Homological algebra, Homework 9

♡

Spectral sequences

The first filtration of the total complex of a bicomplex For a bicomplex (B, d', d'') consider the first decreasing filtration $'F$ by sub-bicomplexes $'F_iB$ where

$$('F_iB)^{pq} = \left\{ \begin{smallmatrix} B^{pq} & if & p\geq 0, \\ 0 & otherwise. \end{smallmatrix} \right\}$$

(on erases the part of B which is on the left from the i^{th} column). The induced filtration on $Tot(B)$ is

$$['F_iTot(B)]^n \stackrel{\text{def}}{=} Tot('F_iB)^n = \oplus_{p+q=n,\ p\geq i} B^{p,q} \subseteq Tot(B)^n = \oplus_{p+q=n} B^{p,q}.$$

The induced filtration on cohomology is

$$'F_p\mathrm{H}^n(Tot\ B) \stackrel{\text{def}}{=} \mathrm{Im}[\mathrm{H}^n(Tot['F_pB]) \to \mathrm{H}^n(Tot[B])],$$

therefore, the cohomology groups are extensions of the graded pieces

$$'Gr_i[\mathrm{H}^n(Tot\ B)] \stackrel{\text{def}}{=} \frac{'F_i\mathrm{H}^n(Tot\ B)}{'F_{i+1}\mathrm{H}^n(Tot\ B)}.$$

Spectral sequence of the first filtration.

Problem 1. Use the first filtration on $Tot(B)$ to construct a spectral sequence $'E$ such that

(1) $'E_0^{p,q} = B^{p,q}$
(2) $'E_1^{p,q} = {''\mathrm{H}}^{p,q}(B)$
(3) $'E_2^{p,q} = {'\mathrm{H}}^p[{''\mathrm{H}}^{\bullet,q}(B)]$
(4) If B is a first quadrant bicomplex (i.e., $B^{pq} = 0$ unless $p, q \geq 0$, then $'E$ is a first quadrant spectral sequence converging to

$$'E_\infty^{p,q} = Gr_p[\mathrm{H}^{p+q}(Tot\ B)].$$

Remark. The lecture notes construct a spectral sequence for any filtration. Here, you are asked to apply this construction to the special case when the complex is of the form $Tot(B)$ and the filtration comes from the first filtration of the bicomplex B.

Problem 2. What is the formula for $'E_r^{pq}$?

Problem 3. Show that if $'E$ degenerates at $'E_2$ then we recover the constituents of $\mathrm{H}^n(Tot\ B)$ from partial cohomologies of B by

$$Gr_\bullet[\mathrm{H}^n(Tot\ B)] \cong \oplus_{p+q=n} {'\mathrm{H}}^p[{''\mathrm{H}}^{\bullet,n-p}(B).$$