Homological algebra, Homework 7

♡

Sheaves

1. Direct image of sheaves. Let $X \xrightarrow{\pi} Y \xrightarrow{\sigma} Z$ be maps of topological spaces.

(a) Let X be a topological space and $a : X \to pt$. Show that the functor of global sections $\Gamma(X, -)$ that takes a sheaf $\mathcal{F}$ to $\Gamma(X, \mathcal{F}) \stackrel{\text{def}}{=} \mathcal{F}(X)$ can be identified with the direct image functor a_*.

(b) Let $X \xrightarrow{\pi} Y \xrightarrow{\sigma} Z$ be maps of topological spaces. Show that for a sheaf $\mathcal{M}$ on X,

$$\sigma_*(\pi_*(\mathcal{M})) \cong (\sigma \circ \pi)_* \mathcal{M}.$$

Ext functors

2. Show that for any ring A and any A-modules M, N,

(1) the functor $\mathrm{Hom}_A(-, M) : \mathfrak{m}(A)^o \to \mathcal{A}b$ is ? exact.
(2) the functor $\mathrm{Hom}_A(N, -) : \mathfrak{m}(A) \to \mathcal{A}b$ is ?? exact.

0.0.1. *Remark.* Their derived functors are called Ext-functors :

$$'\mathrm{Ext}^i_{\Bbbk}(N, M) \stackrel{\text{def}}{=} L^i\mathrm{Hom}(-, M)\ (N) \quad \text{and} \quad ''\mathrm{Ext}^i_{\Bbbk}(N, M) \stackrel{\text{def}}{=} R^i\mathrm{Hom}(N, -)\ (M).$$

We will prove in class that $'Ext ='' Ext$.

Group cohomology III

3. Extensions of groups. Let $(G, \cdot)$ be a group and let abelian group $(A, +)$ be a module for G and let $f \in Z^2(G, A)$ be a two cocycle.

Show that

(1) $G \ni g \mapsto f(1, g) \in A$ is constant, call this constant α.
(2) On the set $E = A \times G$ define operation $*$ by

$$(a, g) * (b, h) \stackrel{\text{def}}{=} (a +^g b + f(g, h), gh).$$

Show that E is a group with the unit $e \stackrel{\text{def}}{=} (\alpha^{-1}, 1_G)$.

(3) Show that there is a canonical group extension.

$$0 \to A \xrightarrow{i} E \xrightarrow{q} G \to 0$$

with a canonical section σ.

0.0.2. *Remarks.* This is really the inverse of the map

$$\textit{Equivalence classes of extension of } G \textit{ by } A \quad \to \quad H^2(G,A),$$

that has been constructed in the preceding homework. So, the meaning of $H^2(G,A)$. is that it is the set of isomorphism classes of extensions of G by a G-module A.

4. Group cohomology as an Ext-functor. Let G be a group and $\Bbbk = \mathbb{Z}[G]$ the group algebra. Show that for any G-module M

(1) $H^i(G,M) \cong \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z},M)$.
(2) Functors $H^i(G,-)$ are the right derived functors of $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},-)$ which is isomorphic to the functor of G-invariants.

0.0.3. *Remarks.* (0) [Hint.] For the first claim recall that

$$\mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z},M) = \ {}'\mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z},M) = \ L^i\mathrm{Hom}_{\mathbb{Z}[G]}(-,M)\ (\mathbb{Z})$$

so it can be calculated using *any* free resolution of the trivial G-module $\mathbb{Z}$. For the second claim recall that ${}'\mathrm{Ext}^i = \ {}''\mathrm{Ext}^i$.

(1) Similarly, the *group homology* can be defined by

$$H_i(G,M) \stackrel{\text{def}}{=} \mathrm{Tor}^i_{\mathbb{Z}[G]}(\mathbb{Z},M),$$

meaning, the left derived functor of of the functor of G-*coinvariants*

$$M \mapsto \mathbb{Z} \otimes_{\mathbb{Z}[G]} M \ \cong \ \frac{M}{\sum_{g\in G}\ (g-1)M}.$$

5. Group cohomology for cyclic groups. Let G be a finite cyclic group and choose some generator γ. Let $N = \sum_{g\in G}\ g \ \in \ \mathbb{Z}[G]$.

(1) Show that the following is a free resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$

$$\cdots \xrightarrow{\gamma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\gamma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\gamma-1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \to 0 \to \cdots$$

(2) Show that for any G-module A
   (a) $H^0(G,A) = \ A^G$.
   (b) $H^{2i-1}(G,A) \ \cong \ \frac{\mathrm{Ker}(A \xrightarrow{N} A)}{(\gamma-1)A}$.
   (c) $H^{2i}(G,A) = \ \frac{A^G}{N\cdot A}$ for $i > 0$.
(3) Calculate $H^\bullet(G,\mathbb{Z})$.