Homological algebra, Homework 8

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Bicomplexes and resolutions of complexes

Bicomplexes. A *bicomplex* in $\mathcal{A}$ is a bigraded object $B = \oplus_{p.q\in\mathbb{Z}} B^{p,q}$ with two differentials $B^{p,q} \xrightarrow{d'} B^{p+1,q}$ and $B^{p,q} \xrightarrow{d''} B^{p,q+1}$, such that $d = d' + d''$ is also a differential. We draw a bicomplex as a two dimensional object:

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow \\
\cdots & \xrightarrow{d'} & B^{-1,2} & \xrightarrow{d'} & B^{0,2} & \xrightarrow{d'} & B^{1,2} & \xrightarrow{d'} & B^{2,2} & \xrightarrow{d'} & B^{3,2} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow \\
\cdots & \xrightarrow{d'} & B^{-1,1} & \xrightarrow{d'} & B^{0,1} & \xrightarrow{d'} & B^{1,1} & \xrightarrow{d'} & B^{2,1} & \xrightarrow{d'} & B^{3,1} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow \\
\cdots & \xrightarrow{d'} & B^{-1,0} & \xrightarrow{d'} & B^{0,0} & \xrightarrow{d'} & B^{1,0} & \xrightarrow{d'} & B^{2,0} & \xrightarrow{d'} & B^{3,0} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow \\
\cdots & \xrightarrow{d'} & B^{-1,-1} & \xrightarrow{d'} & B^{0,-1} & \xrightarrow{d'} & B^{1,-1} & \xrightarrow{d'} & B^{2,-1} & \xrightarrow{d'} & B^{3,-1} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow & & {\scriptstyle d''}\Big\uparrow \\
\cdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \cdots .
\end{array}$$

So, B^{pq} has horizontal position p and height q, and d' is a horizontal differential while d'' is a vertical differential.

Problem 1. Show that for the horizontal differential d' and the vertical differential d'', $d = d' + d''$ is a differential iff d', d'' "anticommute", i.e., $d'd'' + d''d' = 0$. □

The cohomology of a bicomplex. The total complex of a bicomplex is the complex $(Tot(B), d)$ with $Tot(B)^n \stackrel{\text{def}}{=} \oplus_{p+q=n} B^{p,q}$. Its cohomology is called the *cohomology of the bicomplex* B.

Partial cohomologies. By taking the "horizontal" cohomology of B we obtain a bigraded object $'\mathrm{H}(B)$ with

$$'\mathrm{H}(B)^{p.q} \stackrel{\text{def}}{=} \mathrm{H}^p(B^{\bullet,q}) = \frac{\mathrm{Ker}(B^{p,q} \xrightarrow{d'} B^{p+1.q})}{Im(B^{p-1,q} \xrightarrow{d'} B^{p.q})}.$$

Problem 2. The vertical differential d'' on B factors to a differential on $'\mathrm{H}(B)$ which we denote again by d'':

$$'\mathrm{H}(B)^{p,q} \xrightarrow{d''} {}'\mathrm{H}(B)^{p,q+1}.$$

$\square$

Remark. Now we can take the "vertical" cohomology of $'\mathrm{H}(B)$ (i.e., with respect to the <u>new</u> d''), and get a bigraded object $''\mathrm{H}('\mathrm{H}(B))$ with

$$''('\mathrm{H}(B))^{p,q} \stackrel{\text{def}}{=} \mathrm{H}^q('\mathrm{H}(B)^{p,\bullet}) = \frac{\mathrm{Ker}['\mathrm{H}(B)^{p,q} \xrightarrow{d''} {}'\mathrm{H}(B)^{p,q+1}]}{Im['\mathrm{H}(B)^{p,q-1} \xrightarrow{d''} {}'\mathrm{H}(B)^{p,q}]}.$$

One defines $''\mathrm{H}(B)$ and $'\mathrm{H}(''\mathrm{H}(B))$ by switching the roles of the first and second coordinates.

Resolutions of complexes. We say that a *right resolution* I of a complex $A \in C(\mathcal{A})$ is any quasi-isomorphism $A \to I$. An *injective resolution* of a complex $A \in C(\mathcal{A})$ is a right resolution $A \to I$ such that all I^n are injective objects of the abelian category $\mathcal{A}$.

A *right bicomplex resolution* of a complex A is a bicomplex $I^{\bullet,\bullet}$ with $I^{pq} = 0$ for $p < 0$, and a map of complexes $\varepsilon : A \to I^{\bullet,0}$ such that in the following diagram

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & I^{-1,2} & \xrightarrow{d'} & I^{0,2} & \xrightarrow{d'} & I^{1,2} & \xrightarrow{d'} & I^{2,2} & \xrightarrow{d'} & I^{3,2} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & I^{-1,1} & \xrightarrow{d'} & I^{0,1} & \xrightarrow{d'} & I^{1,1} & \xrightarrow{d'} & I^{2,1} & \xrightarrow{d'} & I^{3,1} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & I^{-1,0} & \xrightarrow{d'} & I^{0,0} & \xrightarrow{d'} & I^{1,0} & \xrightarrow{d'} & I^{2,0} & \xrightarrow{d'} & I^{3,0} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & A^{-1} & \xrightarrow{d'} & A^{0} & \xrightarrow{d'} & A^{1} & \xrightarrow{d'} & A^{2} & \xrightarrow{d'} & A^{3} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & 0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & \cdots
\end{array}$$

the columns are resolutions of terms in the complex A. A right bicomplex resolution I s said to be *injective* if all terms I^{pq} are injective.

Problem 3. [Resolutions of complexes.] Let $\mathcal{A}$ be an abelian category with enough injectives. Then any $A \in C^+(\mathcal{A})$ has an injective resolution. More precisely,

(a) Any $A \in C(\mathcal{A})$ has an injective bicomplex resolution I.

(b) Such resolution can be chosen to be "split" in the sense that for

$$0 \to B^n(A) \to Z^n(A) \to H^n(A) \to 0$$

there exist injective resolutions $\mathcal{B}^n, \mathcal{H}^n, \mathcal{Z}^n$ of B^n, H^n, Z^n such that $\mathcal{Z}^n \cong \mathcal{B}^n \oplus \mathcal{H}^n$ and $I^{n,\bullet} \cong \mathcal{Z}^n \oplus \mathcal{B}^{n+1}$.

(c) If $A \in C^+(\mathcal{A})$, for any split injective bicomplex resolution (I, ε) of A, the canonical map $A \xrightarrow{\tilde{\varepsilon}} Tot(I)$ is an injective resolution of A. $\square$