}[I]$, a perverse sheaf $\mathcal{K}_\chi^\alpha$ on $\mathcal{Z}_\chi^\alpha$ smooth along the fine Schubert stratification;

c) For any $\beta, \gamma \in \mathbb{N}[I], \ \varepsilon > 0$, a *factorization isomorphism*

$$\mathcal{K}_\chi^{\beta+\gamma}|_{\mathcal{Z}^{\beta,\gamma}_{\chi,U_\varepsilon,\Upsilon_\varepsilon}} \xrightarrow{\sim} \mathcal{K}_\chi^\beta|_{\mathcal{Z}^\beta_{\chi,U_\varepsilon}} \boxtimes \mathcal{IC}^\gamma_{\chi-\beta}|_{\mathcal{Z}^\gamma_{\chi-\beta,\Upsilon_\varepsilon}}$$

satisfying the *associativity constraints* as in [**?**], §§3,4. We spare the reader the explicit formulation of these constraints.

9.3.2. *Snops (a precise definition).* Since at the moment the conjecture 8.2 is unavailable we will provide an ugly provisional substitute of the Definition 9.3. Namely, recall that $\mathcal{Z}^\alpha = \sqcup_{\alpha\geq\beta\geq\gamma} \overset{\circ}{\mathcal{Z}}{}^\gamma \times (\mathbb{C}^*)^{\beta-\gamma}$. We introduce an open subvariety

$$\ddot{\mathcal{Z}}^\alpha = \bigsqcup_{\alpha\geq\beta\geq\gamma\gg 0} \overset{\circ}{\mathcal{Z}}{}^\gamma \times (\mathbb{C}^*)^{\beta-\gamma}$$

The union is taken over sufficiently dominant γ, i.e. such that $\langle\gamma, i'\rangle > 10$ for any $i \in I$. Certainly, if α itself is not sufficiently dominant, $\ddot{\mathcal{Z}}^\alpha$ may happen to be empty. We have the fine Schubert stratification

$$\ddot{\mathcal{Z}}^\alpha = \bigsqcup_{w\in\mathcal{W}_f,\ \Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\geq\beta\geq\gamma\gg 0} \overset{\circ}{\mathcal{Z}}{}^\gamma_w \times (\mathbb{C}^*)^{\beta-\gamma}_\Gamma$$

with smooth strata (see the Lemma 8.2.1).

Now we can repeat the Definition 9.3 replacing $\mathcal{Z}_\chi^\alpha$ by $\ddot{\mathcal{Z}}_\chi^\alpha$. Thus in 9.3 b) we have to restrict ourselves to sufficiently dominant α, and in 9.3 c) β has to be sufficiently dominant as well.

9.3.3. In what follows we use the Definition 9.3. The reader unwilling to believe in the Conjecture 8.2 will readily substitute the conjectural Definition 9.3 with the provisional working Definition 9.3.2.

9.4. **Irreducible and (co)standard snops $\mathcal{L}(w,\chi)$, $\mathcal{M}^!(w,\chi) = \mathcal{M}(w,\chi)$, $\mathcal{M}^*(w,\chi) = \mathcal{D}\mathcal{M}(w,\chi))$ for $\chi \in Y,\ w \in \mathcal{W}_f$.**

9.4.1. Let us describe a snop $\mathcal{L}(w,\chi)$ for $\chi \in Y,\ w \in \mathcal{W}_f$.

a) The support of $\mathcal{L}(w,\chi)$ is χ.

b) $\mathcal{L}(w,\chi)^\alpha_\chi$ is the irreducible IC-extension $\mathcal{IC}(\overline{\mathcal{Z}}^\alpha_{w,\chi}) = j_{!*}\mathcal{IC}(\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi})$ of the perverse IC-sheaf on the Schubert stratum $\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi} \subset \overset{\bullet}{\mathcal{Z}}{}^\alpha_\chi$. Here j stands for the affine open embedding $\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi} \hookrightarrow \overline{\mathcal{Z}}^\alpha_w$.

In particular, $\mathcal{IC}(\overline{\mathcal{Z}}^\alpha_{w_0,\chi}) = \mathcal{IC}^\alpha_\chi$.

c) Evidently, $\overline{\mathcal{Z}}^\beta_{w,\chi,U_\varepsilon}$ (resp. $\overline{\mathcal{Z}}^{\beta,\gamma}_{w,\chi,U_\varepsilon,\Upsilon_\varepsilon}$) is open in $\overline{\mathcal{Z}}^\beta_{w,\chi}$ (resp. $\overline{\mathcal{Z}}^\alpha_{w,\chi}$) for any $\beta+\gamma=\alpha$. Moreover, $\overline{\mathcal{Z}}^{\beta,\gamma}_{w,\chi,U_\varepsilon,\Upsilon_\varepsilon} = \overline{\mathcal{Z}}^\beta_{w,\chi,U_\varepsilon} \times \mathcal{Z}^\gamma_{\chi-\beta,\Upsilon_\varepsilon}$. This induces the desired factorization isomorphism.

9.4.2. If we replace in 9.4.1b) above $j_{!*}\mathcal{IC}(\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi})$ by $j_!\mathcal{IC}(\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi}) =: \mathcal{M}(w,\chi)^\alpha_\chi$ (resp. $j_*\mathcal{IC}(\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi}) =: \mathcal{DM}(w,\chi)^\alpha_\chi$) we obtain the snop $\mathcal{M}(w,\chi)$ (resp. $\mathcal{DM}(w,\chi)$).

9.5. **The abelian category $\widetilde{\mathcal{PS}}$ of snops.**

9.5.1. *The stabilization action of* $\mathbb{N}[I]$ *on snops.* Given a snop $\mathcal{K}$ with support χ, and $\eta \geq \chi,\ \alpha \in \mathbb{N}[I]$, we define a sheaf $'\mathcal{K}^\alpha_\eta$ on $\mathcal{Z}^\alpha_\chi$ as follows. We set $\gamma \overset{\text{def}}{=} \eta - \chi$. If $\alpha \geq \gamma$ we set

$$'\mathcal{K}^\alpha_\eta \overset{\text{def}}{=} (\sigma^{\alpha-\gamma,\gamma}_\chi)_* \mathcal{K}^{\alpha-\gamma}_\chi$$

(for the definition of σ see 9.1). Otherwise we set $'\mathcal{K}^\alpha_\eta \overset{\text{def}}{=} 0$.

It is easy to see that the factorization isomorphisms for $\mathcal{K}$ induce similar isomorphisms for $'\mathcal{K}$, and thus we obtain a snop $'\mathcal{K}$ with support $\eta \geq \chi$.

9.5.2. Given two snops $\mathcal{F},\mathcal{K}$ we will define the morphisms $\operatorname{Hom}(\mathcal{F},\mathcal{K})$ as follows. Let $\eta \in Y$ be such that $\eta \geq \chi(\mathcal{F}),\chi(\mathcal{K})$. For $\alpha = \beta+\gamma \in \mathbb{N}[I]$ we consider the following composition:

$$\vartheta^{\beta,\gamma}_\eta:\ \operatorname{Hom}_{\mathcal{Z}^\alpha_\eta}('\mathcal{F}^\alpha_\eta,{}'\mathcal{K}^\alpha_\eta) \longrightarrow \operatorname{Hom}_{\mathcal{Z}^{\beta,\gamma}_{U_\varepsilon,\Upsilon_\varepsilon}}('\mathcal{F}^\alpha_\eta|_{\mathcal{Z}^{\beta,\gamma}_{U_\varepsilon,\Upsilon_\varepsilon}},{}'\mathcal{K}^\alpha_\eta|_{\mathcal{Z}^{\beta,\gamma}_{U_\varepsilon,\Upsilon_\varepsilon}}) \overset{\sim}{\longrightarrow}$$

$$\operatorname{Hom}_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}\times\mathcal{Z}^\gamma_{\eta-\beta,\Upsilon_\varepsilon}}('\mathcal{F}^\beta_\eta|_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}}\boxtimes\mathcal{IC}^\gamma_{\eta-\beta}|_{\mathcal{Z}^\gamma_{\eta-\beta,\Upsilon_\varepsilon}},{}'\mathcal{K}^\beta_\eta|_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}}\boxtimes\mathcal{IC}^\gamma_{\eta-\beta}|_{\mathcal{Z}^\gamma_{\eta-\beta,\Upsilon_\varepsilon}}) = \operatorname{Hom}_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}}('\mathcal{F}^\beta_\eta|_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}},{}'\mathcal{K}^\beta_\eta|_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}})$$

(the second isomorphism is induced by the factorization isomorphisms for $'\mathcal{F}$ and $'\mathcal{K}$, and the third equality is just Künneth formula).

