Homological algebra, Homework 6

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Koszul complex II

1. Koszul complex. Let V be a vector space over a field $\Bbbk$. Let $I = V{\cdot}S(V)$ be the ideal in SV generated by V. Prove that

(1) There is a canonical isomorphism $SV)/I \cong \Bbbk$. It makes $\Bbbk$ into a module for $S(V)$.
(2) If we extend the Koszul complex for V to a sequence of maps

$$\cdots \xrightarrow{d^{-k-1}} \overset{k}{\wedge} V \otimes S(V) \xrightarrow{d^{-k}} \cdots \xrightarrow{d^{-3}} \overset{2}{\wedge} V \otimes S(V) \xrightarrow{d^{-2}} \overset{1}{\wedge} V \otimes S(V) \xrightarrow{d^{-1}} \overset{0}{\wedge} V \otimes S(V) \xrightarrow{d^{0}} \overset{0}{\wedge} \Bbbk \to 0 \to \cdots$$

where d^0 is the quotient map $\overset{0}{\wedge} V \otimes S(V) = \Bbbk \otimes S(V) \cong S(V) \twoheadrightarrow S(V)/I \cong \Bbbk$. This is a complex, call it $\mathcal{K}$.
(3) The complex $\mathcal{K}$ is homotopically equivalent to 0.
  (a) Koszul complex K is a sum of complexes $K = \oplus_{n\geq 0}\ K(n)$ where $K(n)^i = \wedge^i V \otimes S^{n-i}V$, i.e., the "total degree" of the two factors is n; in the sense that $K^i = \oplus_i\ K(n)^i$ and the differential d_K preserves each $K(n)$.
  (b) Let $\varepsilon : K^i \to K^{i-1}$ by

$$\varepsilon(v_1 \wedge \cdots \wedge v_p \otimes u_1 \cdots u_n) = \sum_i v_1 \wedge \cdots \wedge v_p \wedge u_i \otimes u_1 \cdots \widehat{u_i} \cdots u_n.$$

  Show that ε preserves each $K(n)$ and on $K(n)$

$$\varepsilon d + d\varepsilon = n.$$

  (c) Let $\mathcal{K}(n) = K(n)$ for $n > 0$ while $\mathcal{K}^0 = (S^0 V \to 0)$ in degrees 0 and 1. For $n > 0$ define $h(n)$ on $\mathcal{K}(n)$ as ε/n and show that it makes $\mathcal{K}(n)$ homotpically equivalent to zero.
  (d) The complex $\mathcal{K}$ is homotopically equivalent to 0.
(4) The Koszul complex is a free resolution of the $S(V)$-module $\Bbbk$.

0.0.1. *Remarks.* (0) If U is the dual vector space V^* then $S(V)$ is the algebra $\mathcal{O}(U)$ of polynomial functions on U and I is the kernel of the evaluation map $ev_0 : \mathcal{O}(U) \to \Bbbk,\ f \mapsto f(0)$. The isomorphism $S(V)/I \xrightarrow{\cong} \Bbbk$ is a factorization of the evaluation map and $S(V)$-module $\Bbbk$ can be viewed as the ring of polynomial functions on the point 0 in U. Now Koszul complex is a way to capture a geometric subobject – point $0 \in U$ – in terms of free modules for the ring of functions on U.

(1) The heart of the problem is to find a homotopy $h^n : \mathcal{K}^n \to \mathcal{K}^{n-1},\ n \in \mathbb{Z}$, such that $dh + hd = 1_{\mathcal{K}}$. Since h decreases the degree of our polynomial functions one is tempted to use derivations. Actually, ε above is the *Euler vector field* $\sum\ x^i \partial_{x^i}$ – the derivative of the the action of the multiplicative group $\Bbbk^*$ on U.

(2) Notice that ε is "the same as d" once we switch the roles of the symmetric and exterior algebra ("except for signs"). This is made precise in *super mathematics* (mathematics with consistent use of a "sign rule").

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Sheafification

Recall that to a presheaf $\mathcal{S}$ on a topological space X we have associated a sheaf $\widetilde{\mathcal{S}}$.