Now we define

$$\operatorname{Hom}(\mathcal{F},\mathcal{K}) \overset{\text{def}}{=} \lim_{\substack{\to\\ \eta}} \lim_{\substack{\leftarrow\\ \alpha}} \operatorname{Hom}_{\mathcal{Z}^\alpha_\eta}('\mathcal{F}^\alpha_\eta,{}'\mathcal{K}^\alpha_\eta)$$

Here the inverse limit is taken over $\alpha \in \mathbb{N}[I]$, the transition maps being $\vartheta_\eta^{\beta,\alpha-\beta}$, and the direct limit is taken over $\eta \in Y$ such that $\eta \geq \chi(\mathcal{F}), \chi(\mathcal{K})$, the transition maps being induced by the obvious isomorphisms $\mathrm{Hom}_{\mathcal{Z}_\eta^\alpha}({}'\mathcal{F}_\eta^\alpha, {}'\mathcal{K}_\eta^\alpha) = \mathrm{Hom}_{\mathcal{Z}_{\eta+\gamma}^{\alpha+\gamma}}({}'\mathcal{F}_{\eta+\gamma}^{\alpha+\gamma}, {}'\mathcal{K}_{\eta+\gamma}^{\alpha+\gamma})$.

9.5.3. With the above definition of morphisms and obvious composition, the snops form a category which we will denote by $\widetilde{\mathcal{PS}}$.

9.6. $\mathcal{PS}$**.** Evidently, the snops $\mathcal{L}(w,\chi)$ are irreducible objects of $\widetilde{\mathcal{PS}}$. It is easy to see that any irreducible object of $\widetilde{\mathcal{PS}}$ is isomorphic to some $\mathcal{L}(w,\chi)$.

We define the category $\mathcal{PS}$ of *finite snops* as the full subcategory of $\widetilde{\mathcal{PS}}$ formed by the snops of finite length. It is an abelian category. We will see later that $\mathcal{M}(w,\chi)$ and $\mathcal{DM}(w,\chi)$ (see 9.4.2) lie in $\mathcal{PS}$ for any w,χ.

One can prove the following very useful technical lemma exactly as in [**?**], 4.7.

9.6.1. *Lemma.* Let $\mathcal{F},\mathcal{K}$ be two finite snops. Let $\eta \geq \chi(\mathcal{F}), \chi(\mathcal{K})$. There exists $\beta \in \mathbb{N}[I]$ such that for any $\alpha \geq \beta$ the canonical maps $\mathrm{Hom}(\mathcal{F},\mathcal{K}) \longrightarrow \mathrm{Hom}_{\mathcal{Z}_\eta^\alpha}({}'\mathcal{F}_\eta^\alpha, {}'\mathcal{K}_\eta^\alpha)$ are all isomorphisms. $\square$

CHAPTER 3. Convolution with affine Grassmannian

10. Plücker models, twisted quasimaps $\mathfrak{Q}$ and parity for IC sheaves

These are three different topics.

In the first part 10.A we write down the Plucker models of $\overline{\mathcal{G}_\lambda}$, Iwahori orbits $\mathcal{G}_{\lambda,w}$ and zastava spaces $\mathcal{Z}^\alpha$.

In the second part 10.B we consider the spaces of twisted quasimaps.

In the third part 10.C we show that the stalks of IC sheaves are the same for the three spaces of quasimaps $\mathcal{Z}^\alpha$, $\mathcal{Q}^\beta$) and the twisted version $\mathfrak{Q}^\beta$. We conjectre that they satisfy a parity vanishing.

Summary.

10.0.1. **[10.B.]** *The spaces of twisted quasimaps.* Let $\mathcal{Y} = Bun_G(\mathcal{C})$ where $\mathcal{C}$ is either a smooth projective curve, the pair $(\mathbb{P}^1, \widehat{\infty})$ (so, $\mathcal{Y} = \widehat{\mathcal{G}}$) or $(\mathbb{P}^1, \mathbb{P}^1 - 0) = (d, d^*)$ (so $\mathcal{Y} = \mathcal{G}$). We define ${}^{Bun_G(\mathcal{C})}\mathcal{B} \supseteq {}^{Bun_G(\mathcal{C})}\overline{\mathcal{B}}$ as spaces over $Bun_G(\mathcal{C}) \times C$ such that for $\mathsf{Q} \in Bun_G(\mathcal{C})$ the restrictions to $\mathsf{Q} \times C$ are $\mathsf{Q}/B \cong \mathsf{Q} \times_G \mathcal{B} \stackrel{\text{def}}{=} {}^{\mathsf{Q}}\mathbb{B}$ and ${}^{\mathsf{Q}}\overline{\mathcal{B}} = \mathsf{Q} \times_G \overline{\mathcal{B}}$.

Then the direct (quasi)images along the projection $Bun_G(\mathcal{C}) \times C \to Bun_G(\mathcal{C})$ are the spaces over $Bun_G(\mathcal{C})$ of (quasi)sections (also called the "twisted (quasi)maps"),

$$\Gamma[C,\ {}^{Bun_G(\mathcal{C})}\mathcal{B}] \quad \text{and} \quad Q\Gamma[C,\ {}^{Bun_G(\mathcal{C})}\mathcal{B}] \stackrel{\text{def}}{=} \Gamma_{gen}[C,\ {}^{Bun_G(\mathcal{C})}\overline{\mathcal{B}}].$$

The fibers at Q are $\Gamma[C,\ {}^{\mathsf{Q}}\mathcal{B}]$ and $Q\Gamma[C,\ {}^{\mathsf{Q}}\mathcal{B}] \stackrel{\text{def}}{=} \Gamma_{gen}[C,\ {}^{\mathsf{Q}}\overline{\mathcal{B}}]$ where Γ_{gen} consists of all sections that visit ${}^{\mathsf{Q}}\mathcal{B}$.

Notation. Over $\overline{\mathcal{G}_\eta} \subseteq \mathcal{G} = Bun_G(\mathbb{P}^1, \mathbb{P}^1 - \infty)$ times $C = \mathbb{P}^1$ we have ${}^{\overline{\mathcal{G}_\eta}}\mathcal{B} \subseteq {}^{\mathcal{G}}\mathcal{B}$. Then over $\overline{\mathcal{G}_\eta} \subseteq \mathcal{G}$ we have the spaces of quasisections

$$\mathcal{GQ}_\eta \stackrel{\text{def}}{=} Q\Gamma(\mathbb{P}^1,\ {}^{\overline{\mathcal{G}_\eta}}\mathcal{B}).$$

In particular as $\overline{\mathcal{G}_\eta}$ is the point given by the trivial torsor, $\mathcal{GQ}_0 \stackrel{\text{def}}{=} Q\Gamma(\mathbb{P}^1, \mathcal{B}) = QMap(\mathbb{P}^1, \mathcal{B}) \stackrel{\text{def}}{=} \mathcal{Q}$ is the untwisted object.

Similarly, over $\widehat{\mathcal{G}} = Bun_G(\mathbb{P}^1, \widehat{\infty})$ times $\mathbb{P}^1$ we have ${}^{\widehat{\mathcal{G}}}\mathcal{B}$ and this gives $\mathfrak{Q} \stackrel{\text{def}}{=} Q\Gamma(\mathbb{P}^1,\ {}^{\widehat{\mathcal{G}}}\mathcal{B})$ over $\widehat{\mathcal{G}}$.

Remark. Notice that a twisted quasimap on a complete curve C (for instance $\mathbb{P}^1$) has a degree $\alpha \in Y^{(6)}$ so we have $Q\Gamma[C,\ {}^{Bun_G(\mathcal{C})}\mathcal{B}] = \sqcup_{\alpha \in Y} Q\Gamma^\alpha[C,\ {}^{Bun_G(\mathcal{C})}\mathcal{B}]$ and in particular $\mathcal{GQ}_\eta = \sqcup_{\alpha \in Y} \mathcal{GQ}_\eta^\alpha$ and $\mathfrak{Q}_\eta = \sqcup_{\alpha \in Y} \mathfrak{Q}_\eta^\alpha$.

[6] For instance a twisted quasimap $\mathbf{L}$ is still a a system of invertible subsheaves $\mathbf{L}_i$ and $\deg(\mathbf{L}) = \sum_{i \in I} \deg(\mathbf{L}_i)$.

10.0.2. [**10.C.**] *Stalks of IC sheaves on $\mathcal{Z}^\alpha, \mathcal{Q}^\alpha, \mathfrak{Q}$.* **(i)** *Relation of stratifications and stalks.* We notice that the (based) quasimap spaces $\mathcal{Z}^\alpha, \mathcal{Q}^\alpha$ as well as the twisted quasimap spaces $\mathfrak{Q}^\alpha$, the IC sheaves are constructible with respect to stratifications indexed by the same data: the type of the defect of a quasimap. Moreover, the IC stalks on the corresponding strata are the same. (However, for $\mathfrak{Q}^\alpha$ these facts are proved later in **??**.]

Remark. For $\mathcal{Q}^\alpha$ the natural stratification also involves the type of the G-torsor, however this turns out not to influence the stalks of $IC(\mathfrak{Q}^\alpha)$.

(ii) *A parity vanishing conjecture for stalks.* Some rsults in the present paper assume this conjecture which is only proved in the second paper SF2.

10.A. Plücker model of $\overline{\mathcal{G}_\lambda}$, $\mathcal{G}_{\lambda,w}$ and $\mathcal{Z}^\alpha$

10.1. Plücker model of $\overline{\mathcal{G}_\lambda}$ on $\mathbb{P}^1$ and on a formal disc.

10.1.1. $\overline{\mathcal{G}_\lambda}$. Let $\mathcal{G}$ be the usual affine Grassmannian $\mathbf{G}((z))/\mathbf{G}[[z]]$. It is the ind-scheme representing the functor of isomorphism classes of pairs $(\mathcal{T}, \tau)$ where $\mathcal{T}$ is a $\mathbf{G}$-torsor on $\mathbb{P}^1$, and τ is its section (trivialization) defined off 0 (see e.g. [**?**]). It is equipped with a natural action of proalgebraic group $\mathbf{G}[[z]]$, and we are going to describe the orbits of this action. It is known (see e.g. *loc. cit.*) that these orbits are numbered by dominant cocharacters $\eta \in Y^+ \subset Y$.

Here $Y^+ \subset Y$ stands for the set of cocharacters η such that $\langle \eta, i' \rangle \geq 0$ for any $i \in I$. For $\eta \in Y^+$ we denote the corresponding $\mathbf{G}[[z]]$-orbit in $\mathcal{G}$ by $\mathcal{G}_\eta$, and we denote its closure by $\overline{\mathcal{G}}_\eta$.

Recall that for a dominant character $\lambda \in X^+$ we denote by V_λ the corresponding irreducible $\mathbf{G}$-module, and we denote by $\mathcal{V}_\lambda$ the trivial vector bundle $V_\lambda \otimes \mathcal{O}_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

10.1.2. *Plucker model of $\overline{\mathcal{G}_\lambda}$ on $\mathbb{P}^1$.* .

Proposition The orbit closure $\overline{\mathcal{G}}_\eta \subset \mathcal{G}$ is the space of collections $(\mathcal{U}_\lambda)_{\lambda \in X^+}$ of vector bundles on $\mathbb{P}^1$ such that

a) $\mathcal{V}_\lambda(-\langle \eta, \lambda \rangle 0) \subset \mathcal{U}_\lambda \subset \mathcal{V}_\lambda(\langle \eta, \lambda \rangle 0)$, or equivalently, $\mathcal{U}_\lambda(-\langle \eta, \lambda \rangle 0) \subset \mathcal{V}_\lambda \subset \mathcal{U}_\lambda(\langle \eta, \lambda \rangle 0)$;

b) $\deg \mathcal{U}_\lambda = \deg \mathcal{V}_\lambda = 0$, or in other words, $\dim \mathcal{V}_\lambda(\langle \eta, \lambda \rangle 0)/\mathcal{U}_\lambda = \langle \eta, \lambda \rangle \dim V_\lambda$;

c) For any surjective G-morphism $\phi: \ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ and the corresponding morphism $\phi: \ \mathcal{V}_\lambda \otimes \mathcal{V}_\mu \longrightarrow \mathcal{V}_\nu$ (hence $\phi: \ \mathcal{V}_\lambda(\langle \eta, \lambda \rangle 0) \otimes \mathcal{V}_\mu(\langle \eta, \mu \rangle 0) \longrightarrow \mathcal{V}_\nu(\langle \eta, \lambda + \mu \rangle 0)$) we have $\phi(\mathcal{U}_\lambda \otimes \mathcal{U}_\mu) = \mathcal{U}_\nu$.

Proof. $\mathbf{G}$-torsor on a curve C is the same as a tensor functor from the category of $\mathbf{G}$-modules to the category of vector bundles on C. □

10.1.3. *Plucker model of $\overline{\mathcal{G}_\lambda}$ on the formal disc d.* Let us give a local version of the above Proposition. Recall that $\mathcal{O} = \mathbb{C}[[z]] \subset \mathcal{K} = \mathbb{C}((z))$. For a finite-dimensional vector space V, a *lattice* $\mathfrak{V}$ in $V \otimes \mathcal{K}$ is an $\mathcal{O}$-submodule of $V \otimes \mathcal{K}$ *commensurable* with $V \otimes \mathcal{O}$, that is, such that $(V \otimes \mathcal{O}) \cap \mathfrak{V}$ is of finite codimension in both $V \otimes \mathcal{O}$ and $\mathfrak{V}$.

Proposition. The orbit closure $\overline{\mathcal{G}}_\eta \subset \mathcal{G}$ is the space of collections $(\mathfrak{U}_\lambda)_{\lambda \in X^+}$ of lattices in $V_\lambda \otimes \mathcal{K}$ such that

a) *Within η distance from the trivial torsor*:

$$z^{\langle \eta,\lambda\rangle}(V_\lambda \otimes \mathcal{O}) \subset \mathfrak{U}_\lambda \subset z^{-\langle \eta,\lambda\rangle}(V_\lambda \otimes \mathcal{O}),$$

or equivalently,

$$z^{\langle \eta,\lambda\rangle}\mathfrak{U}_\lambda \subset V_\lambda \otimes \mathcal{O} \subset z^{-\langle \eta,\lambda\rangle}\mathfrak{U}_\lambda.$$

b) (*The degree of lattices is zero.*) $\dim(z^{-\langle \eta,\lambda\rangle}(V_\lambda \otimes \mathcal{O})/\mathfrak{U}_\lambda) = \langle \eta, \lambda\rangle \dim V_\lambda$;

c) *The Plucker (Tannakian) conditions on lattices.* For any surjective G-morphism $\phi :$ $V_\lambda \otimes V_\mu \longrightarrow V_\nu$ and the corresponding morphism $\phi :\ (V_\lambda \otimes \mathcal{O}) \otimes (V_\mu \otimes \mathcal{O}) \longrightarrow (V_\nu \otimes \mathcal{O})$ (hence $\phi :\ z^{-\langle \eta,\lambda\rangle}(V_\lambda \otimes \mathcal{O}) \otimes z^{-\langle \eta,\mu\rangle}(V_\mu \otimes \mathcal{O}) \longrightarrow z^{-\langle \eta,\lambda+\mu\rangle}(V_\nu \otimes \mathcal{O})$), we have $\phi(\mathfrak{U}_\lambda \otimes \mathfrak{U}_\mu) = \mathfrak{U}_\nu$. □

10.1.4. *Towards replacing Plucker by Vinberg.*

Lemma. (?) For $(\mathsf{P}, \iota) \in \overline{\mathcal{G}_\eta}$ the $G\times H$-torsor $\mathsf{P}(-\eta)$ descends(?) to a torsor for $V^*(G)$ and we notice that it maps to the trivial $V(G)$-torsor $V(G)$. ?

10.2. **Plücker model of Iwahori orbits $\mathcal{G}_{\lambda,w},\ \lambda \in X_*(T)_+,\ w \in W_f$.** Let $\mathbf{I} \subset \mathbf{G}[[z]]$ be the Iwahori subgroup; it is formed by all $g(z) \in \mathbf{G}[[z]]$ such that $g(0) \in \mathbf{B} \subset \mathbf{G}$. We will denote by $\mathcal{P}(\mathcal{G}, \mathbf{I})$ the category of perverse sheaves on $\mathcal{G}$ with finite-dimensional support, constant along $\mathbf{I}$-orbits. The stratification of $\mathcal{G}$ by $\mathbf{I}$-orbits is a certain refinement of the stratification $\mathcal{G} = \sqcup_{\eta\in Y^+}\mathcal{G}_\eta$. Namely, each $\mathcal{G}_\eta$ decomposes into $\mathbf{I}$-orbits numbered by $\mathcal{W}_f/\mathcal{W}_\eta$ where $\mathcal{W}_\eta$ stands for the stabilizer of η in $\mathcal{W}_f$. For $w \in \mathcal{W}_f/\mathcal{W}_\eta$ we will denote the corresponding $\mathbf{I}$-orbit by $\mathcal{G}_{w,\eta}$. Let us introduce a Plücker model of $\mathcal{G}_{w,\eta}$.