2. Sheafification functor.

(1) Construct a canonical map of presheaves $\mathcal{S} \xrightarrow{q_{\mathcal{S}}} \widetilde{\mathcal{S}}$.
(2) Show that the sheafification construction $\mathcal{S} \to \widetilde{\mathcal{S}}$ is a functor from presheaves on X to sheaves on X

$$pre\mathcal{S}heaves \ni \mathcal{S} \mapsto \widetilde{\mathcal{S}} \in \mathcal{S}heaves.$$

(3) In the light of a categorical framework in (2), what kind of animal is q?

3. Sheafification as a left adjoint. Prove hat the sheafification functor $pre\mathcal{S}heaves \ni \mathcal{S} \mapsto \widetilde{\mathcal{S}} \in \mathcal{S}heaves$, is the left adjoint of the inclusion $\mathcal{S}heaves \subseteq pre\mathcal{S}heaves$, i.e, for any presheaf $\mathcal{S}$ and any sheaf $\mathcal{F}$ there is a natural identification

$$\iota_{\mathcal{S}} : \ \mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}}, \mathcal{F}) \xrightarrow{\cong} \mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S}, \mathcal{F}).$$

Explicitly, the bijection is given by $(\iota_{\mathcal{S}})_* \alpha = \ \alpha \circ \iota_{\mathcal{S}}$, i.e., $(\widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F}) \mapsto (\mathcal{S} \xrightarrow{\iota_{\mathcal{S}}} \widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F})$.

4. Direct image of sheaves. Let $X \xrightarrow{\pi} Y$ be a map of topological spaces.

(a) Show that for a sheaf $\mathcal{M}$ on X, formula

$$\pi_*(\mathcal{M}) \ (V) \overset{\text{def}}{=} \mathcal{M}(\pi^{-1}V),$$

defines a sheaf $\pi_* \mathcal{M}$ on Y.

(b) Show that this gives a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

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Group cohomology

An *extension of groups* is a short exact sequence of groups $0 \to A \xrightarrow{i} E \xrightarrow{q} G \to 0$ with A abelian. (We call it an *extension of G by A*.) We say that

- Extension *splits* if there is a map of groups $s : G \to E$ such that $q \circ s = 1_G$.
- Two extensions $0 \to A \xrightarrow{i} E_p \xrightarrow{q} G \to 0$, $p = 1, 2$, are *equivalent* if there is an isomorphism $\phi : E_1 \to E_2$ such that the diagram

$$\begin{array}{ccccc} 0 \to A & \xrightarrow{i} & E_1 & \xrightarrow{q} & G \to 0 \\ {\scriptstyle =}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle =}\downarrow \\ 0 \to A & \xrightarrow{i} & E_2 & \xrightarrow{q} & G \to 0 \end{array}$$

commutes (i.e., extension groups E_i are isomorphic in a way that is compatible with relations to A and G).

5. Let $0 \to A \xrightarrow{i} E_p \xrightarrow{q} G \to 0$ be a group extension. Show that

(1) A is naturally a G-module.
(2) Any set theoretic section $\sigma : G \to E$ defines a two cocycle $\widetilde{\sigma} \in Z^2(G, A)$ by

$$\widetilde{\sigma}(g, h) \stackrel{\text{def}}{=} \sigma(g)\sigma(h)\sigma(gh)^{-1}.$$

In other words

$$^{a}\widetilde{\sigma}(b, c) - \widetilde{\sigma}(ab, c) + \widetilde{\sigma}(a, bc) - \widetilde{\sigma}(a, b) = 0.$$

(3) (a) Two sections σ, τ differ by a function $\beta : G \to A$ by $\tau(g) = \beta(g)\sigma(g)$.
(b) If $\tau = \beta\sigma$ as above then $\widetilde{\tau} - \widetilde{\sigma} = d\beta \in B^2(G, A)$.
(4) An extension of G by A defines a class in $H^2(G, A)$.
(5) Equivalent extensions define the same class in $H^2(G, A)$.

0.0.2. *Remarks.* (0) Here we use multiplicative notation in G and E, but additive in A to emphasize that it is abelian.[1]

(1) Elements of cohomology groups are often called classes to indicate that these are cosets of Z^i modulo B^i.

[1] You can also use the multiplicative notation in A if it is less confusing.