10.2.1. *Partial flag variety $\mathcal{B}_\eta \subseteq \mathcal{G}_\eta$.* For $\eta \in Y^+$ let $I_\eta \subset I$ be the set of all i such that $\langle \eta, i'\rangle = 0$ (thus for $i \notin I_\eta$ we have $\langle \eta, i'\rangle > 0$). Then $\mathcal{W}_\eta$ is generated by the simple reflections $\{s_i, i \in I_\eta\}$. Let $\mathbf{P}(I_\eta)$ be the corresponding parabolic subgroup (e.g. for $I_\eta = \emptyset$ we have $\mathbf{P}(I_\eta) = \mathbf{B}$, while for $I_\eta = I$ we have $\mathbf{P}(I_\eta) = \mathbf{G}$). Let $\mathcal{B}(I_\eta) = \mathbf{G}/\mathbf{P}(I_\eta)$ be the corresponding partial flag variety. The $\mathbf{B}$-orbits on $\mathcal{B}(I_\eta)$ are naturally numbered by $\mathcal{W}_f/\mathcal{W}_\eta :\ \mathcal{B}(I_\eta) = \sqcup_{w\in\mathcal{W}_f/\mathcal{W}_\eta}\mathcal{B}(I_\eta)_w$. The Plücker embedding realizes $\mathcal{B}$ as a closed subvariety in $\prod_{i\in I}\mathbb{P}(V_{\omega_i})$. Its image under the projection $\prod_{i\in I}\mathbb{P}(V_{\omega_i}) \longrightarrow \prod_{i\notin I_\eta}\mathbb{P}(V_{\omega_i})$ exactly coincides with $\mathcal{B}(I_\eta)$.

10.2.2. *Lemma-Definition.* a) For $\eta = \sum_{i\in I} n_i i, i \in I_\eta$, and $(\mathcal{U}_\lambda)_{\lambda\in X^+} \in \mathcal{G}_\eta$ we have $\mathcal{U}_{\omega_i} \subset \mathcal{V}_{\omega_i}((n_i-1)0)$;

b) For $\eta = \sum_{i\in I} n_i i, i \notin I_\eta$, and $(\mathcal{U}_\lambda)_{\lambda\in X^+} \in \mathcal{G}_\eta$ we have $\dim(\mathcal{U}_{\omega_i} + \mathcal{V}_{\omega_i}((n_i-1)0)/\mathcal{V}_{\omega_i}((n_i-1)0)) = 1$;

c) Thus $\mathcal{U}_{\omega_i}, i \notin I_\eta$, defines a line L_i in $\mathcal{V}_{\omega_i}(n_i 0)/\mathcal{V}_{\omega_i}((n_i-1)0) = V_{\omega_i}$. This collection of lines $(L_i)_{i\notin I_\eta} \in \prod_{i\notin I_\eta} \mathbb{P}(V_{\omega_i})$ satisfies the Plücker conditions and thus gives a point in $\mathcal{B}(I_\eta)$;

d) We will denote by $\mathbf{r}$ the map $\mathcal{G}_\eta \longrightarrow \mathcal{B}(I_\eta)$ defined in c);

e) For $w \in \mathcal{W}_f/\mathcal{W}_\eta$ we have $\mathcal{G}_{w,\eta} = \mathbf{r}^{-1}(\mathcal{B}(I_\eta)_w)$. □

10.2.3. *Indexation of Iwahori orbits $\mathcal{G}_{\lambda,w}$ by the T-fixed points $w\lambda$.* For $\theta \in Y$ we consider the corresponding homomorphism $\theta: \mathbb{C}^* \longrightarrow \mathbf{H} \subset \mathbf{G}$ as a formal loop $\theta(z) \in \mathbf{G}((z))$. It projects to the same named point $\theta(z) \in \mathbf{G}((z))/\mathbf{G}[[z]] = \mathcal{G}$. There is a natural bijection between the set of $\theta(z)$, $\theta \in Y$, and the set of Iwahori orbits: each Iwahori orbit $\mathcal{G}_{w,\eta}$ contains exactly one of the above points, namely, the point $w\eta(z)$.

10.3. **Plücker model of the Zastava space $\mathbf{Z}^\alpha$ (a subspace of the global Grassmannian).** Recall the Beilinson-Drinfeld avatar $\mathbf{Z}^\alpha$ of the Zastava space $\mathcal{Z}^\alpha$ (see 6.2). In this subsection we will give a Plücker model of $\mathbf{Z}^\alpha$.

Proposition. $\mathbf{Z}^\alpha$ is the the space of pairs $(D, (\mathfrak{U}_\lambda)_{\lambda\in X^+})$ where $D \in \mathbb{A}^\alpha$ is an I-colored effective divisor, and $(\mathfrak{U}_\lambda)_{\lambda\in X^+}$ is a collection of vector bundles on $\mathbb{P}^1$ such that

a) $\mathcal{V}_\lambda(-\infty D) \subset \mathfrak{U}_\lambda \subset \mathcal{V}_\lambda(+\infty D)$;

b) $\mathcal{V}_\lambda^{\mathbf{N}} \subset \mathfrak{U}_\lambda \supset \mathcal{V}_\lambda^{\mathbf{N}_-}(-\langle D,\lambda\rangle)$ (notations of 3.4.1), the first inclusion being a *line subbundle* (and the second an invertible subsheaf);

c) $\deg \mathfrak{U}_\lambda = 0$;

d) For any surjective G-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ and the corresponding morphism $\phi: \mathcal{V}_\lambda \otimes \mathcal{V}_\mu \longrightarrow \mathcal{V}_\nu$ (hence $\phi: \mathcal{V}_\lambda(+\infty D) \otimes \mathcal{V}_\mu(+\infty D) \longrightarrow \mathcal{V}_\nu(+\infty D)$) we have $\phi(\mathfrak{U}_\lambda \otimes \mathfrak{U}_\mu) = \mathfrak{U}_\nu$.

Proof. Obvious. □

10.3.1. *Remark.* Recall the isomorphism $\varpi^{-1}\xi: \mathbf{Z}^\alpha \xrightarrow{\sim} \mathcal{Z}^\alpha$ constructed in section 7. Let us describe it in terms of 10.3. The Lemma 7.2.4 says that there is a unique system of isomorphisms $\iota_\lambda: \mathfrak{U}_\lambda \xrightarrow{\sim} \mathcal{V}_\lambda$, $\lambda \in X^+$, identical at $\infty \in \mathbb{P}^1$ and compatible with tensor multiplication. Then $\varpi^{-1}\xi(D, (\mathfrak{U}_\lambda)_{\lambda\in X^+}) = (\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda)_{\lambda\in X^+}$ where $\mathfrak{L}_\lambda = \iota_\lambda(\mathcal{V}_\lambda^{\mathbf{N}_-}(-\langle D,\lambda\rangle))$.

10.B. "Quasimaps" $\mathfrak{Q}$ with values in twisted $\mathcal{B}$-bundles

Summary. "Quasimaps" $\mathfrak{Q}^\alpha$ with values in twisted $\mathcal{B}$-bundles as B-torsors

10.3.2. *The moduli ${}^{Bun_G(C)}\mathcal{Q}$ of G-twisted quasimaps.* A G-torsor P over a curve C has a flag variety P/B which can be viewed as the P-twist ${}^{\mathsf{P}}\mathcal{B} \overset{\text{def}}{=} \mathsf{P}\times_G\mathcal{B}$ of the flag variety of G. It has a canonical extension to the stack

$$ {}^{\mathsf{P}}\overline{\mathcal{B}} \overset{\text{def}}{=} \mathsf{P}\times_G\overline{\mathcal{B}} = \ \mathsf{P}\times_G\overline{\mathcal{B}} = \ \mathsf{P}\times_G[(G/N)^{aff}]/ = \ [(\mathsf{P}/N)^{aff}]/H. $$

c in the sense that

By a P-twisted quasimap from C to ${}^{\mathsf{P}}\mathcal{B}$ we mean a *quasisection* of ${}^{\mathsf{P}}\mathcal{B} \to C$ (or a C-*quasipoint* of ${}^{\mathsf{P}}\mathcal{B}$), by which we mean a section of ${}^{\mathsf{P}}\overline{\mathcal{B}}$ which is generic in the sense that it does hit ${}^{\mathsf{P}}\mathcal{B}$.

The space of P-twisted quasimaps is denoted ${}^{\mathsf{P}}\mathcal{Q} \overset{\text{def}}{=} \ Q\Gamma(C,\ {}^{\mathsf{P}}\mathcal{B})$.(7) These are the fibers of ${}^{Bun}\mathcal{Q} \to Bun_G(C)$ for the moduli ${}^{Bun}\mathcal{Q}$ of pairs (P, f) of a G-torsor P on C and a quasisection f of ${}^{\mathsf{P}}\mathcal{B}$.

10.3.3. *The moduli of twisted quasimaps ${}^{Bun_G(C)}\mathcal{Q}$ as a partial compactification of $Bun_B(C)$.* An actual section $\Gamma[C,\ {}^{\mathsf{P}}\mathcal{B}] = \Gamma[C,\ \mathsf{P}/B]$ is the same as a B-reduction of P. So, the moduli ${}^{Bun}\check{Q} \to Bun_G(C)$ for of pairs (P, f) of a G-torsor P on C and a section f of ${}^{\mathsf{P}}\mathcal{B}$ is the same as $Bun_B(C)$.

10.3.4. *The degree decomposition $\mathfrak{Q} = \sqcup_{\alpha\in Y}\ \mathfrak{Q}^\alpha$ of twisted quasimap.* A twisted quasimap $\mathbf{L}$ is still a system of invertible sheaves $\mathbf{L}_\lambda$ and deg $\mathbf{L} = \alpha$ if $\deg(\mathbf{L}_\lambda) = -\langle\alpha, \lambda\rangle$.

10.3.5. *Two versions ${}^{\mathcal{G}}\mathcal{Q}$ (over $\mathcal{G}$) and $\mathfrak{Q} \overset{def}{=} {}^{\widehat{\mathcal{G}}}\mathcal{Q}$ (over $\widehat{\mathcal{G}}$).* We will use two versions of this.

$$ \begin{array}{ccccc} {}^{\overline{\mathcal{G}_\eta}}\mathcal{Q}^\alpha = \ \mathcal{G}\mathcal{Q}^\alpha_\eta & \xrightarrow{\subseteq} & {}^{\mathcal{G}}\mathcal{Q} = \ \mathcal{G}\mathcal{Q} & \xrightarrow{\subseteq} & \mathfrak{Q} \overset{\text{def}}{=} {}^{\widehat{\mathcal{G}}}\mathcal{Q} = \ \widehat{\mathcal{G}}\mathcal{Q} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathcal{G}_\eta} & \xrightarrow{\subseteq} & \mathcal{G} & \xrightarrow{\subseteq} & \widehat{\mathcal{G}} \end{array} $$

- We are really interested in (a version of) ${}^{\mathcal{G}}\mathcal{Q}$ where we replace $Bun_G(C)$ with a local object $\mathcal{G}$. The effect is that the quasimap is twisted only near one point a.

  The version here means that instead of $\mathcal{G} = \mathcal{G}(G)$ we are using the loop Grassmannian $\mathcal{G}(V(G))$ of the Vinbers semigroup $V(G)$ of G. The connected components of $\mathcal{G}(V(G))$ are just the closures $\overline{\mathcal{G}_\eta}$ of $G_{\mathcal{O}}$-orbits in $\mathcal{G}$. For this reason we are really onterested in the first column rather than the second.
- Actually, we will impose trivialization on $\widehat{\infty}$, i.e., pairs (P, ϕ) in the thick Grassmannian $\widehat{\mathcal{G}} = H^1[\mathbb{P}^1, \widehat{\infty}; G]$ (here denoted $\mathfrak{M}$). The version $\mathfrak{Q} \overset{\text{def}}{=} \ {}^{\widehat{\mathcal{G}}}\mathcal{Q}$ uses $C = \mathbb{P}^1$ and torsors endowed with a trivialization anear∞.

7 The notation is symbolic since there is no map $\mathcal{Q} \to \ {}^{\mathsf{P}}\mathcal{Q}$.

Something of this form is essential for the notion if based twisted quasimaps on $(\mathbb{P}^1,\infty)$ since the twisted quasimap is not twisted near ∞ so the condition $f(\infty)-\mathfrak{b}_-$ makes sense.(8)

10.3.6. *The thick Grassmannaian $\widehat{\mathcal{G}}$.* In 10.4 we notice a technical result. Each connected component $\overline{\mathcal{G}_\eta}$ of the loop Grassmannian of the Vinberg semigroup $V(G)$ has a *canonical* (Kasiwara-Tanisaki) embedding into a finite dimensional smooth variety $\underline{\mathfrak{M}}^\eta$.(9)

10.4. **The thick Grassmannian $\mathfrak{M}=H^1[(\mathbb{P}^1,\widehat{\infty}),G]=\widehat{\mathcal{G}}$ and the embeddings $\overline{\mathcal{G}_\eta}\hookrightarrow\underline{\mathfrak{M}}(\eta)$.** Let $\mathfrak{M}$ be the scheme representing the functor of isomorphism classes of $\mathbf{G}$-torsors on $\mathbb{P}^1$ equipped with trivialization in the formal neighborhood of $\infty\in\mathbb{P}^1$ (see [**?**] and [**?**]).

10.4.1. *Stratification $\mathfrak{M}_\eta=\widehat{\mathcal{G}}^\eta$ by isomorphism types of* $\mathbf{G}$*-torsors.* The scheme $\mathfrak{M}$ is stratified by the locally closed subschemes $\mathfrak{M}_\eta$: $\mathfrak{M}=\sqcup_{\eta\in Y^+}\mathfrak{M}_\eta$ according to the isomorphism types of $\mathbf{G}$-torsors. Namely, due to Riemann's classification, for a $\mathbf{G}$-torsor $\mathcal{T}$ and any $\lambda\in X^+$ the associated vector bundle $\mathcal{V}_\lambda^\mathcal{T}$ decomposes as a direct sum of line bundles $\mathcal{O}(r_k^\lambda)$ of well-defined degrees $r_1^\lambda\geq\ldots\geq r_{\dim V_\lambda}^\lambda$. Then $\mathcal{T}$ lies in the stratum $\mathfrak{M}_\eta$ iff $r_1^\lambda=\langle\eta,\lambda\rangle$.

10.4.2. *A neighborhood $\mathfrak{M}^\eta$ of $\mathfrak{M}_\eta$ and its quotient $\underline{\mathfrak{M}}^\eta$.* For any $\eta\in Y^+$ the union of strata $\mathfrak{M}^\eta:=\sqcup_{Y^+\ni\chi\leq\eta}\mathfrak{M}_\chi$ (the "anticlosure" of $\mathfrak{M}_\eta$) forms an open subscheme of $\mathfrak{M}$. This subscheme is a projective limit of schemes of finite type, all the maps in projective system being fibrations with affine fibers.

Moreover, $\mathfrak{M}^\eta$ is equipped with a free action of a prounipotent group $\mathbf{G}^\eta$ (a congruence subgroup in $\mathbf{G}[[z^{-1}]]$) such that the quotient $\underline{\mathfrak{M}}^\eta$ is a smooth scheme of finite type. The theory of perverse sheaves on $\mathfrak{M}$ smooth along the stratification by $\mathfrak{M}_\eta$ is developed in [**?**]. We will refer the reader to this work, and will freely use such perverse sheaves, e.g. $\mathcal{IC}(\mathfrak{M}_\eta)$.

10.4.3. $\mathbf{i}:\ \mathcal{G}\hookrightarrow\mathfrak{M}$ *and the embeddings $\overline{\mathcal{G}}_\eta\hookrightarrow\mathfrak{M}^\eta\longrightarrow\underline{\mathfrak{M}}^\eta$.* Restricting a trivialization of a $\mathbf{G}$-torsor from $\mathbb{P}^1-0$ to the formal neighborhood of $\infty\in\mathbb{P}^1$ we obtain the closed embedding $\mathbf{i}:\ \mathcal{G}\hookrightarrow\mathfrak{M}$. The intersection of $\mathfrak{M}_\eta$ and $\mathcal{G}_\chi$ is nonempty iff $\eta\leq\chi$, and then it is transversal. Thus, $\overline{\mathcal{G}}_\eta\subset\mathfrak{M}^\eta$. According to [**?**], the composition $\overline{\mathcal{G}}_\eta\hookrightarrow\mathfrak{M}^\eta\longrightarrow\underline{\mathfrak{M}}^\eta$ is a closed embedding.

8 Actually, for this we would only need to trivialize the torsor on the point ∞?

9 First, $\mathcal{G}$ embedds into $\widehat{\mathcal{G}}=\cup_\eta\ \widehat{\mathcal{G}}^\eta$ and $\overline{\mathcal{G}_\eta}$ falls into the "anticlosure" $\mathfrak{m}(\eta)$ of $\mathfrak{M}_\eta$ (the smallst I-invariant neighborhood of $\mathfrak{M}_\eta$). Then $\mathfrak{M}^\eta$ is invariant uder a (free!) action of a prounipotent group $\mathbf{G}^\eta$ (a subgroup of $\mathbf{G}[[z^{-1}]]$) such that the quotient $\underline{\mathfrak{M}}^\eta$ is a smooth scheme of finite type.

10.5. **"Quasimaps" $\mathfrak{Q}^\alpha$ with values in twisted $\mathcal{B}$-bundles as B-torsors.** We twist the flag variety by a G-torsor over a curve C. This gives the spaces $Bun\mathcal{Q} \to Bun_G(C)$ of a G-torsor $\mathcal{T}$ and a quasimap $C \to \mathcal{B}^{\mathcal{T}}$ (really a *quasisection* of $\mathcal{B}^{\mathcal{T}}$).

Here we replace Bun with $\widehat{\mathcal{G}} = H^1[\mathbb{P}^1, \widehat{\infty}; G]$ (denoted $\mathfrak{M}$). This gives a version $\mathfrak{Q}$ of $Bun_G\mathcal{Q}$. [Later we will use $\mathcal{G}$ instead of Bun_G, giving a version $\mathcal{G}\mathcal{Q}$ of $Bun\mathcal{Q}$.]

10.5.1. *A G-torsor $\mathcal{T}$ over C produces a twisted C-form $\mathcal{B}^{\mathcal{T}}$ of the flag variety.* Here, $\mathcal{B}^{\mathcal{T}} \stackrel{\text{def}}{=} \check{T} \times_G \mathcal{B}$.

In Plucker language a $\mathbf{G}$-torsor $\mathcal{T}$ twists an irreducible $\mathbf{G}$-module V_λ into the associated vector bundle $\mathcal{V}_\lambda^{\mathcal{T}}$.

10.5.2. *Quasimaps into twisted $\mathcal{B}$-bundles.* For a G-torsor $\check{T}$ over C (here $\mathbb{P}^1$) we consider the spaces $\overset{\circ}{\mathcal{Q}}{}^\alpha(\mathcal{B}^{\mathcal{T}}) \subseteq \mathcal{Q}^\alpha(\mathcal{B}^{\mathcal{T}})$ of (quasi)maps from C to $\mathcal{B}^{\mathcal{T}}$ of a given degree α.

We start with the version when $\mathcal{T}$ is in $\mathfrak{M}$, i.e., it is trivialized on $\widehat{\infty}$. This gives spaces $\overset{\circ}{\mathfrak{Q}}{}^\alpha \subseteq \mathfrak{Q}^\alpha$ over $\mathfrak{M}$.

In Plucker language we define (as in 3.2 and 3.3) for *arbitrary* $\alpha \in Y$ the scheme $\overset{\circ}{\mathfrak{Q}}{}^\alpha$ (resp. $\mathfrak{Q}^\alpha$) representing the functor of isomorphism classes of pairs $(\mathcal{T}, (\mathfrak{L}_\lambda)_{\lambda \in X^+})$ where

- $\mathcal{T}$ is a $\mathbf{G}$-torsor trivialized in the formal neighborhood of $\infty \in \mathbb{P}^1$, and
- $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda^{\mathcal{T}}$, $\lambda \in X^+$, is a collection of line subbundles (resp. invertible subsheaves) of degree $\langle -\alpha, \lambda \rangle$ satisfying the Plücker conditions (cf. *loc. cit.*).

The evident projection $\overset{\circ}{\mathfrak{Q}}{}^\alpha \longrightarrow \mathfrak{M}$ (resp. $\mathfrak{Q}^\alpha \longrightarrow \mathfrak{M}$) will be denoted by $\overset{\circ}{\mathbf{p}}$ (resp. $\mathbf{p}$). The open embedding $\overset{\circ}{\mathfrak{Q}}{}^\alpha \hookrightarrow \mathfrak{Q}^\alpha$ will be denoted by $\mathbf{j}$. Clearly, $\mathbf{p}$ is projective, and $\overset{\circ}{\mathbf{p}} = \mathbf{p} \circ \mathbf{j}$.

10.5.3. *$\overset{\circ}{\mathfrak{Q}}{}^\alpha$ as B-torsors over C.* $\overset{\circ}{\mathfrak{Q}}{}^\alpha$ is the stack of degree $-\alpha$ $\mathbf{B}$-reductions of $\mathbf{G}$-torsors, trivialized on $\widehat{\infty}$. Equivalently, this is the stack of $\mathbf{B}$-torsors $\mathcal{T}_B$ of degree $-\alpha$ with a trivialization of $(B \hookrightarrow G)_* \mathcal{T}_B$ over $\widehat{\infty}$. Then $\mathfrak{Q}^\alpha$ is its relative compactification.

10.C. Equality of IC stalks on $\mathcal{Z}^\alpha, \mathcal{Q}^\alpha, \mathfrak{Q}$ and a parity vanishing conjecture for stalks

Summary.

10.5.4. *Relation of stratifications and stalks on $\mathcal{Z}^\alpha, \mathcal{Q}^\alpha, \mathfrak{Q}$.* We notice that the (based) quasimap spaces $\mathcal{Z}^\alpha, \mathcal{Q}^\alpha$ as well as the twisted quasimap spaces $\mathfrak{Q}^\alpha$, the IC sheaves are constructible with respect to stratifications indexed by the same data: the type of the defect of a quasimap. Moreover, the IC stalks on the corresponding strata are the same. (However, for $\mathfrak{Q}^\alpha$ these facts are proved later in **??**.]

Remark. For $\mathcal{Q}^\alpha$ the natural stratification also involves the type of the G-torsor, however this turns out not to influence the stalks of $IC(\mathfrak{Q}^\alpha)$.

10.5.5. *A parity vanishing conjecture for stalks.* Some results in the present paper assume this conjecture which is only proved in the second paper SF2.

10.6. **IC-sheaf $\mathcal{IC}(\mathfrak{Q}^\alpha)$.**

10.6.1. *Perverse sheaves on $\mathcal{P}(\mathfrak{Q}^\alpha, \mathfrak{S})$ for a stratification $\mathfrak{S}$ above the stratification $\widehat{G}^\eta$ of* $\widehat{G} = \mathfrak{M}$. The free action of prounipotent group $\mathbf{G}^\eta$ on $\mathfrak{M}^\eta$ lifts to the free action of $\mathbf{G}^\eta$ on the open subscheme $\mathbf{p}^{-1}(\mathfrak{M}^\eta) \subset \mathfrak{Q}^\alpha$.

The quotient is a scheme of finite type $\underline{\mathfrak{Q}}^{\alpha,\eta}$ equipped with the projective morphism $\mathbf{p}$ to $\underline{\mathfrak{M}}^\eta$.

There exists a $\mathbf{G}[[z^{-1}]]$-invariant stratification $\mathfrak{S}$ of $\mathfrak{Q}^\alpha$ such that $\mathbf{p}$ is stratified with respect to $\mathfrak{S}$ and the stratification $\mathfrak{M} = \sqcup_{\eta\in Y^+}\mathfrak{M}_\eta$. One can define perverse sheaves on $\mathfrak{Q}^\alpha$ smooth along $\mathfrak{S}$ following the lines of [**?**].

This gurantees the existence of the IC-sheaf $\mathcal{IC}(\mathfrak{Q}^\alpha)$.

10.7. **The stratification of $\mathfrak{Q}^\alpha$ by type of a G-torsor and defect of a quasimap.** Following 3.4.2 we introduce a decomposition of $\mathfrak{Q}^\alpha$ into a disjoint union of locally closed subschemes according to the isomorphism types of $\mathbf{G}$-torsors and defects of invertible subsheaves:

$$\mathfrak{Q}^\alpha \;=\; \bigsqcup_{\beta\le\alpha}^{\eta\in Y^+} \overset{\circ}{\mathfrak{Q}}{}^\beta_\eta \times C^{\alpha-\beta}$$

where $C = \mathbb{P}^1$ and $\overset{\circ}{\mathfrak{Q}}{}^\beta_\eta = \overset{\circ}{\mathbf{p}}{}^{-1}(\mathfrak{M}_\eta) \subset \overset{\circ}{\mathfrak{Q}}{}^\beta$.
[10]

[10] **Extra**:

The map $\mathbf{p}:\ \mathfrak{Q}^\alpha \longrightarrow \mathfrak{M}$ is stratified with respect to the above stratifications. Note that $\overset{\circ}{\mathbf{p}}:\ \overset{\circ}{\mathfrak{Q}}{}^\alpha \longrightarrow \mathfrak{M}$ is open iff $\alpha \in \mathbb{N}[I]$. In general, let α^+ denote the unique representative of $\mathcal{W}_f$-orbit of $-\alpha$ in Y^+, that is, $\alpha^+ = Y^+ \cap \mathcal{W}_f(-\alpha)$.

Exercise. $\overset{\circ}{\mathbf{p}}(\overset{\circ}{\mathfrak{Q}}{}^\alpha) = \sqcup_{-\alpha\le\eta\le\alpha^+}\mathfrak{M}_\eta$.

(11)

10.8. **The corresponding stalks of IC sheaves are the same for $\mathcal{Z}^\alpha, \mathcal{Q}^\alpha$ and the twisted version $\mathfrak{Q}^\alpha$. Conjecturally they are of one parity.** We notice that $\mathcal{Z}^\alpha$, $\mathcal{Q}^\beta$) a$\mathfrak{Q}^\beta$ have stratifications parametrized by the same data. We notice that the stalks are the same in all three cases.(12) Moreover, there is a *parity vanishing conjecture* for these stalks.

10.8.1. *The stalk $\mathcal{IC}^{\alpha-\beta}_\Gamma$ of $\mathcal{IC}^\alpha$ on $\overset{\circ}{\mathcal{Z}}{}^\gamma \times (\mathbb{C}^*)^{\beta-\gamma}_\Gamma$.* The Goresky-McPherson sheaf $\mathcal{IC}^\alpha$ on $\mathcal{Z}^\alpha$ is smooth along stratification

$$\mathcal{Z}^\alpha = \bigsqcup_{\Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\ge\beta\ge\gamma\ge 0} \overset{\circ}{\mathcal{Z}}{}^\gamma \times (\mathbb{C}^*)^{\beta-\gamma}_\Gamma$$

(cf. 8.1.5). It is evidently constant along strata, so its stalk at a point in $\overset{\circ}{\mathcal{Z}}{}^\gamma \times (\mathbb{C}^*)^{\beta-\gamma}_\Gamma$ depends on the stratum only. Moreover, due to factorization property, it depends not on $\alpha \geq \beta$ but only on their difference $\alpha-\beta \in \mathbb{N}[I]$. We will denote it by $\mathcal{IC}^{\alpha-\beta}_\Gamma$. In case $\mathbf{G} = SL_n$ these stalks were computed in [**?**].

11 **Extra**:

10.7.1. $\alpha+$. Let α_+ denote the minimal dominant coweight such that $-\alpha \le \alpha_+$. For instance, if $\alpha \in \mathbb{N}[I]$ then $\alpha_+ = 0$. According to the above exercise, the stratum $\mathfrak{M}_{\alpha_+}$ is open in $\overset{\circ}{\mathbf{p}}(\overset{\circ}{\mathfrak{Q}}{}^\alpha)$. Hence the stratum $\overset{\circ}{\mathfrak{Q}}{}^\alpha_{\alpha_+}$ is open in $\overset{\circ}{\mathfrak{Q}}{}^\alpha$ (and in $\mathfrak{Q}^\alpha$). To unburden the notations, we will denote $\overset{\circ}{\mathfrak{Q}}{}^\alpha_{\alpha_+}$ by $\widetilde{\mathfrak{Q}}^\alpha$. The open embedding $\widetilde{\mathfrak{Q}}^\alpha \hookrightarrow \mathfrak{Q}^\alpha$ will be denoted by $\tilde{\mathbf{j}}$. The evident projection $\widetilde{\mathfrak{Q}}^\alpha \longrightarrow \mathfrak{M}_{\alpha_+}$ will be denoted by $\tilde{\mathbf{p}}$.

10.7.2. *Lemma.* $\tilde{\mathbf{p}}:\ \widetilde{\mathfrak{Q}}^\alpha \longrightarrow \mathfrak{M}_{\alpha_+}$ is smooth.

Proof. Let $\theta_0 \in X^+$ stand for the highest root, and let $\mathfrak{g} = V_{\theta_0}$ be the adjoint representation of $\mathbf{G}$. For a $\mathbf{G}$-torsor $\mathcal{T}$ we denote the associated adjoint vector bundle on $\mathbb{P}^1$ by $\mathfrak{g}^\mathcal{T} = V^\mathcal{T}_{\theta_0}$. If $\mathcal{T}$ is equipped with a $\mathbf{B}$-reduction $\mathcal{T}_\mathbf{B} \subset \mathcal{T}$, we can consider the associated adjoint vector bundle $\mathfrak{b}^{\mathcal{T}_\mathbf{B}} \subset \mathfrak{g}^\mathcal{T}$. The exact sequence of sheaves $0 \longrightarrow \mathfrak{b}^{\mathcal{T}_\mathbf{B}} \longrightarrow \mathfrak{g}^\mathcal{T} \longrightarrow \mathfrak{g}^\mathcal{T}/\mathfrak{b}^{\mathcal{T}_\mathbf{B}} \longrightarrow 0$ gives rise to the long exact sequence of cohomology:

$$0 \to H^0(C,\mathfrak{b}^{\mathcal{T}_\mathbf{B}}) \to H^0(C,\mathfrak{g}^\mathcal{T}) \to H^0(C,\mathfrak{g}^\mathcal{T}/\mathfrak{b}^{\mathcal{T}_\mathbf{B}}) \to H^1(C,\mathfrak{b}^{\mathcal{T}_\mathbf{B}}) \to H^1(C,\mathfrak{g}^\mathcal{T}) \to H^1(C,\mathfrak{g}^\mathcal{T}/\mathfrak{b}^{\mathcal{T}_\mathbf{B}}) \to 0$$

The cokernel of differential of $\mathbf{p}$ at $\mathcal{T}_\mathbf{B}$ equals $H^1(C,\mathfrak{g}^\mathcal{T}/\mathfrak{b}^{\mathcal{T}_\mathbf{B}})$.

Now suppose $\alpha_+ = 0$, i.e. $\alpha \in \mathbb{N}[I]$. Then $\mathfrak{g}^\mathcal{T}$ is a trivial vector bundle, hence $H^1(C,\mathfrak{g}^\mathcal{T}) = 0$, hence $H^1(C,\mathfrak{g}^\mathcal{T}/\mathfrak{b}^{\mathcal{T}_\mathbf{B}}) = 0$ and $\mathbf{p}$ (and $\tilde{\mathbf{p}}$) is smooth at $\mathcal{T}_\mathbf{B}$.

In general, there is an alternative argument. The stack $\overset{\circ}{\mathfrak{Q}}{}^\alpha$ is smooth, hence its open substack $\widetilde{\mathfrak{Q}}^\alpha$ is smooth as well. All the fibers $\tilde{\mathbf{p}}:\ \widetilde{\mathfrak{Q}}^\alpha \longrightarrow \mathfrak{M}_{\alpha_+}$ are isomorphic. If they were not smooth, $\widetilde{\mathfrak{Q}}^\alpha$ would not be smooth either. □

10.7.3. *Conjecture.* $\overset{\circ}{\mathbf{p}}:\ \overset{\circ}{\mathfrak{Q}}{}^\alpha \longrightarrow \sqcup_{-\alpha\le\eta\le\alpha^+}\mathfrak{M}_\eta$ is smooth.

12 The "reason" that the stalks do not depend on position in $\widehat{\mathcal{G}}$ is that $\widehat{\mathcal{G}}$ is smooth. This will change later when we replace $\widehat{\mathcal{G}}$ with $\mathcal{G}$.

10.8.2. *The stalks of $\mathcal{IC}(\mathcal{Q}^\beta)$ are the same as for $\mathcal{IC}(\mathcal{Z}^\alpha)$.* Recall (see 3.4.2) that $\mathcal{Q}^\beta,\ \beta \in \mathbb{N}[I]$, is stratified by the type of defect:

$$\mathcal{Q}^\beta = \bigsqcup_{\Gamma\in\mathfrak{P}(\beta-\gamma)}^{\beta\geq\gamma\geq 0} \overset{\circ}{\mathcal{Q}}{}^\gamma \times C_\Gamma^{\beta-\gamma}$$

The Goresky-McPherson sheaf $\mathcal{IC}(\mathcal{Q}^\beta)$ on $\mathcal{Q}^\beta$ is constant along the strata. It is immediate to see that its stalk at any point in the stratum $\overset{\circ}{\mathcal{Q}}{}^\gamma \times C_\Gamma^{\beta-\gamma}$ is isomorphic, up to a shift, to $\mathcal{IC}_\Gamma^0$. In particular, it depends on the defect only.

10.8.3. *The stalks of $\mathcal{IC}(\mathfrak{Q}^\beta)$ are the same as for $\mathcal{Z}^\alpha$ and $\mathcal{Q}^\alpha$.* .

Proposition. a) The Goresky-McPherson sheaf $\mathcal{IC}(\mathfrak{Q}^\beta)$ on $\mathfrak{Q}^\beta,\ \beta \in Y$, is constant along the locally closed subschemes

$$\mathfrak{Q}^\beta = \bigsqcup_{\Gamma\in\mathfrak{P}(\beta-\gamma)}^{\beta\geq\gamma} \overset{\circ}{\mathfrak{Q}}{}^\gamma \times C_\Gamma^{\beta-\gamma}$$

b) The stalk of $\mathcal{IC}(\mathfrak{Q}^\beta)$ at any point in the $\overset{\circ}{\mathfrak{Q}}{}^\gamma \times C_\Gamma^{\beta-\gamma}$ is isomorphic, up to a shift, to $\mathcal{IC}_\Gamma^0$.

Proof. Will be given in **??**. □

10.9. **The parity vanishing conjecture.** Let $\phi \in \mathfrak{Q}^\beta$. The stalk $\mathcal{IC}(\mathfrak{Q}^\beta)_\phi$ is a graded vector space.

Conjecture. *(Parity vanishing)* Nonzero graded parts of $\mathcal{IC}(\mathfrak{Q}^\beta)_\phi$ appear in cohomological degrees of the same parity.

10.9.1. *Remark.* In case $\mathbf{G} = SL_n$ the conjecture follows from the Proposition 10.8.3 and [**?**] 2.5.2. In the general case the conjecture follows from the unpublished results of G.Lusztig. The proof is written in the paper SF2.

11. The convolution diagrams $\mathcal{GQ}^\alpha_\eta$ $(\alpha \in Y, \eta \in Y^+)$, (space of quasimaps into twisted $\mathcal{B}$-bundles)

Summary.

11.0.1. *Intuition: The group theoretic origin of the convolution.* The idea is that on $\boldsymbol{\mathcal{S}}$ which is supposed to be some version of $G_\mathcal{K}/N_\mathcal{K}$, we consider sheaves with I and $G_\mathcal{O}$-equivariance and they carry the convolution actions of $\mathcal{P}_{I^2}(G_\mathcal{K}) = \mathcal{P}_I(\mathcal{F})$ and $\mathcal{P}_{G_\mathcal{O}{}^2}(G_\mathcal{K}) = \mathcal{P}_{G_\mathcal{O}}(\mathcal{G})$. The version that we consider is the "pairing""

$$\mathcal{P}_I(\mathcal{G}) \times \mathcal{P}_{G_\mathcal{O}}(\boldsymbol{\mathcal{S}}) \rightarrow \mathcal{P}_I(\boldsymbol{\mathcal{S}}).$$

It is based on the correspondence

$$I\backslash G_\mathcal{K}/G_\mathcal{O} \times G_\mathcal{O}\backslash\boldsymbol{\mathcal{S}} \leftarrow I\backslash G_\mathcal{K} \times_{G_\mathcal{O}} \backslash\boldsymbol{\mathcal{S}} \rightarrow \boldsymbol{\mathcal{S}}$$

which is an $I\backslash\mathcal{G}$-bundle over $\boldsymbol{\mathcal{S}}$.

Here, $\boldsymbol{\mathcal{S}}$ (or really two versions of it) are realized as spaces of (based) quasimaps $\mathcal{Z}$ and $\mathcal{Q}$.

11.0.2. *The correspondence* $\overline{\mathcal{G}_\eta} \overset{\mathbf{p}}{\leftarrow} Q\Gamma^\alpha[\mathbb{P}^1, {}^{\overline{\mathcal{G}_\eta}}\mathcal{B}] \overset{\mathbf{q}}{\rightarrow} \mathcal{Q}^{\alpha+\eta} = Q\Gamma^{\alpha+\eta}(\mathbb{P}^1, \mathcal{B})$. In 11.2 one defines $\mathbf{q}$ as the *η-untwisting map*. The idea is that $\overline{\mathcal{G}_\eta}$ is a connected component of the loop Grassmannian $\mathcal{G}(V(G))$ for the Vinberg semigroup $V(G)$. The $-\eta$ twist in the direction of the Cartan group $H_a \subseteq V^*(G) = G \times_{Z(G)} H_a$ takes (in a certain sense) any torsor $\mathsf{P} \in \overline{\mathcal{G}_\eta}$ into the trivial G-torsor (?) and the P-twisted quasimaps of degree α into ordinary quasimaps of degree $\alpha + \eta$.

The local version of the convolution diagram... In

xxx

11.1. Moduli ${}^{\overline{\mathcal{G}_\eta}}\mathcal{Q}^\alpha = \mathcal{GQ}^\alpha_\eta$ of quasimaps of degree α twisted by a torsor in $\overline{\mathcal{G}_\eta}$.

For $\alpha \in Y$ and a dominant cocharacter $\eta \in Y^+ \subset Y$ we define the *convolution diagram* $\mathcal{GQ}^\alpha_\eta$.

11.1.1. *The Plucker definition of* ${}^{\overline{\mathcal{G}_\eta}}\mathcal{Q}^\alpha = \mathcal{GQ}^\alpha_\eta$. This is the space of collections $(\mathcal{U}_\lambda, \mathfrak{L}_\lambda)_{\lambda \in X^+}$ of vector bundles with invertible subsheaves such that

a) $(\mathcal{U}_\lambda)_{\lambda \in X^+} \in \overline{\mathcal{G}}_\eta$, or in other words, $(\mathcal{U}_\lambda)_{\lambda \in X^+}$ satisfies the conditions 10.1.2 a)-c);

b) $\mathfrak{L}_\lambda \subset \mathcal{U}_\lambda$ has degree $-\langle\alpha, \lambda\rangle$;

c) For any surjective $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ we have (cf. 10.1.2 c) $\phi(\mathcal{L}_\lambda \otimes \mathcal{L}_\mu) = \mathcal{L}_\nu$;

d) For any $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\phi(\mathcal{L}_\lambda \otimes \mathcal{L}_\mu) = 0$.

11.1.2. *The open part $\overset{\circ}{\mathcal{G}\mathcal{Q}}{}^{\alpha}_{\eta}$ given by twisted maps.* Let us denote by $\overset{\circ}{\mathcal{G}\mathcal{Q}}{}^{\alpha}_{\eta}$ the open subvariety in $\mathcal{G}\mathcal{Q}^{\alpha}_{\eta}$ formed by all the collections $(\mathcal{U}_\lambda, \mathfrak{L}_\lambda)$ such that $\mathfrak{L}_\lambda$ is a *line subbundle* in $\mathcal{U}_\lambda$ for any $\lambda \in X^+$. The open embedding $\overset{\circ}{\mathcal{G}\mathcal{Q}}{}^{\alpha}_{\eta} \hookrightarrow \mathcal{G}\mathcal{Q}^{\alpha}_{\eta}$ will be denoted by $\mathbf{j}$.

11.2. **The correspondence $\overline{\mathcal{G}}_\eta \leftarrow \ \mathcal{G}\mathcal{Q}^{\alpha}_{\eta} \rightarrow \ \mathcal{Q}^{\alpha+\eta}$.** Here it will be important that we are not using all of $\mathcal{G}$ but just a finite piece $\overline{\mathcal{G}_\eta}$. This allows us to untwist the twisted quasimaps by the shift $(-\langle\eta,\lambda\rangle 0)$ which works for shifts from $\overline{\mathcal{G}_\eta}$.

11.2.1. *Twists.* Recall from 10.1.2 that the Plucker realization of the orbit closure $\overline{\mathcal{G}}_\eta \subset \mathcal{G}$ is the space of collections $(\mathcal{U}_\lambda)_{\lambda\in X^+}$ of vector bundles on $\mathbb{P}^1$ such that

- a) $\mathcal{U}_\lambda$ differs from the constant vector bundle $\mathcal{V}_\lambda$ by at most the $\pm\langle\eta,\lambda\rangle 0$ twist:
$$\mathcal{U}_\lambda(-\langle\eta,\lambda\rangle 0) \subset \mathcal{V}_\lambda \subset \mathcal{U}_\lambda(\langle\eta,\lambda\rangle 0).$$
- b) The degree of $\mathcal{U}_\lambda$ is the same as for $\mathcal{V}_\lambda$, i.e., 0. (Equivalently, $\dim \mathcal{V}_\lambda(\langle\eta,\lambda\rangle 0)/\mathcal{U}_\lambda = \langle\eta,\lambda\rangle \dim V_\lambda$).
- c) $\mathcal{U}_\lambda$'s satisfy a Plucker condition:
  For any surjective G-morphism $\phi: \ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ and the corresponding morphism $\phi: \ \mathcal{V}_\lambda \otimes \mathcal{V}_\mu \longrightarrow \mathcal{V}_\nu$ (hence $\phi: \ \mathcal{V}_\lambda(\langle\eta,\lambda\rangle 0) \otimes \mathcal{V}_\mu(\langle\eta,\mu\rangle 0) \longrightarrow \mathcal{V}_\nu(\langle\eta,\lambda+\mu\rangle 0))$ we have $\phi(\mathcal{U}_\lambda \otimes \mathcal{U}_\mu) = \mathcal{U}_\nu$.

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